Zeta Functions and Chaos

Audrey Terras

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Abstract: The zeta functions of Riemann, Selberg and Ruelle are briefly introduced along with some others. The Ihara zeta function of a finite graph is our main topic. We consider two determinant formulas for the Ihara zeta, the Riemann hypothesis, and connections with random matrix theory and quantum chaos.

1 Introduction

This paper is an expanded version of lectures given at M.S.R.I. in June of 2008. It provides an introduction to various zeta functions emphasizing zeta functions of a finite graph and connections with random matrix theory and quantum chaos.

Section 2. Three Zeta Functions

For the number theorist, most zeta functions are multiplicative generating functions for something like primes (or prime ideals). The Riemann zeta is the chief example. There are analogous functions arising in other fields such as Selberg's zeta function of a Riemann surface, Ihara's zeta function of a finite connected graph. We will consider the Riemann hypothesis for the Ihara zeta function and its connection with expander graphs.

Section 3. Ruelle's zeta function of a Dynamical System, A Determinant Formula, The Graph Prime Number Theorem.

The first topic is the Ruelle zeta function which will be shown to be a generalization of the Ihara zeta. A determinant formula is proved for the Ihara zeta function. Then we prove the graph prime number theorem.

Section 4. Edge and Path Zeta Functions and their Determinant Formulas, Connections with Quantum Chaos.

We define two more zeta functions associated to a finite graph - the edge and path zetas. Both are functions of several complex variables. Both are reciprocals of polynomials in several variables, thanks to determinant formulas. We show how to specialize the path zeta to the edge zeta and then the edge zeta to the original Ihara zeta. The Bass proof of Ihara's determinant formula for the Ihara zeta function is given. The edge zeta allows one to consider graphs with weights on the edges. This is of interest for work on quantum graphs. See Smilansky [42] or Horton, Stark and Terras [23].

Lastly we consider what the poles of the Ihara zeta have to do with the eigenvalues of a random matrix. That is the sort of question considered in quantum chaos theory. Physicists have long studied spectra of Schrödinger operators and random matrices thanks to the implications for quantum mechanics where eigenvalues are viewed as energy levels of a system. Number theorists such as A. Odlyzko have found experimentally that (assuming the Riemann hypothesis) the high zeros of the Riemann zeta function on the line $\text{Re}(s) = 1/2$ have spacings that behave like the eigenvalues of a random Hermitian matrix. Thanks to our two determinant formulas we will see that the Ihara zeta function, for example, has connections with spectra of more that one sort of matrix.

References [50] and [51] may be helpful for more details on some of these matters. The first is some introductory lectures on quantum chaos given at Park City, Utah in 2002. The second is a draft of a book on zeta functions of graphs.

2 Three Zeta Functions

2.1 Riemann's Zeta Function

Riemann's zeta function for $s \in \mathbb{C}$ with $\text{Re}(s) > 1$ is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=prime} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

In 1859 Riemann extended the definition of zeta to an analytic function in the whole complex plane except for a simple pole at $s = 1$. He also showed that there is a **functional equation**

$$\Lambda(s) = \pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s) = \Lambda(1-s). \tag{1}$$

The **Riemann hypothesis** (or **RH**) says that the non-real zeros of $\zeta(s)$ (equivalently those with $0 < \text{Re}(s) < 1$) are on the line $\text{Re}(s) = \frac{1}{2}$. It is equivalent to giving an explicit error term in the prime number theorem stated below. The Riemann

hypothesis has been checked to 10^{13}-th zero. (October 12th 2004), by Xavier Gourdon with the help of Patrick Demichel. See Ed Pegg Jr.'s website for an article called the Ten Trillion Zeta Zeros:

$$http://www.maa.org/editorial/mathgames.$$

Proving (or disproving) the Riemann hypothesis is one of the 1 million dollar problems on the Clay Math. Institute website.

There is a duality between the primes and the zeros of zeta, given analytically through the Hadamard product formula as various sorts of explicit formulas. See Davenport [11] and Murty [36]. Such results lead to the **prime number theorem** which says

$$\# \{p = \text{prime} \mid p \leq x\} \sim \frac{x}{\log x}, \quad \text{as} \ x \to \infty.$$

The spacings of high zeros of zeta have been studied by A. Odlyzko

$$www.dtc.umn.edu/\~odlyzko/doc/zeta.htm$$

who has found that experimentally they look like the spacings of the eigenvalues of random Hermitian matrices (GUE). We will say more about this in the last section. See also Conrey [10].

Exercise 1 *Use Mathematica to do a plot of the Riemann zeta function.* **Hint.** *Mathematica has a command to give you the Riemann zeta function. It is Zeta[s].*

There are many other kinds of zeta function. One is the Dedekind zeta of an algebraic number field F such as $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$, where primes are replaced by prime ideals $\mathfrak{p}$ in the ring of integers O_F (which is $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$, if $F = \mathbb{Q}(\sqrt{2})$). Define the **norm of an ideal** of O_F to be $N\mathfrak{a} = |O_F/\mathfrak{a}|$. Then the **Dedekind zeta function** is defined for $\text{Re} \, s > 1$ by

$$\zeta(s, F) = \prod_{\mathfrak{p}} \left(1 - N\mathfrak{p}^{-s}\right)^{-1},$$

where the product is over all prime ideals of O_F. The Riemann zeta function is $\zeta(s, \mathbb{Q})$.

Hecke gave the analytic continuation of the Dedekind zeta to all complex s except for a simple pole at $s = 1$. And he found the functional equation relating $\zeta(s, F)$ and $\zeta(1 - s, F)$. The value at 0 involves the interesting number h_F=the class number of O_F which measures how far O_F is from having unique factorization into prime numbers rather than prime ideals $(h_{\mathbb{Q}(\sqrt{2})} = 1)$. Also appearing in $\zeta(0, F)$ is the regulator which is a determinant of logarithms of units (i.e., elements $u \in O_F$ such that $u^{-1} \in O_F$). For $F = \mathbb{Q}(\sqrt{2})$, the regulator is $\log\left(1 + \sqrt{2}\right)$. The formula is

$$\zeta(0, F) = \frac{-hR}{w}, \tag{2}$$

where w is the number of roots of unity in F ($w = 2$ for $F = \mathbb{Q}(\sqrt{2})$. One has $\zeta(0, \mathbb{Q}) = -\frac{1}{2}$. See Stark [43] for an introduction to this subject meant for physicists.

2.2 The Selberg Zeta Function

This zeta function is associated to a compact (or finite volume) Riemannian manifold. Assuming M has constant curvature -1, it can be realized as a quotient of the **Poincaré upper half plane**

$$H = \{x + iy \mid x, y \in \mathbb{R}, \ y > 0\}.$$

The **Poincaré arc length** element is

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

which can be shown invariant under fractional linear transformation

$$z \longrightarrow \frac{az + b}{cz + d}, \quad \text{where} \ a, b, c, d \in \mathbb{R}, \ ad - bc > 0.$$

It is not hard to see that **geodesics** which are curves minimizing the Poincaré arc length are half lines and semicircles in H orthogonal to the real axis. Calling these geodesics straight lines creates a model for non-Euclidean geometry since Euclid's 5th postulate fails. There are infinitely many geodesics through a fixed point not meeting a given geodesic.

The **fundamental group** Γ of M acts as a discrete group of distance-preserving transformations. The favorite group of number theorists is the **modular group** $\Gamma = SL(2, \mathbb{Z})$ of 2×2 matrices of determinant one and integer entries or the quotient $\overline{\Gamma} = \Gamma/\{\pm I\}$. However the Riemann surface $M = SL(2, \mathbb{Z})\backslash H$ is not compact, although it does have finite volume.

Selberg defined primes in the compact Riemannian manifold $M = \Gamma\backslash H$ to be primitive closed geodesics C in M. Here **primitive** means you only go around the curve once.

Define the **Selberg zeta function,** for $\mathrm{Re}(s)$ sufficiently large, as

$$Z(s) = \prod_{[C]} \prod_{j \geq 1} \left(1 - e^{-(s+j)\nu(C)} \right).$$

The product is over all primitive closed geodesics C in $M = \Gamma\backslash H$ of Poincaré length $\nu(C)$. By the Selberg trace formula (which we do not discuss here), there is a duality between the lengths of the primes and the spectrum of the Laplace operator on M. Here

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Moreover one can show that the Riemann hypothesis (suitably modified to fit the situation) can be proved for Selberg zeta functions of compact Riemann surfaces.

Exercise 2 *Show that $Z(s+1)/Z(s)$ has a product formula which is more like that for the Riemann zeta function.*

The closed geodesics in $M = \Gamma\backslash H$ correspond to geodesics in H itself. One can show that the endpoints of such geodesics in $\mathbb{R}$ (the real line = the boundary of H) are fixed by hyperbolic elements of Γ; i.e., the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with trace $a + d > 2$. Primitive closed geodesics correspond to hyperbolic elements that generate their own centralizer in Γ.

Some references for this subject are Selberg [41] and Terras [48].

2.3   The Ihara Zeta Function

We will see that the Ihara zeta function of a graph has similar properties to the preceding zetas. A good reference for graph theory is Biggs [7].

First we must figure out what primes in graphs are. Recalling what they are for manifolds, we expect that we need to look at closed paths that minimize distance. What is distance? It is the number of oriented edges in a path.

