Summary of lectures of the fourth week.

- If the position vector of a moving particle at time t is $\vec{r}(t)$, then

  ① velocity: $\vec{v}(t) = \vec{r}\,'(t)$.

  ② acceleration: $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

  ③ Speed : $s(t) = \|\vec{v}(t)\| = \|\vec{r}\,'(t)\|$.

  ④ Total distance traveled over $a \le t \le b$ is

$$\int_a^b s(t)\,dt = \int_a^b \|\vec{r}\,'(t)\|\,dt .$$

- Length of a curve with vector parametrization

$$\vec{r}(t), \qquad a \le t \le b,$$

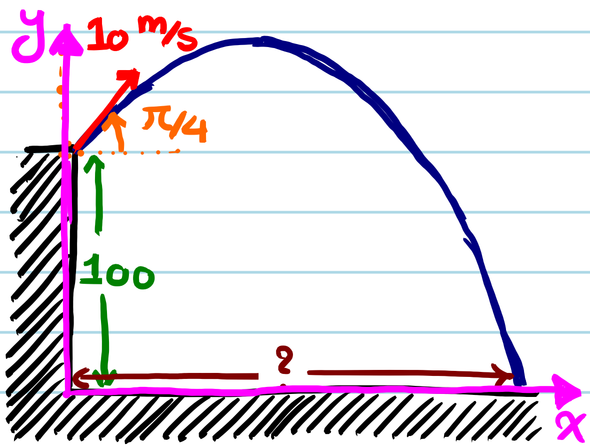
is $\displaystyle\int_a^b \|\vec{r}\,'(t)\|\,dt$.

- Using the fundamental theorem of calculus, we have

  ① $\displaystyle \vec{r}(t) - \vec{r}(t_0) = \int_{t_0}^{t} \vec{v}(u)\,du$

  ② $\displaystyle \vec{v}(t) - \vec{v}(t_0) = \int_{t_0}^{t} \vec{a}(u)\,du$

Exp. A stone is thrown from top of a cliff with the initial speed $10\ m/s$ and the initial angle $\frac{\pi}{4}$. The height of the cliff is $100\ m$. Where does the stone hit the ground?



Solution.

By Newton's law we know that the acceleration of this particle is constant

$$\vec{a}(t) = \langle 0, -g \rangle$$

where $g \approx 10\ m/s^2$.

Therefore $\vec{v}(t) - \vec{v}(0) = \int_0^t \langle 0, -g \rangle \, du$

$$= \langle \int_0^t 0 \, du, -\int_0^t g \, du \rangle$$

$$= \langle 0, -gu \Big|_0^t \rangle$$

$$= \langle 0, -gt \rangle.$$

$$\vec{v}(0) = \|\vec{v}(0)\| \langle \cos\theta, \sin\theta \rangle$$

$$= S(0)\,\langle \cos\theta, \sin\theta\rangle$$

$$= 10\,\langle \cos \pi/4 , \sin \pi/4\rangle = \langle 5\sqrt{2}, 5\sqrt{2}\rangle.$$

Hence $\vec{v}(t) = \langle 5\sqrt{2}, 5\sqrt{2}\rangle + \langle 0, -gt\rangle$

$$= \langle 5\sqrt{2}, 5\sqrt{2} - gt\rangle.$$

So $\vec{r}(t) - \vec{r}(0) = \int_0^t \vec{v}(u)\,du$

(Warning: here we are using the dummy variable $\underline{\underline{u}}$. Do not forget to write the function in terms of u.)

$$= \int_0^t \langle 5\sqrt{2}, 5\sqrt{2} - gu\rangle\,du$$

$$= \langle \int_0^t 5\sqrt{2}\,du, \int_0^t 5\sqrt{2} - gu\,du\rangle$$

$$= \langle 5\sqrt{2}\,u\Big|_0^t, \left(5\sqrt{2}u - \frac{gu^2}{2}\right)\Big|_0^t\rangle$$

$$= \langle 5\sqrt{2}\,t,\ 5\sqrt{2}\,t - \tfrac{1}{2}gt^2\rangle$$

$\vec{r}(0) = \langle 0, 100\rangle.$

So $\vec{r}(t) = \langle 5\sqrt{2}\,t,\ 100 + 5\sqrt{2}\,t - 5t^2\rangle.$

Stone hits the ground exactly when y-component is zero. Hence

$$-5t^2 + 5\sqrt{2}\,t + 100 = 0$$

$$\Rightarrow \quad t^2 - \sqrt{2}\, t - 20 = 0$$

$$\Rightarrow \quad \left(t^2 - \sqrt{2}\, t + \left(\frac{\sqrt{2}}{2}\right)^2\right) - \left(\frac{\sqrt{2}}{2}\right)^2 - 20 = 0$$

