

# Finite Analogs of Maass Wave Forms

Audrey Terras

ABSTRACT. Orthogonality relations for finite Eisenstein series are obtained. It is shown that there exist finite analogs of Maass wave forms which are not Eisenstein series.

## 1. Introduction

Thanks to work of people like E. Hecke, holomorphic automorphic forms on the Poincaré upper half plane  $H$  for groups like the modular group  $SL(2, \mathbb{Z})$  of  $2 \times 2$  integer matrices of determinant 1 have long been important to number theorists. Since the work of H. Maass, eigenforms of the Poincaré Laplacian  $\Delta$  on  $H$  have enlarged our toolkit of useful functions on  $H$  to include Maass wave forms. We discussed both of these types of modular forms and their uses in [7]. In this paper  $H$  is replaced with a finite analog and  $\Delta$  is replaced with adjacency operators on finite upper half plane graphs, as discussed in [8]. This paper is a continuation of my joint paper with Tony Shaheen [6]. It has also benefitted from a recent discussion with Tony Shaheen. The main topic is that of the orthogonality relations for the Eisenstein series discussed in [6].

The **finite “upper” half plane**  $H_q$  is attached to a finite field  $\mathbb{F}_q$  with  $q$  elements. We assume  $q$  is an odd number. Then we take a fixed non-square  $\delta \in \mathbb{F}_q$ , and define

$$H_q = \left\{ z = x + y\sqrt{\delta} \mid x, y \in \mathbb{F}_q, y \neq 0 \right\}.$$

The finite upper half plane is considered in detail in Terras [8], Chapter 19. See also Shaheen [5].

Recall that an element of the **general linear group**

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{F}_q)$$

has entries in  $\mathbb{F}_q$  and non-zero determinant. Then there is the **group action** by  $g \in G = GL(2, \mathbb{F}_q)$  on  $z \in H_q$  given by:

$$gz = \frac{az + b}{cz + d} \in H_q.$$

---

2000 *Mathematics Subject Classification.* Primary 11M99, 05C25, 05C50.

*Key words and phrases.* Finite upper half plane, Finite Eisenstein series, Finite Maass Wave Forms, Gauss Sums, Kloosterman Sums, Inner Products of Finite Eisenstein Series.

The finite upper half plane  $H_q$  can be identified with  $G/K$ , where  $G = GL(2, \mathbb{F}_q)$  and  $K$  is the subgroup of  $k \in G$  such that  $k\sqrt{\delta} = \sqrt{\delta}$ .  $K$  is the finite analog of the orthogonal group  $O(2, \mathbb{R})$  and can be shown to be isomorphic to the multiplicative group  $\mathbb{F}_q(\sqrt{\delta})^*$ .

Define for  $z = x + y\sqrt{\delta}$ , with  $\in \mathbb{F}_q$ , the **conjugate**  $\bar{z} = x + y\sqrt{\delta} = z^q$  and the **norm**  $Nz = N_{\mathbb{F}_q(\sqrt{\delta})/\mathbb{F}_q} = z\bar{z}$ . The norm has the usual properties:

$$N : \mathbb{F}_q(\sqrt{\delta}) \rightarrow \mathbb{F}_q, \quad \text{and} \quad N(zw) = Nz Nw, \quad \text{for } z, w \in \mathbb{F}_q(\sqrt{\delta}).$$

Then the the “**distance**” between  $z, w \in H_q$  is defined to be

$$(1.1) \quad d(z, w) = \frac{N(z - w)}{\operatorname{Im} z \operatorname{Im} w}.$$

It is easily checked that for  $g \in GL(2, \mathbb{F}_q)$ ,  $d(gw, gw) = d(z, w)$ .