First suppose that X is a finite connected unoriented graph. Thus it is a collection of vertices and edges. Usually we assume the graph is not a cycle or a cycle with hair (i.e., degree 1 vertices). Thus Figure 1 is a bad graph. We do allow our graphs to have loops and multiple edges however.
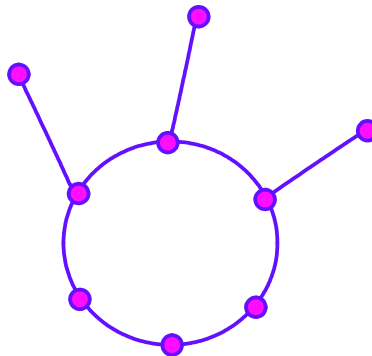

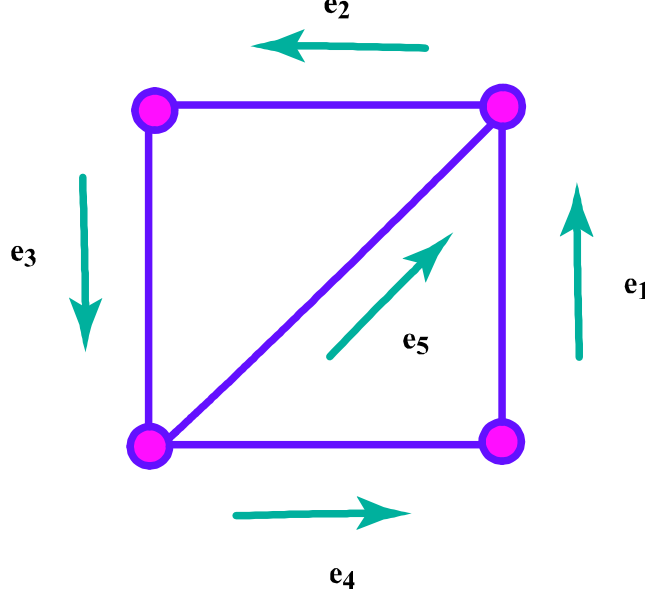Figure 1: This is an example of a bad graph for zeta functions.

Figure 2: We choose an arbitrary orientation of the edges of a graph. Then we label the inverse edges via $e_{j+|E|} = e_j^{-1}$, for $j = 1, ...5.$.

Let E be the set of unoriented (or undirected) edges of X and V the set of vertices. We orient (or direct) the edges arbitrarily and label them $e_1, e_2, \ldots, e_{|E|}$. An example is shown in Figure 2. Then we label the inverse edges (meaning the edge with the opposite orientation) $e_{j+|E|} = e_j^{-1}$, for $j = 1, ..., |E|$. The oriented edges give an alphabet which we use to form words representing the paths in our graph.

Now we can define primes in the graph X. They correspond to closed geodesics in compact manifolds. They are equivalence classes $[C]$ of tailless primitive closed paths C. We define these last adjectives in the next paragraph.

A path or walk $C = a_1 \cdots a_s$, where a_j is an oriented or directed edge of X, is said to have a **backtrack** if $a_{j+1} = a_j^{-1}$, for some $j = 1, ..., s-1$. A path $C = a_1 \cdots a_s$ is said to have a **tail** if $a_s = a_1^{-1}$. The **length** of $C = a_1 \cdots a_s$ is $s = \nu(C)$. A **closed path** means the starting vertex is the same as the terminal vertex. The closed path $C = a_1 \cdots a_s$ is called a **primitive or prime path** if it has no backtrack or tail and $C \neq D^f$, for $f > 1$. For the path $C = a_1 \cdots a_s$, the **equivalence class** $[C]$ means the following

$$[C] = \{a_1 \cdots a_s, \ a_2 \cdots a_s a_1, \ \ldots, \ a_s a_1 \cdots a_{s-1}\}.$$

That is, we call two prime paths **equivalent** if we get one from the other by changing the starting point. A **prime** in the graph X is an equivalence class $[C]$ of prime paths.

Examples of Primes in a Graph.

For the graph in Figure 2, we have primes $[C] = [e_2 e_3 e_5]$, $[D] = [e_1 e_2 e_3 e_4]$, $E = [e_1 e_2 e_3 e_4 e_1 e_{10} e_4]$. Here $e_{10} = e_5^{-1}$ and the lengths of these primes are: $\nu(C) = 3$, $\nu(D) = 4$, $\nu(E) = 7$. We have infinitely many primes since $E_n = [(e_1 e_2 e_3 e_4)^n e_1 e_{10} e_4]$ is prime for all $n \geq 1$. But we don't have unique factorization into primes. The only non-primes are powers of primes.

Definition 3 *The **Ihara zeta function** is defined for $u \in \mathbb{C}$, with $|u|$ sufficiently small by*

$$\zeta(u, X) = \prod_{[P]} \left(1 - u^{\nu(P)}\right)^{-1}, \tag{3}$$

where the product is over all primes $[P]$ in X. Recall that $\nu(P)$ denotes the length of P.

Exercise 4 *How small should $|u|$ be for convergence of $\zeta(u, X)$?*
   Hint. See formula (12) below for $\log \zeta(u, X)$.

4

There are two determinant formulas for the Ihara zeta function (see formulas (4) and (8) below). The first was proved in general by Bass [5] and Hashimoto [17] as Ihara really considered a special case (that each vertex has the same **degree**; i.e. the same number of oriented edges coming out of the vertex) and in fact was considering p-adic groups and not graphs. Moreover the degree had to be $1 + p^e$, where p is a prime number.

The (vertex) **adjacency matrix** A of X is a $|V| \times |V|-$ matrix whose ij entry is the number of directed edges from vertex i to vertex j. The matrix Q is defined to be a diagonal matrix with jth diagonal entry -1+degree of jth vertex. If there is a loop at a vertex, it contributes 2 to the degree.

Then we have the **Ihara determinant formula**

$$\zeta(u, X)^{-1} = (1 - u^2)^{r-1} \det(I - \mathrm{A} + Qu^2). \tag{4}$$

Here r is the rank of the fundamental group of the graph. This is $r = |E| - |V| + 1$. In Section 4 we will give a version of Bass's proof of this formula.

In the case of regular graphs, one can prove the formula using the Selberg trace formula for the graph realized as a quotient $\Gamma \backslash T$, where T is the universal covering tree of the graph and Γ is the fundamental group of the graph. A graph T is a **tree** if it is a connected graph without any closed backtrackless paths. For a tree to be regular, it must be infinite. We will discuss covering graphs in the last section of this paper. For a discussion of the Selberg trace formula on $\Gamma \backslash T$, see the last chapter of Terras [49].

Figure 3 shows part of the 4-regular tree T_4. As the tree is infinite, we cannot put the whole thing on a page. It can be identified with the 3-adic quotient $SL(2, \mathbb{Q}_3)/SL(2, \mathbb{Z}_3)$. A finite 4-regular graph X is a quotient of T_4 modulo the fundamental group of X.
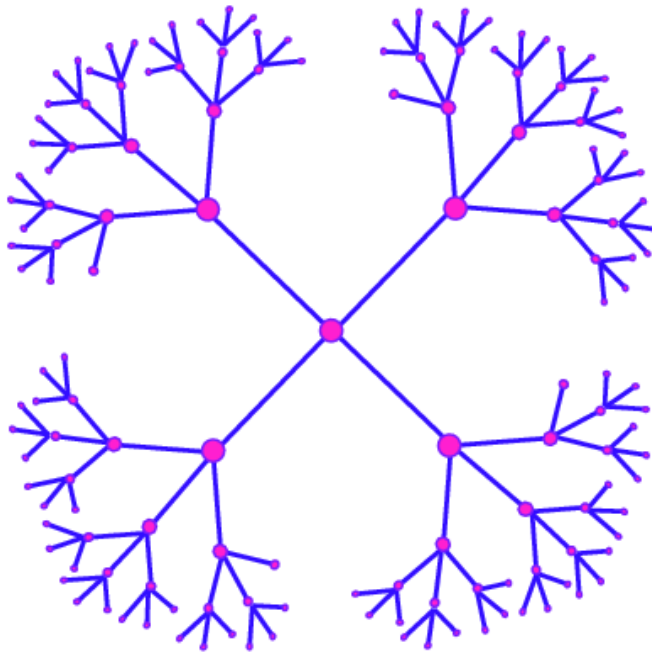

Figure 3: Part of the 4-regular tree is pictured. It is infinite.

Example 5 *The **tetrahedron graph** K_4 is the complete graph on 4 vertices and its zeta function is given by*

$$\zeta(u, K_4)^{-1} = \left(1 - u^2\right)^2 (1 - u) (1 - 2u) \left(1 + u + 2u^2\right)^3.$$

Example 6 *Let $X = K_4 - e$ be the graph obtained from K_4 by deleting an edge e. See Figure 2. Then*

$$\zeta(u, X)^{-1} = \left(1 - u^2\right) (1 - u) \left(1 + u^2\right) \left(1 + u + 2u^2\right) \left(1 - u^2 - 2u^3\right).$$

Exercise 7 *Compute the Ihara zeta functions of your favorite graphs; e.g., the cube, the icosahedron, the buckyball or soccer ball graph.*

Exercise 8 *Obtain a functional equation for the Ihara zeta function of a $(q+1)$-regular graph. It will relate $\zeta(u, X)$ and $\zeta(1/qu, X)$.*

Hint. Use the Ihara determinant formula (4). There are various possible answers to this question. One answer is:
$$\Lambda_X(u) = (1 - u^2)^{r-1+\frac{n}{2}} \left(1 - q^2 u^2\right)^{\frac{n}{2}} \zeta_X(u) = (-1)^n \Lambda_X(\tfrac{1}{qu}).$$

In the special case of a $(q+1)$-regular graph the substitution $u = q^{-s}$ makes the Ihara zeta more like Riemann zeta. That is we set $f(s) = \zeta(q^{-s}, X)$ when X is $(q+1)$-regular. Then the functional equation relates $f(s)$ and $f(1-s)$. See Exercise 8.

The **Riemann Hypothesis** for Ihara's zeta function of a $(q+1)$-regular graph says that

$$\zeta(q^{-s}, X) \text{ has no poles with } 0 < \operatorname{Re} s < 1 \text{ unless } \operatorname{Re} s = \frac{1}{2}. \tag{5}$$

It turns out (using the Ihara determinant formula again) that the Riemann Hypothesis means that the graph is **Ramanujan**; i.e., the non-trivial spectrum of the adjacency matrix of the graph is contained in the spectrum of the adjacency operator on the universal covering tree which is the interval $[-2\sqrt{q}, 2\sqrt{q}]$. This definition comes from the paper of Lubotzky, Phillips and Sarnak [30] who showed that for each fixed degree of the form $p^e + 1$, p =prime, there is a family of Ramanujan graphs X_n with $|V(X_n)| \to \infty$. Ramanujan graphs are of interest to computer scientists because they provide efficient communication networks. The graph is a good expander.