$$\Rightarrow \quad \left(t - \frac{\sqrt{2}}{2}\right)^2 = \frac{41}{2}$$

$$\Rightarrow \quad t - \frac{\sqrt{2}}{2} = \pm\sqrt{\frac{41}{2}}$$

$t > 0$

$$\Rightarrow \quad t = \frac{\sqrt{2}}{2} + \frac{\sqrt{82}}{2}$$

So the x-component is

$$5\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{82}}{2}\right) = 5\left(1 + \sqrt{41}\right).$$

[And the y-component is $\underline{0}$.]

FUNCTIONS OF TWO OR MORE VARIABLES.

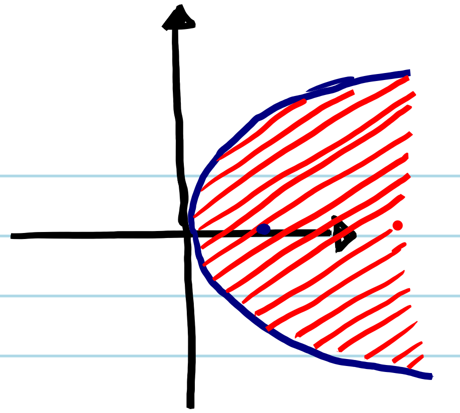
Exp. Find the domain of the following functions:

ⓐ $f(x,y) = \sqrt{x - y^2}$.

ⓑ $f(x,y) = \ln(4 - x^2 - y^2) + \sqrt{x}$.

Solution. ⓐ $f(x,y)$ is defined at the point (x,y) if and only if $x \geq y^2$. So its domain is the following region:

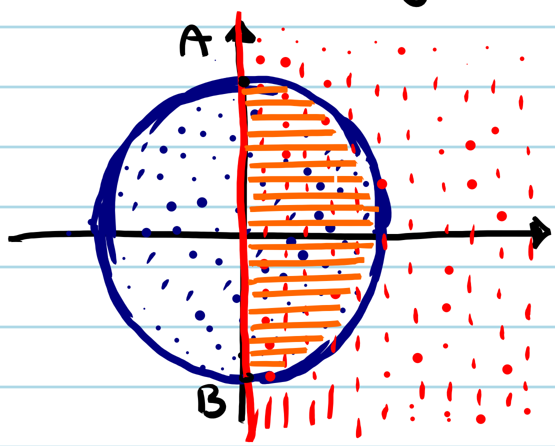
(1st: sketh $x=y^2$.
2nd: pick a pt and check the inequality.)



$y^2=x$ is included.

ⓑ $f(x,y)$ is defined at the point (x,y) if and only if

$$\begin{cases} 4-x^2-y^2>0 \\ x\geq 0 \end{cases}$$



- Segment AB included
- the half circle NOT included.

Graph of a two-variable function

The set of points $(x,y,f(x,y))$ where (x,y) is in the domain of f is called the graph of f. We usually write it as $z=f(x,y)$.

In order to sketch graph of a function, we usually use the horizontal and vertical level curves. Namely we look at three families of curves:

1. Curves of intersection: $z=c$ and $f(x,y)=c$.
2. Curves of intersection: $x=c$ and $z=f(c,y)$
3. Curves of intersectio: $y=c$ and $z=f(x,c)$

When the level curves $f(x,y)=c$ are sketched in the xy-plane, it is called the Contour map of $z=f(x,y)$.

Exp. Sketch the Contour map and the graph of the following functions:

ⓐ $f(x,y)=1-2x-y$.

ⓑ $f(x,y)=x^2+\left(\frac{y}{2}\right)^2$.

ⓒ $f(x,y)=x^2-y^2$.

Solution ⓐ $z=1-2x-y \Rightarrow 2x+y+z=1$

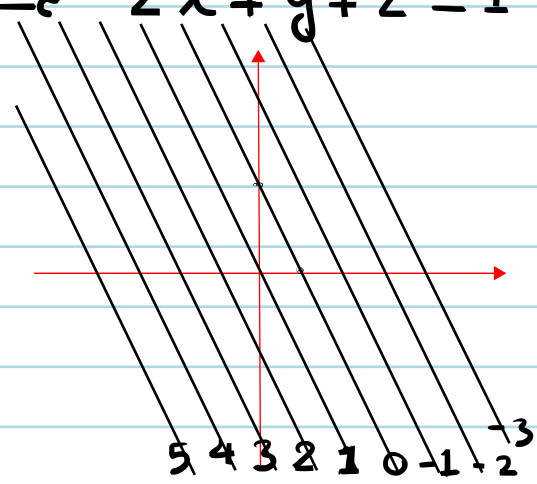
It is a plane.