For  $a \in \mathbb{F}_q$ , define the **finite upper half plane graphs**  $X_q(\delta, a)$  to have vertices the elements of  $H_q$  and an edge between  $z$  and  $w$  in  $H_q$  iff  $d(z, w) = a$ . For  $a \neq 0, 4\delta$ , the graph  $X_q(\delta, a)$  is  $(q+1)$ -regular, connected, and Ramanujan. See Terras [8] for more information. Ramanujan graphs are of interest to computer scientists because they provide efficient communications networks. They are of interest to number theorists because their Ihara zeta functions satisfy the analog of the Riemann hypothesis (see [9]). Chai and Li [2] have proved some interesting connections of the spectra of the finite upper half plane graphs and modular forms of function fields.

The  $GL(2, \mathbb{F}_q)$ -invariant operators on  $H_q$  analogous to the non-Euclidean Laplacian on the Poincaré upper half plane  $H$  are the **adjacency operators** on the finite upper half plane graphs defined for  $a \in \mathbb{F}_q$  by

$$(1.2) \quad A_a f(z) = \sum_{\substack{w \in H_q \\ d(z, w) = a}} f(w).$$

These operators generate a commutative algebra of operators.

Since one can view  $H_q$  as a finite symmetric space  $G/K$  as in Terras [8, Chapter 19], the adjacency operators  $A_a$  correspond to  $G$ -invariant differential operators (such as the Poincaré Laplacian on  $H$ ) on a symmetric space. This subject may also be studied from the point of view of association schemes (see Bannai and Ito [1]) or Hecke operators (see Krieg [3]).

Roughly a (complex-valued) **modular form** on the finite upper half plane  $H_q$  is a function  $f : H_q \rightarrow \mathbb{C}$  having an invariance property for a subgroup  $\Gamma$  of  $GL(2, \mathbb{F}_q)$ . Here we consider  $\Gamma = GL(2, \mathbb{F}_p)$ , with  $q = p^r, r > 1$ . Such modular forms which are eigenfunctions of the adjacency operators for the finite upper half plane graphs attached to  $H_q$  are analogs of Maass wave forms on the Poincaré upper half plane.

Now we consider one of our finite upper half plane analogs of the Maass Eisenstein series. Let  $\Gamma$  be a subgroup of  $GL(2, \mathbb{F}_q)$  and let  $\chi$  be a multiplicative character on  $\mathbb{F}_q^*$ . Define the **Eisenstein “series”** for  $z \in H_q$  as

$$(1.3) \quad E_{\chi, \Gamma}(z) = \sum_{\gamma \in \Gamma} \chi(\operatorname{Im}(\gamma z)).$$

$E_{\chi, \Gamma}$  is an eigenfunction of the adjacency operators of finite upper half plane graphs if it does not vanish identically on  $H_q$ . To see when this happens, recall

that  $\mathbb{F}_q^*$  is a cyclic group with generator  $g$ . See Terras [8] and the references given there for more information about finite fields. Thus a **multiplicative character**  $\chi$  of  $\mathbb{F}_q^*$  has the following form for integers  $a, b$ :

$$(1.4) \quad \chi_a(g^b) = e^{\frac{2\pi i ab}{q-1}}, \quad 0 \leq a, b \leq q-2.$$

We will find that for  $q = p^r$ , with  $q > 2$  and  $r > 1$ ,  $E_{\chi, GL(2, \mathbb{F}_q)}$  is non-zero if and only if  $\chi = \chi_a$  when  $a$  is a multiple of  $(p-1)$ . This last condition is equivalent to saying that  $\chi_a|_{\mathbb{F}_p^*} = 1$ . In the paper [6] we could only prove this non-vanishing result when  $r > 2$ , but here the case  $r = 2$  will be shown to work in the same way as the rest using an evaluation of inner products of the Eisenstein series.

Fourier expansions of Maass wave forms on  $H$  involve  $K$ -Bessel functions. In the finite case under consideration the Bessel functions will be replaced with Kloosterman sums. Thus we need to define these sums as well as Gauss sums in order to state the Fourier expansion of the Eisenstein series  $E_{\chi, GL(2, \mathbb{F}_p)}(z)$ .