Exercise 9 *Show that for a $(q+1)$-regular graph the Riemann Hypothesis is equivalent to saying that the graph is Ramanujan; i.e. if λ is an eigenvalue of the adjacency matrix A of the graph such that $|\lambda| \neq q + 1$, then $|\lambda| \leq 2\sqrt{q}$.*

Hint. Use the Ihara determinant formula (4).

What is an expander graph? There are 4 ideas.

1) There is a spectral property of some matrix associated to our finite graph X. Choose one of the three matrices below:
    a) (vertex) adjacency matrix A
    b) Laplacian: $D - A$ or $I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$,
              D=diagonal matrix of degrees of vertices.
    c) edge adjacency matrix W_1 to be defined in the next Section.
    According to Lubotzky [31] for a graph to be Ramanujan, the spectrum of the adjacency matrix for X
    should be inside the spectrum of the analogous operator on the universal covering tree of X. One could
    ask for the analogous property of the other operators such as the Laplacian or the edge adjacency matrix.
2) X behaves like a "random graph".
3) Information is passed quickly in the gossip network based on X. The graph has a large expansion constant.
    This is defined by formula (6) below.
4) The random walker on graph the gets lost FAST.

Definition 10 *For sets of vertices S, T of X, define*

$$E(S, T) = \{e \,|\, e \text{ is edge of } X \text{ with one vertex in } S \text{ and the other vertex in } T\}.$$

Definition 11 *If S is a set of vertices of X, we say the **boundary** is $\partial S = E(S, X - S)$.*

Definition 12 *A graph X with vertex set V and $n = |V|$ has **expansion constant***

$$h(X) = \min_{\{S \subset V \,|\, |S| \leq \frac{n}{2}\}} \frac{|\partial S|}{|S|}. \tag{6}$$

The expansion constant is an analog of the Cheeger constant for differentiable manifolds. References for these things include: Chung [9] and Hoory, Lineal and Wigderson [19] and Terras [49] and [51]. Chung [9] gives relations between the expansion constant and the **spectral gap** $\lambda_X = \min\{\lambda_1, 2 - \lambda_{n-1}\}$ if $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of $I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$. She proves that $2h_X \geq \lambda_X \geq \frac{h_X^2}{2}$. This is an analog of the Cheeger inequality in differential geometry. She also connects these inequalities with webpage search algorithms of the sort used by Google.

The possible locations of poles u of $\zeta(u, X)$ of a $(q+1)$-regular graph can be found in Figure 4. The poles satisfying the Riemann hypothesis are those on the circle of radius $1/\sqrt{q}$. Any non-trivial pole; i.e., $u \neq \pm 1$, $\pm 1/q$, which is not on that circle is a non-RH pole. In the $(q+1)$-regular graph case, $1/q$ is always the closest pole of the Ihara zeta to the origin.
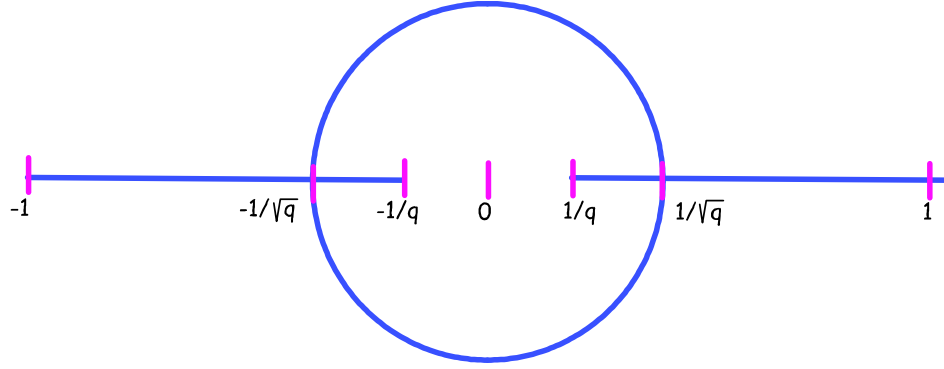
Figure 4: Possible locations of poles of zeta for a $(q+1)$-regular graph.

Exercise 13 *Show that Figure 4 correctly locates the position of possible poles of the Ihara zeta function of a $(q+1)$-regular graph.*

Hint. Use the Ihara determinant formula (4).

The **Alon conjecture** for regular graphs says that the RH is **approximately** true for "most" regular graphs. See Friedman [13] for a proof. See Steven J. Miller et al [34] for experiments leading to the conjecture that the percent of regular graphs exactly satisfying the RH approaches 27% as the number of vertices approaches infinity. The argument involves the Tracy-Widom distribution from random matrix theory.

In his Ph. D. thesis [37], Derek Newland performed graph analogs of Odlyzko's experiments on the spacings of imaginary parts of zeros of Riemann zeta. See Figure 5 and Figures 9 and 8 below.
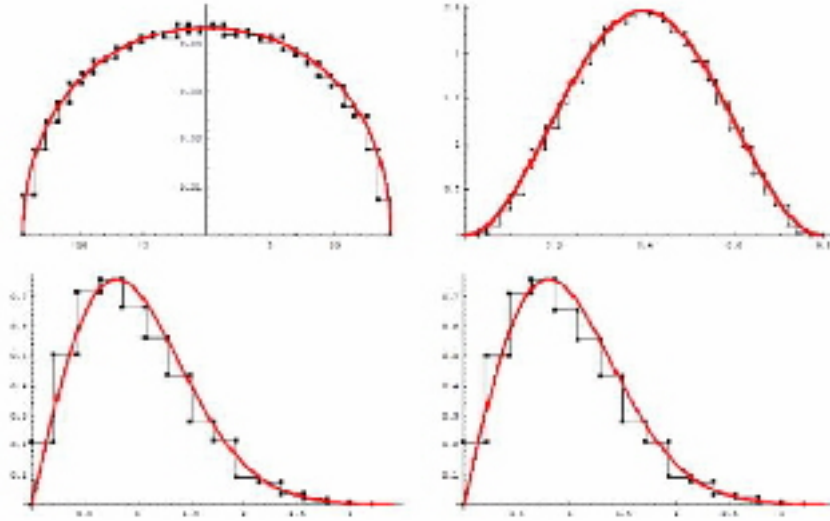

Figure 5: Taken from Newland [37]. For a pseudo-random regular graph with degree 53 and 2000 vertices, generated by Mathematica, the top row shows the distributions of the eigenvalues of the adjacency matrix on the left and imaginary parts of the Ihara zeta poles on the right. The bottom row contains their respective level spacings. The red line on the bottom left is the Wigner surmise for the GOE $y = (\frac{\pi x}{2})e^{-\frac{\pi x^2}{4}}$.

An obvious question is: "What is the meaning of the RH for irregular graphs?" To understand this we need a definition.

Definition 14 $R_X = R$ is the **radius of the largest circle of convergence** of the Ihara zeta function.

As a power series in the complex variable u, the Ihara zeta function has non-negative coefficients. Thus, by a classic theorem of Landau, both the series and the product defining $\zeta_X(u)$ will converge absolutely in a circle $|u| < R_X$ with a singularity (pole of order 1 for connected X) at $u = R_X$. See Apostol [2], p. 237 for Landau's theorem.

Define the **spectral radius** of a matrix M to be $max\{|\lambda| \mid$ for any eigenvalue λ of $M\}$. We will see in the next section that by the Perron-Frobenius theorem in linear algebra, $1/R_X$ is the spectral radius or Perron-Frobenius eigenvalue of the edge adjacency matrix W_1 which will be defined at the beginning of the next section. See Horn and Johnson [20] for the Perron-Frobenius theorem. To apply the Perron-Frobenius theorem, one must show that the edge adjacency matrix of a graph (under our usual assumptions) satisfies the necessary hypotheses. See Stark and Terras [46]. It is interesting to see that the quantity R_X can be viewed from two points of view - complex analysis and linear algebra.

For a $(q+1)$-regular graph $R_X = 1/q$. If the graph is not regular, one sees by experiment that generally there is no functional equation. Thus when we make the change of variables $u = R^s$ in our zeta, the critical strip $0 \leq \operatorname{Re} s \leq 1$ is too large. We should only look at half of it and our Riemann Hypothesis becomes:

The Graph Theory RH for Irregular Graphs:

$$\zeta(u,X) \text{ is pole free in} \qquad R < |u| < \sqrt{R} \,. \tag{7}$$

If the graph is $(q+1)$-regular (by the functional equations), this is equivalent to the Riemann Hypothesis stated earlier in formula (5).

To investigate this we need to define the edge adjacency matrix W_1, which is found in the next section. We will consider examples in the last section.

Exercise 15 *Consider the graph $X = K_4 - e$ from Exercise 6. Show that the poles of $\zeta(u,X)$ are not invariant under the map $u \to R/u$. This means there is no functional equation of the sort that occurs for regular graphs in Exercise 8. Do the poles satisfy the Riemann hypothesis? Do they satisfy a weak RH meaning that $\zeta(u,X)$ is pole free in $R < |u| < 1/\sqrt{q}$?*

3 Ruelle's Zeta Function of a Dynamical System, A Determinant Formula, Graph Prime Number Theorem.

3.1 The Edge Adjacency Matrix of a Graph and Another Determinant Formula for the Ihara Zeta

In this section we consider some zeta functions that arise from those in algebraic geometry. For the Ihara zeta function, we prove a simple determinant formula. We also consider a proof of the graph theory prime number theorem.

Definition 16 *The **edge adjacency matrix** W_1 is defined to be the $2|E| \times 2|E|$ matrix with i,j entry 1 if edge i feeds into edge j (meaning that the terminal vertex of edge i is the initial vertex of edge j) provided that edge i is not the inverse of edge j.*

We will soon prove a second determinant formula for the Ihara zeta function:

$$\zeta(u,X)^{-1} = \det\left(I - uW_1\right). \tag{8}$$

Corollary 17 *The poles of the Ihara zeta are the reciprocals of the eigenvalues of W_1.*

Recall that R is the radius of convergence of the product defining the Ihara zeta function. By the corollary it is the reciprocal of the spectral radius or Perron-Frobenius eigenvalue of W_1 as well as the closest pole of zeta to the origin. It is necessarily positive. See the last chapter of Horn and Johnson [20] or see Apostol [2], p. 237 for Landau's theorem which implies the same thing.