Contour map

$0=1-2x-y$

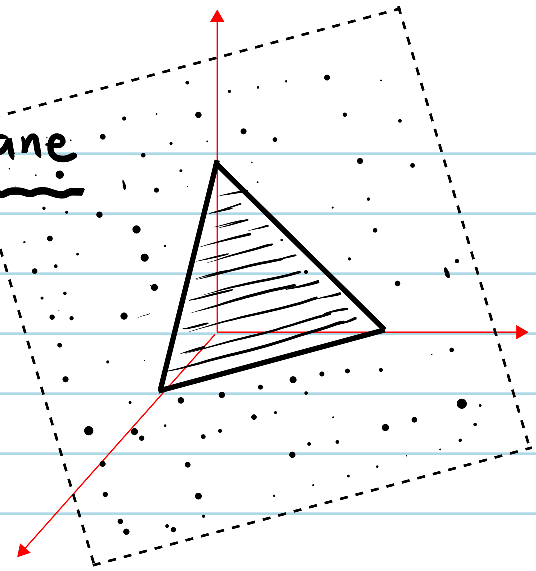
$1=1-2x-y$

$2=1-2x-y$



Intersection with the xy-plane

$0 = 1 - 2x - y$



yz-plane : $z = 1 - y$

xz-plane : $z = 1 - 2x$

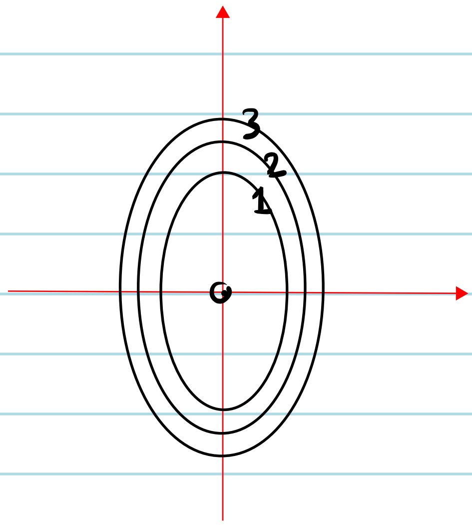
ⓑ $z = x^2 + \left(\frac{y}{2}\right)^2$

$0 = x^2 + \left(\frac{y}{2}\right)^2 \Rightarrow x = y = 0$

$1 = x^2 + \left(\frac{y}{2}\right)^2 \Rightarrow$ ellipse

$2 = x^2 + \left(\frac{y}{2}\right)^2$

$3 = x^2 + \left(\frac{y}{2}\right)^2$



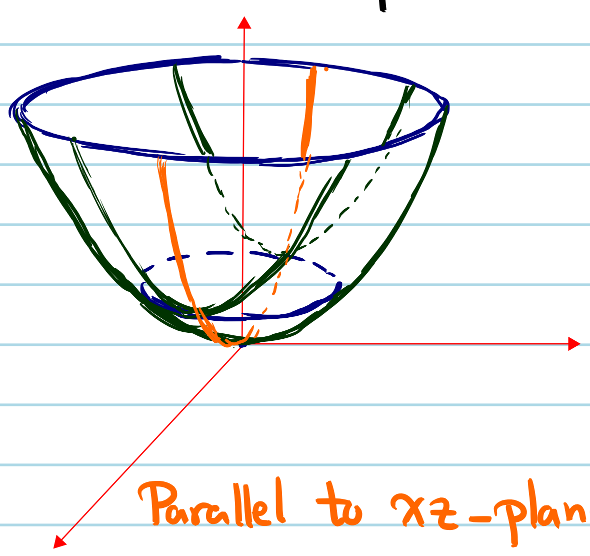
Contour map.

Horizontal level curves

$0 = x^2 + \left(\frac{y}{2}\right)^2$

$1 = x^2 + \left(y/2\right)^2$

$4 = x^2 + \left(y/2\right)^2$



Parallel to yz-plane

$z = \left(\frac{y}{2}\right)^2$

$z = (\pm 1)^2 + \left(\frac{y}{2}\right)^2$

Parallel to xz-plane

$z = x^2$.