Let  $\Psi$  be an **additive character** of  $\mathbb{F}_q$ , with  $q = p^r$ ,  $p$  prime. Then  $\Psi = \Psi_b$  where

$$(1.5) \quad \Psi_b(u) = e^{\frac{2\pi i Tr(bu)}{p}}, \quad \text{for } b, u \in \mathbb{F}_q.$$

Here the **trace** in the exponent is

$$(1.6) \quad Tr(u) = Tr_{\mathbb{F}_q/\mathbb{F}_p}(u) = u + u^p + u^{p^2} + \cdots + u^{p^{r-1}}.$$

This has the usual properties of the trace on field extensions:

$$Tr(u+v) = Tr(u) + Tr(v) \quad \text{and} \quad Tr_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \rightarrow \mathbb{F}_p.$$

Let  $\Psi$  be an additive character and  $\chi$  a multiplicative character. Define the **Gauss sum** by

$$(1.7) \quad \Gamma(\chi, \Psi) = \Gamma_q(\chi, \Psi) = \sum_{t \in \mathbb{F}_q^*} \chi(t)\Psi(t).$$

The Gauss sum is analogous to the gamma function.

If  $\chi$  and  $\Psi_b$  are not trivial, then

$$(1.8) \quad \Gamma(\chi, \Psi_b) = \chi(b^{-1})\Gamma(\chi, \Psi_1).$$

Another basic result says that if  $\chi$  and  $\Psi$  are not trivial, then

$$(1.9) \quad |\Gamma_q(\chi, \Psi)| = \sqrt{q}.$$

The **Kloosterman sum** for  $a, b \in \mathbb{F}_q$  is defined by

$$(1.10) \quad K_\Psi(\chi|a, b) = \sum_{t \in \mathbb{F}_q^*} \chi(t)\Psi(at + bt^{-1}).$$

The Kloosterman sum is analogous to the K-Bessel function. These sums have great importance in number theory. See Sarnak [4] who discusses the connection with modular forms and “Kloostermania”.

Define the **quadratic residue character**  $\lambda$  on  $\mathbb{F}_q^*$  by

$$(1.11) \quad \lambda(y) = \begin{cases} 1, & \text{if } y \text{ is a square in } \mathbb{F}_q^*; \\ -1, & \text{if } y \text{ is not a square in } \mathbb{F}_q^*; \\ 0, & \text{if } y = 0. \end{cases}$$

Note that if  $\chi_a$  is as in (1.4), then  $\chi_{(q-1)/2} = \lambda$ .

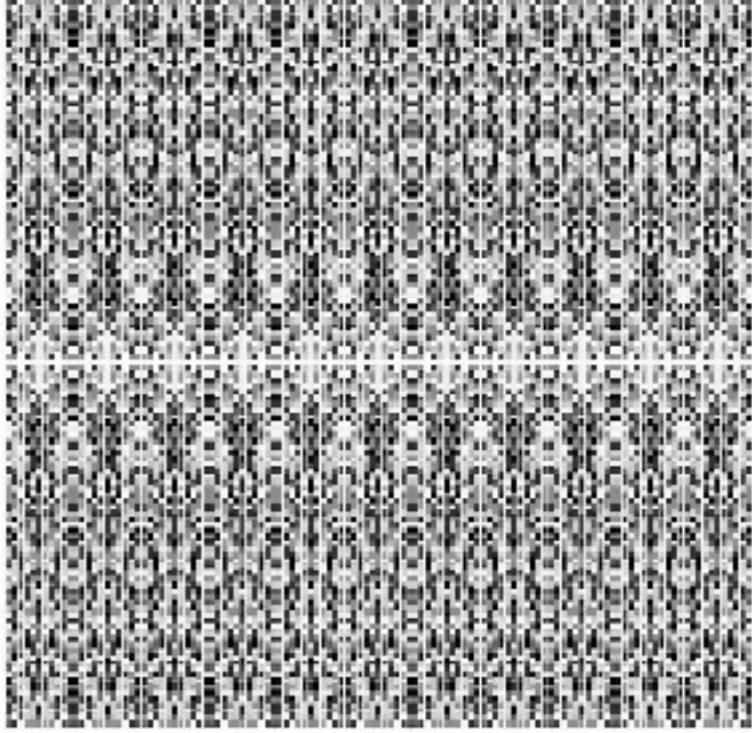


FIGURE 1. Tessellation of the finite upper half plane over the field with 121 elements under  $GL(2, \mathbb{F}_{11})$ . The fundamental domain itself has 11 small squares, each having a different shade of grey (or color).