There are many proofs of formula (8). We give the dynamical systems version here. There is another related proof in Section 4.1 and in [51].

3.2 Ruelle zeta (aka Dynamical Systems Zeta or Smale Zeta)

References are Ruelle [39] or Bedford et al [6] or Lagarias [27]. Ruelle's motivation for his definition came partially from a paper of M. Artin and B. Mazur [3]. They were in turn inspired by the zeta function of a projective non-singular algebraic variety V of dimension n over a a finite field with q elements. If N_m denotes the number of points of V with coordinates in the degree m extension field of k, the **zeta function of a variety V over a finite field** is

$$Z_V(u) = \exp\left(\sum_{m \geq 1} \frac{N_m u^m}{m}\right). \tag{9}$$

Example of varieties are given by taking solutions of polynomial equations over finite fields; e.g., $x^2 + y^2 = 1$ and $y^2 = x^3 + ax + b$. You actually have to look at the homogeneous version of the equations in projective space. For more information on these zeta functions, see Lorenzini [29] p. 280 or Rosen [38].

Let F be the **Frobenius map** taking a point on the variety with coordinates x_i to the point with coordinates x_i^q. Here q is the number of elements in the finite field k. Define $Fix(F^m) = \{x \in M \mid F^m(x) = x\}$. One sees that $N_m = |Fix(F^m)|$.

Weil conjectured that zeta satisfies a functional equation relating the values $Z_V(u)$ and $Z_V\left(\frac{1}{q^n u}\right)$. He also conjectured that

$$Z_V(u) = \prod_{j=0}^{2n} P_j(u)^{(-1)^{j+1}},$$

where the P_j are polynomials with zeros of absolute value $q^{-j/2}$. Weil proved the conjectures for the case of curves ($n = 1$). The proof was later simplified. See Rosen [38]. For general n, the Weil conjectures were proved by Deligne. Moreover, the P_j have a cohomological meaning as $\det\left(1 - u F^*|_{H^j(V)}\right)$. Here the Frobenius has induced an action on the ℓ-adic étale cohomology. The case that $n = 1$ is very similar to that of the Ihara zeta function for a $(q + 1)$-regular graph.

Artin and Mazur replace the Frobenius of V with a diffeomorphism f of a smooth compact manifold M such that its iterates f^k all have isolated fixed points. They defined the **Artin-Mazur zeta function** by

$$\zeta(u) = \exp\left(\sum_{m \geq 1} \frac{u^m}{m} |Fix(f^m)|\right). \tag{10}$$

The Ruelle zeta function involves a function $f : M \to M$ on a compact manifold M. Assume $Fix(f^m)$ is finite for all $m \geq 1$. The (first type of) **Ruelle zeta** is defined for a matrix valued function $\varphi : M \to \mathbb{C}^{d \times d}$ by

$$\zeta(u) = \exp\left\{\sum_{m \geq 1} \frac{u^m}{m} \sum_{x \in Fix(f^m)} Tr\left(\prod_{k=0}^{m-1} \varphi\left(f^k(x)\right)\right)\right\}. \tag{11}$$

Here we consider only the special case that $d = 1$ and φ is identically 1, when formula (11) looks exactly like formula (10).

Let I be a finite non-empty set (**our alphabet**). For a graph X, I is the set of directed edges. The transition matrix t is a matrix of zeros and ones with indices in I. In the case of a graph X, t is the 0,1 edge adjacency W_1 from Definition 16.

Since $I^{\mathbb{Z}}$ is compact, the following subset is closed:

$$\Lambda = \left\{(\xi_k)_{k \in \mathbb{Z}} \mid t_{\xi_k \xi_{k+1}} = 1, \text{ for all } k\right\}.$$

In the graph case, $t = W_1$ and $\xi \in \Lambda$ corresponds to a path without backtracking.

A continuous function $\tau : \Lambda \to \Lambda$ such that $\tau(\xi)_k = \xi_{k+1}$ is called a **subshift of finite type**. In the graph case, this shifts the path right, assuming the paths go from left to right.

Then we can find a new formula for the Ihara zeta function which shows that it is a Ruelle zeta. To understand this formula, we need a definition.

Definition 18 $N_m = N_m(X)$ *is the **number of closed paths of length** m **without backtracking and tails in the graph** X.*

From Definition 3 of the Ihara zeta, we can prove in the next paragraph that

$$\log \zeta(u, X) = \sum_{m \geq 1} \frac{N_m}{m} u^m. \tag{12}$$

Compare this formula with formula (9) defining the zeta function of a projective variety over a finite field.

To prove formula (12), take the logarithm of Definition 3 where the product is over primes $[P]$ in the graph X:

$$
\begin{aligned}
\log \zeta(u, X) &= \log \left(\prod_{\substack{[P] \\ \text{prime}}} \left(1 - u^{\nu(P)}\right)^{-1} \right) = -\sum_{[P]} \log\left(1 - u^{\nu(P)}\right) \\
&= \sum_{[P]} \sum_{j \geq 1} \frac{1}{j} u^{j\nu(P)} = \sum_{P} \sum_{j \geq 1} \frac{1}{j\nu(P)} u^{j\nu(P)} = \sum_{P} \sum_{j \geq 1} \frac{1}{\nu(P^j)} u^{\nu(P^j)} \\
&= \sum_{\substack{C \text{ closed} \\ \text{backtrackless} \\ \text{tailless path}}} \frac{1}{\nu(C)} u^{\nu(C)} = \sum_{m \geq 1} \frac{N_m}{m} u^m.
\end{aligned}
$$

Here we have used the power series for $\log(1 - x)$ to see the third equality. Then the fourth equality comes from the fact that there are $\nu(P)$ elements in the equivalence class $[P]$, for any prime $[P]$. The sixth equality is proved using the fact that any closed backtrackless tailless path C in the graph is a power of some prime path P. The last equality comes from Definition 18 of N_m.

If the subshift of finite type τ is as defined above for the graph X, we have

$$|Fix(\tau^m)| = N_m. \tag{13}$$

It follows from this result and formula (12) that the Ihara zeta is a special case of the Ruelle zeta.

Next we claim that

$$N_m = Tr(W_1^m). \tag{14}$$

To see this, set $B = W_1$, with entries b_{ef}, for oriented edges e, f. Then

$$Tr(W_1^m) = Tr(B^m) = \sum_{e_1, \ldots, e_m} b_{e_1 e_2} b_{e_2 e_3} \cdots b_{e_m e_1},$$

where the sum is over all oriented edges of the graph. The b_{ef} are 0 unless edge e feeds into edge f without backtracking; i.e., the terminal vertex of e is the initial vertex of f and $f \neq e^{-1}$. Thus $b_{e_1 e_2} b_{e_2 e_3} \cdots b_{e_m e_1} = 1$ means that the path $C = e_1 e_2 \cdots e_m$ is closed, backtrackless, tailless of length m.

It follows that:

$$
\begin{aligned}
\log \zeta(u, X) &= \sum_{m \geq 1} \frac{u^m}{m} Tr(W_1^m) = Tr \left(\sum_{m \geq 1} \frac{u^m}{m} W_1^m \right) \\
&= Tr \left(\log \left(I - uW_1\right)^{-1} \right) = \log \det \left(I - uW_1\right)^{-1}.
\end{aligned}
$$

Here we have used formula (14) and the continuous linear property of trace. Then we need the power series for the matrix logarithm and the following exercise.

Exercise 19 *Show that* $\exp Tr(A) = \det(\exp A)$, *for any matrix A. To prove this, you need to know that there is a non-singular matrix B such that $BAB^{-1} = T$ is upper triangular. See your favorite linear algebra book.*

This proves formula (8) for the Ihara zeta function which says $\zeta(u, X) = \det\left(I - uW_1\right)^{-1}$. This is known as the Bowen-Lanford theorem for subshifts of finite type in the context of Ruelle zeta functions.

3.3 Graph Prime Number Theorem

Next we prove the graph prime number theorem. This requires two definitions and a theorem.

Definition 20 *The **prime counting function** is*

$$\pi(n) = \#\left\{\text{primes } [P] \,|\, n = \nu(P) = \text{length of } P\right\}.$$

Definition 21 *The **greatest common divisor of the prime path lengths** is*

$$\Delta_X = g.c.d.\left\{\nu(P) \,|\, [P] \text{ prime of } X\right\}.$$

Kotani and Sunada [26] prove the following theorem. Their proof makes heavy use of the Perron-Frobenius theorem from linear algebra. A proof can also be found in [51].

Theorem 22 *(Kotani and Sunada.) Assume, as usual, that the graph X is connected, has fundamental group of rank $r > 1$, and has no degree 1 vertices.*
1) Every pole u of $\zeta_X(u)$ satisfies $R_X \leq |u| \leq 1$, with R_X from Definition 14, and

$$q^{-1} \leq R_X \leq p^{-1}. \tag{15}$$

2) For a graph X, if $q + 1$ is the maximum degree of X and $p + 1$ is the minimum degree of X, then every non-real pole u of $\zeta_X(u)$ satisfies the inequality

$$q^{-1/2} \leq |u| \leq p^{-1/2}. \tag{16}$$

3) The poles of ζ_X on the circle $|u| = R_X$ have the form $R_X e^{2\pi i a / \Delta_X}$, where $a = 1, ..., \Delta_X$. Here Δ_X is from Definition 21.