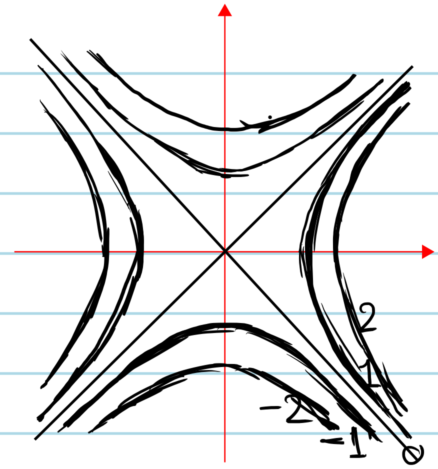
(This is called an elleptic paraboloid.)

Ⓒ $z = x^2 - y^2$.

$0 = x^2 - y^2 \Rightarrow y = \pm x$

$1 = x^2 - y^2$ (intersects the x-axis at $(\pm 1, 0)$.)

$2 = x^2 - y^2$



Contour map.

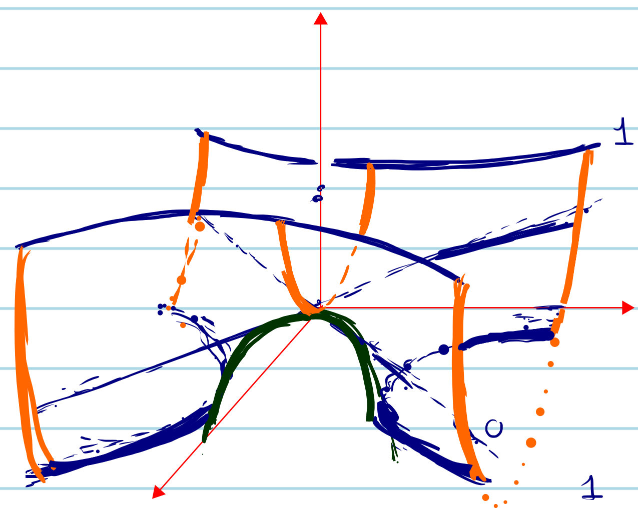
$-1 = x^2 - y^2$ (does NOT intersect the x-axis)

Horizontal level curves

$0 = x^2 - y^2$

$1 = x^2 - y^2$

$-1 = x^2 - y^2$



(Saddle-shaped surface)

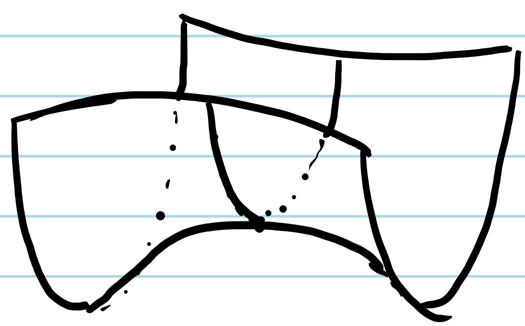
Parallel to yz-plane

$z = -y^2$

Parallel to xz-plane

$z = x^2$

$z = x^2 - (\pm 2)^2$



Three variable functions

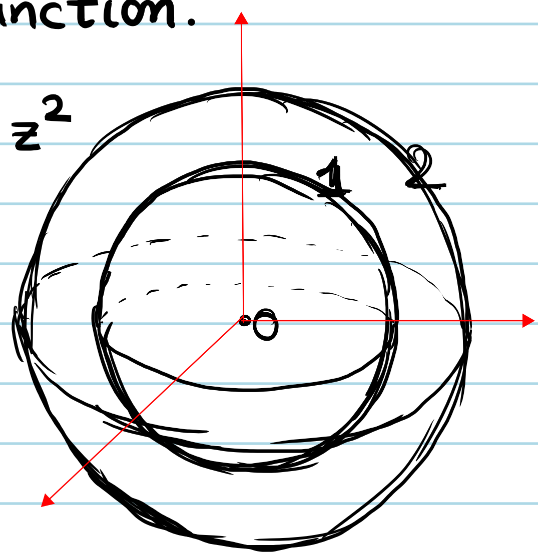
We can use level surfaces to gain an intuition about a three variable function.

Exp. $f(x,y,z) = x^2+y^2+z^2$

$0 = x^2+y^2+z^2$

$1 = x^2+y^2+z^2$

$2 = x^2+y^2+z^2$.



Exp. $f(x,y,z) = x^2+y^2-z^2$

$0 = x^2+y^2-z^2$ (cone)

$1 = x^2+y^2-z^2$ (one-sheeted hyperboloid)

$-1 = x^2+y^2-z^2$ (two-sheeted hyperboloid.)