One of the main results of [6] is the **Fourier expansion of the finite Eisenstein series**  $E_{\chi, GL(2, \mathbb{F}_q)}(z)$ , for  $\chi = \chi_a$  as in (1.4), given when  $z = x + y\sqrt{\delta}$ , where  $x, y \in \mathbb{F}_q$ , by

$$(1.12) \quad \begin{aligned} \frac{1}{p(p-1)^2} E_{\chi, GL(2, \mathbb{F}_q)}(z) &= \chi(y) + \frac{p}{q} \frac{\Gamma(\lambda, \Psi_1)\Gamma(\lambda\chi, \Psi_1)}{\Gamma(\chi, \Psi_1)} \chi^{-1}(-\delta y)\lambda(-\delta) \\ &+ \frac{p}{q} \frac{\Gamma(\lambda, \Psi_1)}{\Gamma(\chi, \Psi_1)} \chi(y) \sum_{\substack{b \in \mathbb{F}_q^* \\ Tr(b)=0}} (\chi\lambda)(b) K_{\Psi_b} \left( \lambda\chi \left| -\delta y^2, -\frac{1}{4} \right. \right) \Psi_b(-x). \end{aligned}$$

If  $a \notin \{0, p-1, 2(p-1), \dots, m(p-1)\}$ , where  $m = \frac{q-1}{p-1} - 1$ , then  $E_{\chi, GL(2, \mathbb{F}_q)}$  is identically 0, as was shown in [6].

## 2. Size of the Fundamental Domain

Our goal is to find a basis for  $L^2(GL(2, \mathbb{F}_p) \backslash H_q)$ , if  $q = p^r, r > 1$ , with  $p$  an odd prime. First we need to find the dimension of the space  $L^2(GL(2, \mathbb{F}_p) \backslash H_q)$ .

This is equivalent to finding the order of the fundamental domain  $GL(2, \mathbb{F}_p) \backslash H_q$  which can also be viewed as the set of orbits of  $\Gamma = GL(2, \mathbb{F}_p)$  in  $H_q$ . When  $q = p^2$  we will see that the order is  $p$ . We have used Mathematica to draw a picture of the tessellation of  $H_{121}$  under  $GL(2, \mathbb{F}_{11})$  obtained by taking any 11 points represented by small squares having different colors or shades of grey as a fundamental domain; i.e., points in the same orbit of  $\Gamma$  have the same color or shade of grey. See Figure 1. In this Figure, the field  $\mathbb{F}_{121}$  is flattened out to appear to be the real line with  $11^2 = 121$  points. Note also that in our picture  $H_{121}$  is actually a union of an upper and a lower half plane. The "real" axis is left out of  $H_{121}$  and appears as a horizontal white line in the center of the figure.

The general result on the order of  $GL(2, \mathbb{F}_p) \backslash H_q$  comes from the Burnside Lemma (proved first by others). That lemma tells us that for a finite group  $\Gamma$  acting on  $H_q$

$$(2.1) \quad |\Gamma \backslash H_q| = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} |\text{Fix}(\gamma)|, \quad \text{where } \text{Fix}(\gamma) = \{z \in H_q \mid \gamma z = z\}.$$

Before computing the terms in this sum let us recall our classification of the conjugacy classes in  $\Gamma = GL(2, \mathbb{F}_p)$ . These are to be found in the following table.

conjugacy classes in $\Gamma$	representatives
<b>central</b>	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \in \mathbb{F}_p^*$
<b>hyperbolic</b>	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a \neq b \in \mathbb{F}_p^*$
<b>parabolic</b>	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, a \in \mathbb{F}_p^*$
<b>elliptic</b>	$\begin{pmatrix} a & b\xi \\ b & a \end{pmatrix}, a, b \in \mathbb{F}_p, b \neq 0, \xi \neq u^2, u \in \mathbb{F}_p$

We must split the computation of the terms in formula (2.1), into 2 cases.