Exercise 23 *Look up the paper of Kotani and Sunada [26] and figure out their proof of the result that we needed in proving the prime number theorem. Another version of this proof can be found in [51].*

Theorem 24 *Graph Prime Number Theorem.* *Suppose that R_X is as in Definition 14. If $\pi(m)$ and Δ_X are as in Definitions 20 and 21, then $\pi(m) = 0$ unless Δ_X divides m. If Δ_X divides m, we have*

$$\pi(m) \sim \Delta_X \frac{R_X^{-m}}{m}, \qquad \text{as} \quad m \to \infty.$$

Proof. If N_m is as in Definition 18, then formula (12) implies we have

$$u\frac{d}{du} \log \zeta_X(u) = \sum_{m \geq 1} N_m u^m. \tag{17}$$

Now observe that the defining formula for the Ihara zeta function can be written as

$$\zeta_X(u) = \prod_{n \geq 1} \left(1 - u^n\right)^{-\pi(n)}.$$

Then

$$u\frac{d}{du} \log \zeta_X(u) = \sum_{n \geq 1} \frac{n\pi(n)u^n}{1 - u^n} = \sum_{m \geq 1} \sum_{d|m} d\pi(d)u^m.$$

Here the inner sum is over all positive divisors of m. Thus we obtain the **relation between N_m and $\pi(n)$**

$$N_m = \sum_{d|m} d\pi(d).$$

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This sort of relation occurs frequently in number theory and combinatorics. It is inverted using the **Möbius function** $\mu(n)$ defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^r, & \text{if } n = p_1 \cdots p_r, \quad \text{for distinct primes } p_i; \\ 0, & \text{otherwise.} \end{cases}$$

Then, by the **Möbius inversion formula** (which can be found in any elementary number theory book),

$$\pi(m) = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) N_d. \tag{18}$$

Next use formula (8) to see that

$$u\frac{d}{du} \log \zeta_X(u) = -u\frac{d}{du} \sum_{\lambda \in Spec(W_1)} \log(1 - \lambda u) = \sum_{\lambda \in Spec(W_1)} \sum_{n \geq 1} (\lambda u)^n.$$

From this, we get the **formula relating N_m and the spectrum of the edge adjacency W_1** :

$$N_m = \sum_{\lambda \in Spec(W_1)} \lambda^m. \tag{19}$$

The dominant terms in this last sum are those coming from $\lambda \in Spec(W_1)$ such that $|\lambda| = R^{-1}$, with $R = R_X$ from Definition 14.

By Theorem 22, the largest absolute value of an eigenvalue λ occurs Δ_X times with these eigenvalues having the form $e^{2\pi i a/\Delta_X} R^{-1}$, where $a = 1, ..., \Delta_X$. we see that

$$\pi(n) \sim \frac{1}{n} \sum_{|\lambda| \text{ maximal}} \lambda^n = \frac{R^{-n}}{n} \sum_{a=1}^{\Delta_X} e^{\frac{2\pi i a n}{\Delta_X}}.$$

The orthogonality relations for exponential sums (see [49]) which are basic to the theory of the finite Fourier transform says that:

$$\sum_{a=1}^{\Delta_X} e^{\frac{2\pi i a n}{\Delta_X}} = \begin{cases} 0, & \text{if } \Delta_X \text{ does not divide } n \\ \Delta_X, & \text{if } \Delta_X \text{ divides } n. \end{cases} \tag{20}$$

The graph prime number theorem follows. ∎

Note that the Riemann hypothesis gives information on the size of the error term in the prime number theorem.

Exercise 25 *Fill in the details in the proof of the graph theory prime number theorem. In particular, prove the orthogonality relations for exponential sums which implies formula (20) above.*

As is the case for the Riemann zeta function, it is clear that the graph theory Riemann hypothesis gives information on the size of the error term in the prime number theorem.

Example 26 *Tetrahedron or K_4.*
   We saw that

$$\zeta_{K_4}(u)^{-1} = \left(1 - u^2\right)^2 (1 - u)(1 - 2u)\left(1 + u + 2u^2\right)^3.$$

From this we find that

$$\begin{aligned} u\frac{d}{du} \log \zeta_X(u) &= \sum_{m \geq 1} N_m u^m \\ &= 24x^3 + 24x^4 + 96x^6 + 168x^7 + 168x^8 + 528x^9 + \cdots. \end{aligned}$$

Then the question becomes what are the corresponding $\pi(n)$? We see that $\pi(3) = \frac{N_3}{3} = 8$; $\pi(4) = \frac{N_4}{4} = 6$; $\pi(5) = N_5 = 0$. Then, because $\pi(1) = \pi(2) = 0$, we have

$$N_6 = \sum_{d|5} d\pi(d) = 3\pi(3) + 6\pi(6),$$

which implies $\pi(6) = 12$.

It follows from the fact that there are paths of lengths 3 and 4 that the greatest common divisor of the lengths of the prime paths $\Delta = 1$.

The poles of zeta for K_4 are $\{1, 1, 1, -1, \frac{1}{2}, a, a, a, b, b, b\}$, where $a = \frac{1+\sqrt{-7}}{4}, b = \frac{1-\sqrt{-7}}{4}$. Then $|a| = |b| = \frac{1}{\sqrt{2}}$. The closest pole of zeta to the origin is $\frac{1}{2}$.

The prime number theorem for K_4 says

$$\pi(m) \sim \frac{2^m}{m}, \quad \text{as} \quad m \to \infty.$$

Example 27 **_Tetrahedron minus an edge_** $X = K_4 - e$.

We saw that

$$\zeta_X(u)^{-1} = \left(1 - u^2\right)\left(1 - u\right)\left(1 + u^2\right)\left(1 + u + 2u^2\right)\left(1 - u^2 - 2u^3\right).$$

From this, we have

$$
\begin{aligned}
u\frac{d}{du}\log\zeta_X(u) &= \sum_{m \geq 1} N_m u^m \\
&= 12x^3 + 8x^4 + 24x^6 + 28x^7 + 8x^8 + 48x^9 + \cdots.
\end{aligned}
$$

It follows that $\pi(3) = 4$; $\pi(4) = 2$; $\pi(5) = 0$; $\pi(6) = 2$. Again $\Delta = g.c.d.$ lengths of primes $= 1$.

The poles of zeta for $K_4 - e$ are $\{1, 1, -1, i, -i, a, b, \alpha, \beta, \overline{\beta}\}$. Here $a = \frac{1+\sqrt{-7}}{4}, b = \frac{1-\sqrt{-7}}{4}$ and $\alpha = R = $ real root of the cubic while $\beta, \overline{\beta}$ are the remaining (non-real) roots of the cubic.

The prime number theorem for $K_4 - e$ becomes for $\frac{1}{\alpha} \cong 1.5$

$$\pi(m) \sim \frac{\alpha^{-m}}{m}, \quad \text{as} \quad m \to \infty.$$

Exercise 28 _Compute the Ihara zeta function of your favorite graph and then use formula (17) to compute the first 5 non-zero N_m. State the prime number theorem explicitly for this graph. Next do the same computations for the graph with one edge removed._

Exercise 29 _a) Show that the radius of convergence of the Ihara zeta function of a $(q+1)$-regular graph is $R = 1/q$._

b) A graph is a **bipartite graph** iff the set of vertices can be partitioned into 2 disjoint sets S, T such that no vertex in S is adjacent to any other vertex in S and no vertex in T is adjacent to any other vertex in T. Assume your graph is non bipartite and prove the prime number theorem using the Ihara determinant formula (4).

c) What happens if the graph is $(q+1)$-regular graph and bipartite?

Exercise 30 _List all the zeta functions you can and what they are good for. There is a website that lists lots of them: www.maths.ex.ac.uk/~mwatkins._

Now that we have the prime number theorem, we can also produce analogs of the explicit formulas of analytic number theory. That is, we seek an analog of Weil's explicit formula for the Riemann zeta function. In Weil's original work he used the result to formulate an equivalent statement to the Riemann hypothesis. See Weil [53].

Our analog of the Von Mangoldt function from elementary number theory is N_m. Using formula (8), we have

$$u\frac{d}{du}\log\zeta(u, X) = -u\frac{d}{du}\sum_{\lambda \in Spec(W_1)}\log\left(1 - \lambda u\right) = \sum_{\lambda \in Spec(W_1)}\frac{\lambda u}{1 - \lambda u} = -\sum_{\rho \text{ pole of } \zeta}\frac{u}{u - \rho}. \tag{21}$$

Then it is not hard to prove the following result following the method of Murty [36], p. 109.

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Proposition 31 *An Explicit Formula.* *Let $0 < a < R$, where R is the radius of convergence of $\zeta(u, X)$. Assume $h(u)$ is meromorphic in the plane and holomorphic outside the circle of center 0 and radius $a - \varepsilon$, for small $\varepsilon > 0$. Assume also that $h(u) = O(|u|^p)$ as $|u| \to \infty$ for some $p < -1$. Also assume that its transform $\widehat{h}_a(n)$ decays rapidly enough for the right hand side of the formula to converge absolutely. Then if N_m is as in Definition 18, we have*

$$\sum_{\rho} \rho h(\rho) = \sum_{n \geq 1} N_n \widehat{h}_a(n),$$

where the sum on the left is over the poles of $\zeta(u, X)$ and

$$\widehat{h}_a(n) = \frac{1}{2\pi i} \oint_{|u|=a} u^n h(u) du.$$

Proof. We follow the method of Murty [36], p. 109. Look at

$$\frac{1}{2\pi i} \oint_{|u|=a} \left\{ u \frac{d}{du} \left(\log \zeta(u, X) \right) \right\} h(u) du.$$

Use Cauchy's integral formula to move the contour over to the circle $|u| = b > 1$. Then let $b \to \infty$. Also use formulas (**??**) and (17). Note that $N_n \sim \frac{\Delta_X}{R_X^m}$, as $m \to \infty$. ∎

Such explicit formulas are basic to work on the pair correlation of complex zeros of zeta (see Montgomery [35]). They can also be viewed as an analog of Selberg's trace formula. See [21], [52] for discussion of Selberg's trace formula for a $q + 1$ regular graph. In these papers various kernels (e.g., Green's, characteristic functions of intervals, heat) were plugged in to the trace formula deducing various things such as McKay's theorem on the distribution of eigenvalues of the adjacency matrix and the Ihara determinant formula for the Ihara zeta. It would be an interesting research project to do the same sort of thing for irregular graphs.

4 Edge and Path Zeta Functions and their Determinant Formulas, Connections with Quantum Chaos

4.1 Proof of Ihara's Determinant Formula for Ihara Zeta

Before we give our version of the Bass proof of formulas (4) and (8), we define a new graph zeta function with many complex variables. We orient and label the edges of our undirected graph as usual.