**Case 1.**  $q = p^r$ ,  $r$  odd and  $\mathbb{F}_p(\sqrt{\delta}) \subsetneq \mathbb{F}_q$ .

If  $\gamma \in \text{center}(\Gamma)$ , then  $\gamma = aI$ ,  $a \in \mathbb{F}_p^*$ , and  $\text{Fix}(\gamma) = H_q$ .

If  $\gamma$  is elliptic, then  $|\text{Fix}(\gamma)| = 2$ . For then  $\gamma$  is similar to  $\begin{pmatrix} a & b\delta \\ b & a \end{pmatrix}$  which fixes only  $\pm\sqrt{\delta}$ .

If  $\gamma$  is parabolic or hyperbolic, it does not fix any point in  $H_q$ .

This implies (using the table on p. 366 of [8]):

$$|\Gamma \backslash H_q| = \frac{1}{(p^2 - 1)(p^2 - p)} \left\{ (p - 1)q(q - 1) + \frac{(p^2 - p)^2}{2} 2 \right\} = \frac{q^2 - q + p^3 - p^2}{p(p^2 - 1)}.$$

**Case 2.**  $q = p^r$ ,  $r$  even and  $\mathbb{F}_p(\sqrt{\delta}) \subseteq \mathbb{F}_q$ .

In this case, the only elements of  $\Gamma$  with fixed points in  $H_q$  are in the center.

Elliptic elements of  $\Gamma$ , similar to  $\begin{pmatrix} a & b\delta \\ b & a \end{pmatrix}$ , fix  $\pm\sqrt{\delta}$  which are not in  $H_q$ . They

are on the analog of the real axis. So now the computation goes as follows:

$$|\Gamma \setminus H_q| = \frac{1}{(p^2 - 1)(p^2 - p)} \{(p-1)q(q-1)\} = p^{r-1} \frac{p^r - 1}{p^2 - 1}.$$

The preceding computations prove the following theorem.

**THEOREM 1.** *If  $q = p^r, r > 1$ , then*

$$|GL(2, \mathbb{F}_p) \setminus H_q| = \begin{cases} p^{r-1} \frac{p^r - 1}{p^2 - 1}, & \text{if } r \text{ is even;} \\ \frac{p(p^{2r-2} - 1) - (p^{r-1} - p^2)}{p^2 - 1}, & \text{if } r \text{ is odd.} \end{cases}$$

In particular, when  $r = 2$ , we find  $|GL(2, \mathbb{F}_p) \setminus H_{p^2}| = p$ . And when  $r = 3$ , we obtain  $|GL(2, \mathbb{F}_p) \setminus H_{p^3}| = \frac{(p^5 - p)}{p^2 - 1} = p(p^2 + 1) = p^3 + p$ . The graphs on  $H_q$  induce graphs on the fundamental domains  $GL(2, \mathbb{F}_p) \setminus H_q$ .

### 3. Inner Products of Eisenstein Series

In this section we will find the inner product of two Eisenstein series on  $H_q$ , something that would require truncation of the Eisenstein series on the usual Poincaré upper half plane. But first we need to define the inner product.