Definition 32 *The **edge matrix** W for graph X is a $2m \times 2m$ matrix with a, b entry corresponding to the oriented edges a and b. This a, b entry is the complex variable w_{ab} if edge a feeds into edge b (i.e., the terminal vertex of a is the starting vertex of b) and $b \neq a^{-1}$ and the a, b entry is 0 otherwise.*

Definition 33 *Given a path C in X, which is written as a product of oriented edges $C = a_1 a_2 \cdots a_s$, the **edge norm** of C is*

$$N_E(C) = w_{a_1 a_2} w_{a_2 a_3} \cdots w_{a_{s-1} a_s} w_{a_s a_1}.$$

*The **edge Ihara zeta function** is*

$$\zeta_E(W, X) = \prod_{[P]} \left(1 - N_E(P) \right)^{-1},$$

where the product is over primes in X. Here assume that all $|w_{ab}|$ are sufficiently small for convergence.

Properties and Applications of Edge Zeta

1) By the definitions, if you set all non-zero variables in W equal to u, the **edge zeta function specializes to the Ihara zeta function**; i.e,

$$\zeta_E(W, X)|_{\substack{0 \neq w_{ab} = u \\ \forall a, b}} = \zeta(u, X). \tag{22}$$

2) If you cut or delete an edge of a graph, you can compute the edge zeta for the new graph with one less edge by setting all variables equal to 0 if the cut or deleted edge or its inverse appear in a subscript.

3) The edge zeta allows one to define a zeta function for a weighted or quantum graph. See Smilansky [42] or Horton et al [23].

4) There is an application of the edge zeta to error correcting codes. See Koetter et al [25].

The following result is a generalization of formula (8).

Theorem 34 *(Determinant Formula for the Edge Zeta).*

$$\zeta_E(W, X) = \det (I - W)^{-1}.$$

We prove the theorem after giving an example.
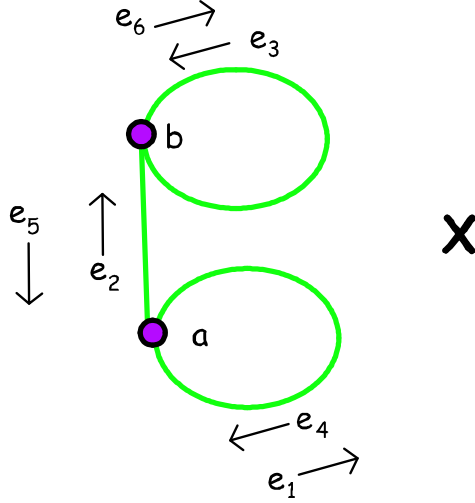
Figure 6: The dumbbell graph

Example. Dumbbell Graph. Figure 6 shows the labeled picture of the dumbbell graph X. For this graph we find that $\zeta_E(W, X)^{-1} =$

$$\det \begin{pmatrix} w_{11} - 1 & w_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & w_{33} - 1 & 0 & w_{35} & 0 \\ 0 & w_{42} & 0 & w_{44} - 1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & w_{66} - 1 \end{pmatrix}.$$

If we cut or delete the vertical edges which are edges e_2 and e_5, we should specialize all the variables with 2 or 5 in them to be 0. This yields the edge zeta function of the subgraph with the vertical edge removed, and incidentally diagonalizes the matrix W. This also diagonalizes the edge matrix W. Of course the resulting graph consists of 2 disconnected loops. So zeta is the product of 2 loop zetas.

Exercise 35 *Do another example computing the edge zeta function of your favorite graph. Then see what happens if you delete an edge.*

Proof of Theorem 34.

Proof. First note that, from the Euler product for the edge zeta function, we have

$$- \log \zeta_E(W, X) = \sum_{[P]} \sum_{j \geq 1} \frac{1}{j} N_E(P)^j.$$

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Then, since there are $\nu(P)$ elements in $[P]$, we have

$$-\log \zeta_E(W, X) = \sum_{\substack{m \geq 1 \\ j \geq 1}} \frac{1}{jm} \sum_{\substack{P \\ \nu(P)=m}} N_E(P)^j.$$

It follows that

$$-\log \zeta_E(W, X) = \sum_C \frac{1}{\nu(C)} N_E(C).$$

This comes from the fact that any closed path C without backtracking or tail has the form P^j for a prime path P. Then by the Exercise below, we see that

$$-\log \zeta_E(W, X) = \sum_{m \geq 1} \frac{1}{m} Tr(W^m).$$

Finally, again using the Exercise below, we see that the right hand side of the preceding formula is $\log \det (I - W)^{-1}$ This proves the theorem. ∎

Exercise 36 *Prove that*

$$\sum_C \frac{1}{\nu(C)} N_E(C) = \sum_{m \geq 1} \frac{1}{m} Tr(W^m) = \log \det(I - W)^{-1}.$$

Hints. 1) For the first equality, you need to think about $Tr(W^m)$ as an m-fold sum of products of w_{ij} in terms of closed paths C of length m just as we did in proving formula (14) above.
2) Exercise 19 says

$$\det(\exp(B)) = e^{Tr(B)}.$$

Then write $\log \left((I - W)^{-1}\right) = B$, using the matrix logarithm (which converges for small w_{ij}), and see that

$$\log \det \left((I - W)^{-1}\right) = Tr(\log(I - W)^{-1}).$$

Theorem 34 gives another proof of formula (8) for the Ihara zeta by specializing all the non-zero w_{ij} to be u.

Next we give a version of Bass's proof of the Ihara determinant formula (4) using formula (8). In what follows, n is the number of vertices of X and m is the number of unoriented edges of X.

First define some matrices. Set $J = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$. Then define the $n \times 2m$ **start matrix** S and the $n \times 2m$ **terminal matrix** T by setting

$$s_{ve} = \begin{cases} 1, & \text{if } v \text{ is starting vertex of oriented edge } e, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$t_{ve} = \begin{cases} 1, & \text{if } v \text{ is terminal vertex of oriented edge } e, \\ 0, & \text{otherwise}. \end{cases}$$

Proposition 37 *Some Matrix Identities* *Using the preceding definitions, the following formulas hold. We write ^tM for the **transpose** of the matrix M.*

Proposition 38 *1) $SJ = T$, $TJ = S$.*
2) If A is the adjacency matrix of X and $Q + I_n$ is the diagonal matrix whose jth diagonal entry is the degree of the jth vertex of X, then $A = S\,{}^tT$, and $Q + I_n = S\,{}^tS = T\,{}^tT$.
3) The 0,1 edge adjacency W_1 from Definition 16 satisfies $W_1 + J = {}^tTS$.

Proof. 1) This comes from the fact that the starting (terminal) vertex of edge e_j is the terminal (starting) vertex of edge $e_{j+|E|}$, according to our edge numbering system.

2) Consider

$$(S \, {}^tT)_{a,b} = \sum_e s_{ae} t_{be}.$$

The right hand side is the number of oriented edges e such that a is the initial vertex and b is the terminal vertex of e, which is the a,b entry of A. Note that $A_{a,a} = 2*$number of loops at vertex a. Similar arguments prove the second formula.

3) We have

$$({}^tTS)_{ef} = \sum_v t_{ve} s_{vf}.$$

The sum is 1 iff edge e feeds into edge f, even if $f = e^{-1}$. ∎

Bass's Proof of the Generalized Ihara Determinant Formula (4).

Proof. In the following identity all matrices are $(n + 2m) \times (n + 2m)$, where the first block is $n \times n$, if n is the number of vertices of X and m is the number of unoriented edges of X. Use the preceding proposition to see that

$$\begin{pmatrix} I_n & 0 \\ {}^tT & I_{2m} \end{pmatrix} \begin{pmatrix} I_n(1 - u^2) & Su \\ 0 & I_{2m} - W_1 u \end{pmatrix}$$
$$= \begin{pmatrix} I_n - Au + Qu^2 & Su \\ 0 & I_{2m} + Ju \end{pmatrix} \begin{pmatrix} I_n & 0 \\ {}^tT - {}^tSu & I_{2m} \end{pmatrix}.$$

Exercise 39 *Check this equality.*

Take determinants to obtain

$$(1 - u^2)^n \det (I - W_1 u) = \det \left(I_n - Au + Qu^2 \right) \det \left(I_{2m} + Ju \right).$$

To finish the proof of formula (4), observe that

$$I + Ju = \begin{pmatrix} I & Iu \\ Iu & I \end{pmatrix}$$

implies

$$\begin{pmatrix} I & 0 \\ -Iu & I \end{pmatrix} (I + Ju) = \begin{pmatrix} I & Iu \\ 0 & I \left(1 - u^2 \right) \end{pmatrix}.$$

Thus $\det (I + Ju) = \left(1 - u^2 \right)^m$. Since $r - 1 = m - n$, for a connected graph, formula (4) follows. ∎

Exercise 40 *Read about quantum graphs and consider the properties of their zeta functions. See [22], [23], [24], as well as the other papers in those volumes. Another reference is Smilansky [42].*

4.2   The Path Zeta Function of a Graph

First we need a few definitions. A **spanning tree** T for graph X means a tree which is a subgraph of X containing all the vertices of X.

The **fundamental group** of a topological space such as our graph X has elements which are closed directed paths starting and ending at a fixed basepoint $v \in X$. Two paths are equivalent iff one can be continuously deformed into the other (i.e., one is homotopic to the other within X, while still starting and ending at v). The product of 2 paths a, b means the path obtained by first going around a then b.

It turns out (by the Seifert-von Kampen theorem, for example) that the fundamental group of graph X is a free group on r generators, where r is the number of edges left out of a spanning tree for X. Let us try to explain this a bit. More information can be found in Hatcher [18] or Massey [32], p. 198, or Gross and Tucker [16].
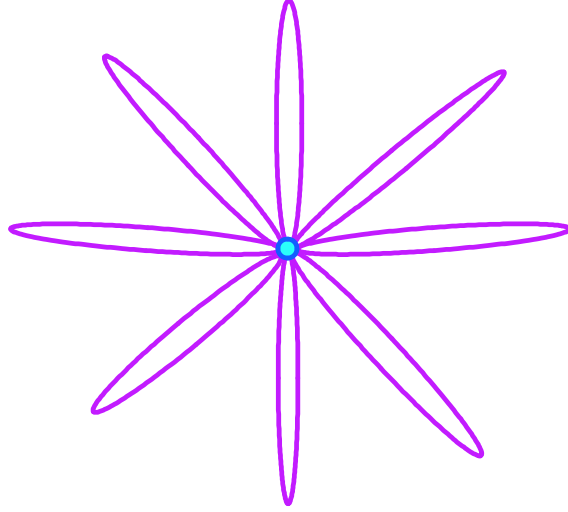
Figure 7: a bouquet of loops

From the graph X construct a new graph $X^{\#}$ by shrinking a spanning tree T of X to a point. The new graph will be a bouquet of r loops as in Figure 7. The fundamental group of X is the same as that of $X^{\#}$. Why? The quotient map $X \to X/T$ is what algebraic topologists call a "homotopy equivalence." This means that intuitively you can continuously deform one graph into the other without changing the topology.

The fundamental group of the bouquet of r loops in Figure 7 is the free group on r generators. The generators are the directed loops! The elements are the words in these loops.

Exercise 41 *Show that $r - 1 = |E| - |V|$.*

Exercise 42 *The **complexity** κ_X of a graph is defined to be the number of spanning trees in X. Use the matrix-tree theorem (see Biggs [7]) to prove that*

$$\left[\frac{d^r}{du^r} \zeta_X^{-1}(u) \right]\Big|_{u=1} = r!(-1)^{r+1} 2^r (r-1) \kappa_X.$$

This is an analog of formula (2) for the Dedekind zeta function of a number field at 0. The complexity is considered to be an analog of the class number of a number field.

Here we look at a zeta function invented by Stark. It has several advantages over the edge zeta. It can be used to compute the edge zeta with smaller determinants. It gives the edge zeta for a graph in which an edge has been fused; i.e., shrunk to one vertex.

Choose a spanning tree T of X. Then T has $|V| - 1 = n - 1$ edges. We call the oriented versions of these **edges left out of the spanning tree** T (or **"deleted" edges** of T) (and their inverses)

$$e_1, \ldots, e_r, e_1^{-1}, \ldots, e_r^{-1}.$$

Call the remaining (oriented) **edges in the spanning tree** T

$$t_1, \ldots, t_{n-1}, t_1^{-1}, \ldots, t_{n-1}^{-1}.$$

Any backtrackless, tailless cycle on X is uniquely (up to starting point on the tree between last and first e_k) determined by the ordered sequence of e_k's it passes through. The free group of rank r generated by the e_k's puts a group structure on backtrackless tailless cycles which is equivalent to the fundamental group of X.

There are 2 elementary reduction operations for paths written down in terms of directed edges just as there are elementary reduction operations for words in the fundamental group of X. This means that if $a_1, ..., a_s$ and e are taken from the e_k's and their inverses, the **2 elementary reduction operations** are:

i) $a_1 \cdots a_{i-1} e e^{-1} a_{i+2} \cdots a_s \cong a_1 \cdots a_{i-1} a_{i+2} \cdots a_s$;

ii) $a_1 \cdots a_s \cong a_2 \cdots a_s a_1$.

Using the 1st elementary reduction operation, each equivalence class of words corresponds to a group element and a word of minimum length in an equivalence class is **reduced** word in group theory language. Since the second operation is equivalent to conjugating by a_1, an equivalence class using both elementary reductions corresponds to a conjugacy class in the fundamental group. A word of minimum length using both elementary operations corresponds to finding words of minimum length in a conjugacy class in the fundamental group. If $a_1, ..., a_s$ are taken from $e_1, ..., e_{2r}$, a word $C = a_1 \cdots a_s$ is of minimum length in its conjugacy class iff $a_{i+1} \neq a_i^{-1}$, for $1 \leq i \leq s-1$ and $a_1 \neq a_s^{-1}$.

This is equivalent to saying that C corresponds to a **backtrackless, tailless** cycle under the correspondence above. Equivalent cycles correspond to conjugate elements of the fundamental group. A conjugacy class $[C]$ is **primitive** if a word of minimal length in $[C]$ is not a power of another word. We will say that a word of minimal length in its conjugacy class is **reduced in its conjugacy class**. From now on, we assume a representative element of $[C]$ is chosen which is reduced in $[C]$.

Definition 43 *The $2r \times 2r$ **path matrix** Z has ij entry given by the complex variable z_{ij} if $e_i \neq e_j^{-1}$ and by 0 if $e_i = e_j^{-1}$.*

The path matrix Z has only one zero entry in each row unlike the edge matrix W from Definition 32 which is rather sparse. Next we imitate the definition of the edge zeta function.

Definition 44 *Define the **path norm** for a primitive path $C = a_1 \cdots a_s$ reduced in its conjugacy class $[C]$, where $a_i \in \left\{ e_1^{\pm 1}, \ldots, e_s^{\pm 1} \right\}$ as*

$$N_P(C) = z_{a_1 a_2} \cdots z_{a_{s-1} a_s} z_{a_s a_1}.$$

*Then the **path zeta** is defined for small $|z_{ij}|$ to be*

$$\zeta_P(Z, X) = \prod_{[C]} \left(1 - N_P(C)\right)^{-1},$$

where the product is over primitive reduced conjugacy classes $[C]$ other than the identity class.

We have similar results to those for the edge zeta.

Theorem 45
$$\zeta_P(Z, X)^{-1} = \det\left(I - Z\right).$$

Proof. Imitate the proof of Theorem 34 for the edge zeta. ∎

The path zeta function is the same for all graphs with the same fundamental group. Next we define a procedure called **specializing the path matrix to the edge matrix** which will allow us to specialize the path zeta function to the edge zeta function. Use the notation above for the edges e_i left out of the spanning tree T and denote the edges of T by t_j. A prime cycle C is first written as a product of generators of the fundamental group and then as a product of actual edges e_i and t_k. Do this by inserting $t_{k_1} \cdots t_{k_s}$ which is the unique non backtracking path on T joining the terminal vertex of e_i and the starting vertex of e_j if e_i and e_j are successive deleted or non-tree edges in C. Now **specialize the path matrix Z to $Z(W)$ with entries**

$$z_{ij} = w_{e_i t_{k_1}} w_{t_{k_1} t_{k_2}} \cdots w_{t_{k_{n-1}} t_{k_s}} w_{t_{k_s} e_j}. \tag{23}$$

Theorem 46 *Using the specialization procedure just given, we have*

$$\zeta_P(Z(W), X) = \zeta_E(W, X).$$

Example 47 _The Dumbbell Again._ _Recall that the edge zeta of the dumbbell graph of Figure 6 was evaluated by a 6×6 determinant. The path zeta requires a 4×4 determinant. Take the spanning tree to be the vertical edge. One finds, using the determinant formula for the path zeta and the specialization of the path to edge zeta:_

$$\zeta_E(W,X)^{-1} = \det \begin{pmatrix} w_{11}-1 & w_{12}w_{23} & 0 & w_{12}w_{26} \\ w_{35}w_{51} & w_{33}-1 & w_{35}w_{54} & 0 \\ 0 & w_{42}w_{23} & w_{44}-1 & w_{42}w_{26} \\ w_{65}w_{51} & 0 & w_{65}w_{54} & w_{66}-1 \end{pmatrix}. \tag{24}$$

If we shrink the vertical edge to a point (which we call **"fusion"** or contraction), the edge zeta of the new graph is obtained by replacing any $w_{x2}w_{2y}$ (for $x,y = 1,3,4,6$) which appear in formula (24) by w_{xy} and any $w_{x5}w_{5y}$ (for $x,y = 1,3,4,6$) by w_{xy}. This gives the zeta function of the new graph obtained from the dumbbell, by fusing the vertical edge.

Exercise 48 _Compute the path zeta function for $K_4 - e$ (the tetrahedron minus one edge) and then specialize it to the edge zeta function of the graph._

Exercise 49 _Write a Mathematica program to specialize the path matrix Z to the matrix $Z(W)$ so that $\zeta_P(Z(W),X) = \zeta_E(W,X)$._

4.3   Connections with Quantum Chaos.

A reference with some background on random matrix theory and quantum chaos is [50]. In section 2 Figure 5 we saw the experimental connections between the statistics of spectra of random real symmetric matrices and the statistics of the imaginary parts of s at the poles of the Ihara zeta function $\zeta(q^{-s}, X)$ for a $(q+1)$-regular graph X. This is analogous to the connection between the statistics of the imaginary parts of zeros of the Riemann zeta function and the statistics of the spectra of random Hermitian matrices. At this point one should look at the figure produced by Bohigas and Giannoni comparing spacings of spectral lines from nuclear physics with those from number theory and billiards. Sarnak added lines from the spectrum of the Poincaré Laplacian on the fundamental domain of the modular group and I added eigenvalues of finite upper half plane graphs. See p. 337 of [50].

Suppose you must arrange the eigenvalues E_i of a random symmetric matrix in decreasing order: $E_1 \geq E_2 \geq \cdots E_n$ and then normalize the eigenvalues so that the mean of the level spacings $E_i - E_{i+1}$ is 1. Wigner's surmise from 1957 says that the normalized level (eigenvalue) spacing histogram is approximated by the function $\frac{1}{2}\pi x \exp\left(\frac{-\pi x^2}{4}\right)$. In 1960 Gaudin and Mehta found the correct distribution function which is close to Wigner's. The correct distribution function is called the GOE distribution. A reference is Mehta [33]. The main property of this distribution is its vanishing at the origin (often called "level repulsion" in the physics literature). This differs in a big way from the spacing density of a Poisson random variable which is e^{-x}.

Many experiments have been performed with spacings of eigenvalues of the Laplace operator for a manifold such as the fundamental domain for a discrete group Γ acting on the upper half plane H or the unit disc. In particular, the experiments of Schmit give the spacings of the eigenvalues of the Laplacian on $\Gamma \backslash H$ for an arithmetic Γ. To define "arithmetic" we must first define **commensurable subgroups** A, B of a group C. This means that $A \cap B$ has finite index both in A and B. Then suppose that Γ is an algebraic group over $\mathbb{Q}$ as in Borel's article in [8] p. 4. One says that Γ is **arithmetic** if there is a faithful rational representation ρ into the general linear group of $n \times n$ non-singular matrices such that ρ is defined over the rationals and $\rho(\Gamma)$ is commensurable with $\rho(\Gamma) \cap GL(n, \mathbb{Z})$. Roughly we are saying that the integers are hiding somewhere in the definition of Γ. See Borel's article in [8] for more information. Arithmetic and non-arithmetic subgroups of $SL(2, \mathbb{C})$ are discussed by Elstrodt, Grunewald, and Mennicke [12].