For  $z \in H_q$  define the **stabilizer** of  $z$  to be  $\Gamma_z = \{\gamma \in \Gamma \mid \gamma z = z\}$ . The **inner product** of functions  $f, g : \Gamma \setminus H_q \rightarrow \mathbb{C}$  is defined by

$$(3.1) \quad (f, g)_{\Gamma \setminus H_q} = \sum_{z \in \Gamma \setminus H_q} \frac{1}{|\Gamma_z|} f(z) \overline{g(z)} = \frac{1}{|\Gamma|} \sum_{z \in H_q} f(z) \overline{g(z)}.$$

This may seem like a worrisome definition since we are really saying  $(f, g)_{\Gamma \setminus H_q} = \frac{1}{|\Gamma|} (f, g)_{H_q}$ . This definition would make no sense for an infinite group  $\Gamma$  acting on the real Poincaré upper half plane  $H$ . The problem with fixed points does not cause so much trouble when you are over  $H$ , since the fixed points in  $H$  for a countable group  $\Gamma$  form a set of measure 0.

To see the 2nd equality in (3.1), note that if  $\mathcal{O}_z = \{\gamma z \mid \gamma \in \Gamma\}$  =the **orbit of  $z$** , we have:

$$\begin{aligned} \sum_{z \in H_q} f(z) \overline{g(z)} &= \sum_{z \in \Gamma \setminus H_q} |\mathcal{O}_z| f(z) \overline{g(z)} \\ &= \sum_{z \in \Gamma \setminus H_q} \frac{|\Gamma|}{|\Gamma_z|} f(z) \overline{g(z)} = |\Gamma| \sum_{z \in \Gamma \setminus H_q} \frac{1}{|\Gamma_z|} f(z) \overline{g(z)}. \end{aligned}$$

Using the notation previously formulated, we see then that if  $\chi$  is a multiplicative character on  $\mathbb{F}_q^*$ ,

$$E_{\chi, \Gamma}(z) = \sum_{\gamma \in \Gamma \setminus H_q} \chi(\operatorname{Im}(\gamma z)),$$

and  $g : \Gamma \setminus H_q \rightarrow \mathbb{C}$ , then

$$(3.2) \quad \begin{aligned} (E_{\chi, \Gamma}, g)_{\Gamma \setminus H_q} &= \frac{1}{|\Gamma|} \sum_{z \in H_q} \sum_{\gamma \in \Gamma} \chi(\operatorname{Im}(\gamma z)) \overline{g(\gamma z)} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_{z \in H_q} \chi(\operatorname{Im}(\gamma z)) \overline{g(\gamma z)} \\ &= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_{z \in H_q} \chi(\operatorname{Im}(z)) \overline{g(z)} = \sum_{z \in H_q} \chi(\operatorname{Im}(z)) \overline{g(z)}. \end{aligned}$$

In the case that  $g$  is itself an Eisenstein series  $E_\eta$ , we have the Fourier expansion

$$E_\eta(z) = \sum_{\substack{b \in \mathbb{F}_q \\ Tr(b)=0}} a_b^\eta(y) \Psi_b(-x).$$

It follows (using the orthogonality relations for additive characters on  $\mathbb{F}_q$ ) that taking  $\Gamma = GL(2, \mathbb{F}_p)$ , we have

$$(3.3) \quad (E_\chi, E_\eta)_{\Gamma \setminus H_q} = \sum_{y \in \mathbb{F}_q^*} \chi(y) \overline{a_0^\eta(y)}.$$

The next theorem is a consequence of this and our earlier Fourier expansion of the Eisenstein series.

**THEOREM 2. (Inner Products of Eisenstein Series).** *Suppose that  $\chi, \eta$  are multiplicative characters of  $\mathbb{F}_q^*$ , the Eisenstein series  $E_\chi, E_\eta$  are as in (1.3) and the inner product is defined by (3.1), then*

$$\frac{1}{p(p-1)^2} (E_\chi, E_\eta)_{\Gamma \setminus H_q} = \sum_{y \in \mathbb{F}_q^*} \chi(y) \overline{\eta(y)} + \kappa_\eta \sum_{y \in \mathbb{F}_q^*} \chi(y) \eta(y),$$

where  $\overline{\kappa_\eta} = \frac{p}{q} \frac{\Gamma(\lambda, \Psi_1) \Gamma(\lambda \eta, \Psi_1)}{\Gamma(\eta, \Psi_1)} (\lambda \eta) (-\delta)$ .

PROOF. Substitute formula (1.12) into formula (3.3) to obtain:

$$\begin{aligned} \frac{1}{p(p-1)^2} (E_\chi, E_\eta)_{\Gamma \setminus H_q} &= \sum_{y \in \mathbb{F}_q^*} \chi(y) \left\{ \overline{\eta(y)} + \frac{p}{q} \frac{\Gamma(\lambda, \Psi_1) \Gamma(\lambda \eta, \Psi_1)}{\Gamma(\eta, \Psi_1)} \eta^{-1}(-\delta y) \lambda (-\delta) \right\}. \end{aligned}$$

The theorem follows.  $\square$

By the orthogonality relations for characters of  $\mathbb{F}_q^*$ , we see that, when  $\eta \neq \eta^{-1}$ , we have  $(E_\chi, E_\eta)_{\Gamma \setminus H_q} = 0$  unless  $\chi = \eta$  or  $\chi = \overline{\eta} = \eta^{-1}$ . Of course it is possible that  $\eta = \eta^{-1} = \lambda$ , with  $\lambda$  as in formula (1.11). In this case both terms in the sum are the same and

$$p^{-1}(p-1)^{-2} (E_\chi, E_\eta)_{\Gamma \setminus H_q} = (1 + \kappa_\eta) \sum_{y \in \mathbb{F}_q^*} \chi(y) \overline{\eta(y)}, \quad \text{when } \eta = \eta^{-1} = \lambda.$$

To know whether this vanishes, we need to evaluate  $\kappa_\lambda$ . We have:

$$\begin{aligned} \kappa_\lambda &= \frac{p}{q} \frac{\overline{\Gamma(\lambda, \Psi_1) \Gamma(\lambda^2, \Psi_1)}}{\Gamma(\lambda, \Psi_1)} (\lambda^2) (-\delta) \\ &= \frac{p}{q} \overline{\Gamma(1, \Psi_1)} = -\frac{p}{q}. \end{aligned}$$

Here we have used the orthogonality relations for additive characters to see that  $\Gamma(1, \Psi_1) = -1$ . Thus if  $q = p^r$ , we have  $1 + \kappa_\lambda = 1 - \frac{1}{p^{r-1}} \neq 0$ , for  $r \geq 2$ . Thus we see that with  $\lambda$  as in formula (1.11):

$$\begin{aligned} p^{-1}(p-1)^{-2}(E_\lambda, E_\lambda)_{\Gamma \setminus H_q} &= \left(1 - \frac{1}{p^{r-1}}\right)(p^r - 1) \\ &= \frac{(p^{r-1} - 1)}{p^{r-1}}(p^r - 1) \neq 0, \quad \text{for } r \geq 2. \end{aligned}$$

**COROLLARY 1.** *Using the notation of the preceding theorem, write the multiplicative characters  $\chi, \eta$  as in formula (1.4). Then we have the Corollary.*

$$(E_{\chi_r}, E_{\chi_s})_{\Gamma \setminus H_q} \neq 0, \quad \text{if } r = s \text{ or } r = -s \pmod{q-1}.$$

It follows that we have **orthogonal Eisenstein series if we take**

$$r, s \in \left\{ p-1, 2(p-1), \dots, \left(\frac{q-1}{2(p-1)}\right)(p-1) \right\}.$$

This means that when  $q = p^2$  and  $\Gamma = GL(2, \mathbb{F}_p)$ , we get  $\frac{p+1}{2}$  of our orthogonal basis for  $L^2(\Gamma \setminus H_q)$ . If  $p = 3$ , this is 2 of the 3 elements of an orthogonal basis. The other is the constant function corresponding to the character with  $r = 0$ . But when  $q = p^2$  and  $p$  is larger than 3, we are missing  $p-1 - \frac{p+1}{2} = \frac{p-3}{2}$  elements of an orthogonal basis for  $L^2(\Gamma \setminus H_q)$ .

Recall that the finite upper half plane graphs have eigenfunctions of the adjacency operators that come from averaging  $\chi(\text{Im } z)$  over  $K$ , where  $\chi$  is a multiplicative character of  $\mathbb{F}_q^*$ . These are spherical functions corresponding to principal series representations of  $GL(2, \mathbb{F}_q)$ . But these do not span all the eigenfunctions of the adjacency operators. The missing eigenfunctions come from discrete series representations of  $GL(2, \mathbb{F}_q)$ . These are the Soto-Andrade spherical functions. Thus one might expect the missing finite wave forms to be constructed by analogs of Poincaré series, for example by averaging Soto-Andrade spherical functions over  $\Gamma$ . Soto-Andrade's formula for the spherical function on  $H_q$  associated to a discrete series representation of  $GL(2, \mathbb{F}_q)$  is as follows. See Terras [8] for the details. Suppose  $\nu \neq \bar{\nu}$  is a multiplicative character of  $\mathbb{F}_q(\sqrt{\delta})^*$ . Let  $\lambda$  denote the character of  $\mathbb{F}_q^*$  defined by formula (1.11). Then the **Soto-Andrade formula for the spherical function**  $h_\nu(z)$  associated to  $\nu$  and an element  $z \in H_q$  is

$$(q+1)h_\nu(z) = \sum_{\substack{w=u+v\sqrt{\delta} \\ Nw=1}} \lambda \left( 2(u-1) + \frac{d(z, \sqrt{\delta})}{\delta} \right) \nu(w).$$

The sum is over  $w = u + v\sqrt{\delta}$  with  $u, v \in \mathbb{F}_q$  and norm  $Nw = N_{\mathbb{F}_q(\sqrt{\delta})/\mathbb{F}_q} w = w^{1+q} = 1$ . If we average  $h_\nu(z)$  over  $\Gamma$  we will certainly obtain an eigenfunction of all the adjacency operators from formula (1.2) which are invariant under  $\Gamma$ , **provided that the result is non-zero**. We leave this question to another time.

### References

- [1] E. Bannai and T. Ito, *Association Schemes, Algebraic Combinatorics I*, Benjamin/Cummings, Menlo Park, CA, 1984.
- [2] Chai, C.-L. and Li, W.-C. W., Character sums, automorphic forms, equidistribution, and Ramanujan graphs, II, Eigenvalues of Terras graphs, *Forum Math.*, 16 (2004), 631-661.
- [3] A. Krieg, *Hecke Algebras, Memoirs of Amer. Math. Soc.*, 435, Amer. Math. Soc., Providence, RI, 1990.
- [4] P. Sarnak, Kloosterman, quadratic forms and modular forms, *Kloosterman Centennial Celebration, Nieuw. Arch. Wiskd.*, 5/1, nr.4 (Dec., 2000), 385-389.
- [5] A. Shaheen, *Finite Planes and Finite Upper Half Planes: Their Geometry, a Trace Formula, Modular Forms and Eisenstein Series*, Ph.D. Thesis, U. C. S. D., 2005.
- [6] T. Shaheen and A. Terras, Fourier expansions of complex-valued Eisenstein series on finite upper half planes, *International J. of Math. and Math. Sciences*, Vol. 2006, Article ID 63918, pp. 1-17.
- [7] A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications, Vol. I*, Springer-Verlag, New York, NY, 1985, 2013.
- [8] A. Terras, *Fourier Analysis on Finite Groups and Applications*, Cambridge University Press, Cambridge, UK, 1999.
- [9] A. Terras, *Zeta Functions of Graphs: A Stroll through the Garden*, Cambridge University Press, Cambridge, UK, 2011.

MATH. DEPT., U.C.S.D., LA JOLLA, CA 92093-0112

*E-mail address:* aterras@ucsd.edu

*URL:* <http://www.math.ucsd.edu/~atterras>