Experiments of Schmit [40] compared spacings of eigenvalues of the Laplacian for arithmetic and non-arithmetic groups acting on the unit disc. Schmit found that the arithmetic group had spacings that were close to Poisson while the non-arithmetic group spacings looked GOE.

Newland [37] did experiments on spacings of poles of the Ihara zeta for regular graphs. When the graph was a certain Cayley graph for an abelian group which we called a Euclidean graph in [37], he found Poisson spacings. When the graph

was random, he found GOE spacings (actually a transform of GOE coming from the relationship between the eigenvalues of the adjacency matrix of the graph and the zeta poles). Figure 8 shows the spacing histogram for the poles of the Ihara zeta for a finite Euclidean graph Euc1999(2,1) as in Chapter 5 of my book [49]. It is a Cayley graph for a finite abelian group. Figure 9 is the spacing histogram for the poles of the Ihara zeta of a random regular graph as given by Mathematica with 2000 vertices and degree 71. The moral is that the spacings for the poles of the Ihara zeta of a Cayley graph of an abelian group look Poisson while, for a random graph, the spacings look GOE. The difference between figures 8 and 9 is similar to that between the spacings of the Laplacian for arithmetic and non-arithmetic groups.
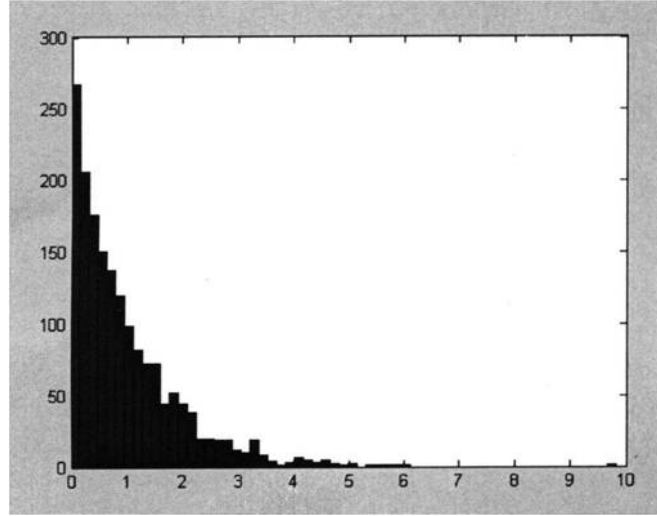
Figure 8: Newland [37] finds the spacings of the poles of the Ihara zeta for a finite euclidean graph $Euc_{1999}(2,1)$ as defined in [49].
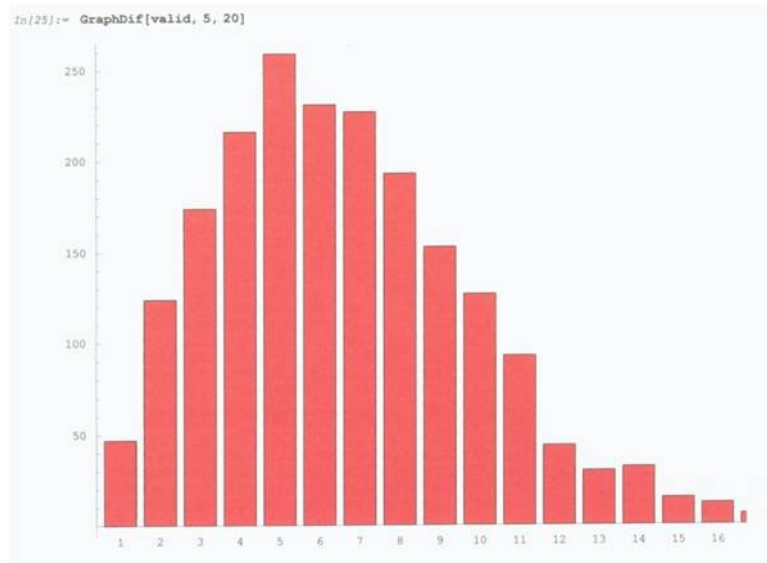
Figure 9: Newland gives the spacings of the poles of the Ihara zeta function of a random regular graph from Mathematica with 2000 vertices and degree 71.

Our plan for the rest of this section is to investigate the spacings of the poles of the Ihara zeta function of a random graph and compare the result with spacings for covering graphs both random and with abelian Galois group. By formula 8, this is essentially the same as investigating the spacings of the eigenvalues of the edge adjacency matrix W_1 from Definition 16. Here, although W_1 is not symmetric, the nearest neighbor spacing can be studied. If the eigenvalues of the matrix are

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$\lambda_i, \ i = 1, ..., 2m,$ we want to look at $v_i = \min\{|\lambda_i - \lambda_j| \, | \, j \neq i\}$. The question becomes: what function best approximates the histogram of the v_i, assuming they are normalized to have mean 1?

References for the study of spacings of eigenvalues of non-symmetric matrices include Ginibre [14], LeBoeuf [28], and Mehta [33]. The **Wigner surmise for non-symmetric matrices** is

$$4\Gamma\left(\frac{5}{4}\right)^4 x^3 \exp\left(-\Gamma\left(\frac{5}{4}\right)^4 x^4\right). \tag{25}$$

Since our matrix W_1 is real and has certain special properties, this may not be the correct Wigner surmise. In what follows some experiments are performed. The following proposition gives some of the properties of W_1.

Proposition 50 *Properties of* W_1. *1)* $W_1 = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$, *where B and C are symmetric real, A is real with transpose* A^T. *The diagonal entries of B and C are 0.*
  2) The sum of the entries of the ith row of W_1 is the degree of the vertex which is the start of edge i.
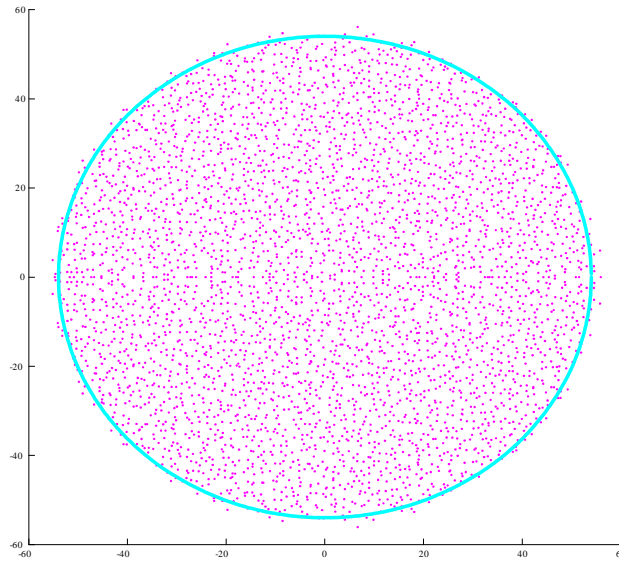
Proof. See Horton [21] or [51]. ∎

Figure 10: A Matlab experiment showing the spectrum of a random 2000×2000 matrix with the properties of W_1 except that the entries are not 0 and 1. The circle has $r = \frac{1}{2}(1 + \sqrt{2})\sqrt{2000}$ rather than $\sqrt{2000}$ as in Girko's circle law.

Our first experiment involves the eigenvalues of a random matrix with block form $\begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$ where B and C are symmetric and 0 on the diagonal. We used Matlab's randn(N) command to get matrices A, B, C with normally distributed entries. There is a result known as the Girko circle law which says that the eigenvalues of a set of random $n \times n$ real matrices with independent entries with a standard normal distribution should be approximately uniformly distributed in a circle of radius $\sqrt{n}$ for large n. References are Bai [4], Girko [15], Tao and Vu [47]. The plot of the eigenvalues of a random matrix with the properties of W_1 in Figure 10. Note the symmetry with respect to the real axis, since our matrix is real. Another interesting fact is that the circle radius is not exactly that which Girko predicts. The spacing distribution for this random matrix is compared with the non-symmetric Wigner surmise in formula (25) in Figure 11.
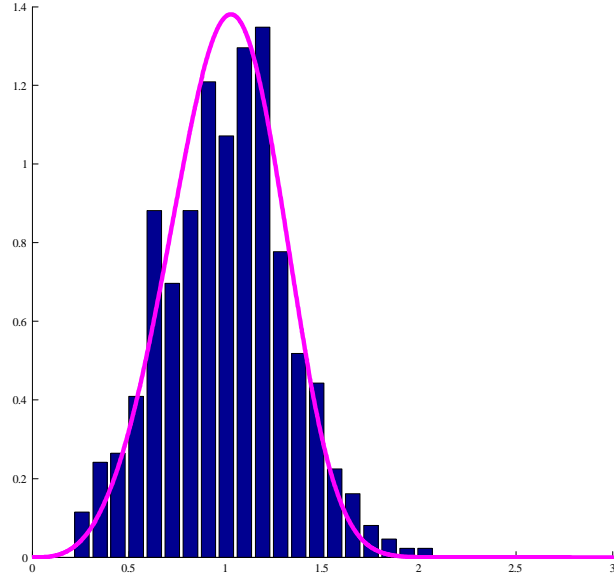
Figure 11: The normalized nearest neighbor spacing for the spectrum of the matrix in Figure 10. The curve is the Wigner surmise from formula (25).

Our next experiments concern the spectra of actual W_1 matrices for graphs. First recall that the eigenvalues of W_1 are the reciprocals of the poles of the Ihara zeta function. You should also recall the graph theory Riemann Hypothesis given in formula (7) as well as Theorem 22 of Kotani and Sunada. Figure 12 shows Ihara zeta poles for three graphs. Figures 13, 15, 17 plot the eigenvalues of the W_1 matrix as well as circles of radius $\sqrt{p} \leq \frac{1}{\sqrt{R}} \leq \sqrt{q}$, where $p+1$ is the minimum degree of vertices of our graph and $q+1$ is the maximum degree. Then R is from Definition 14 and $1/R$ is the Perron-Frobenius eigenvalue of W_1. Theorem 22 of Kotani and Sunada says the spectra cannot ever fill up a circle. They must lie in an annulus.
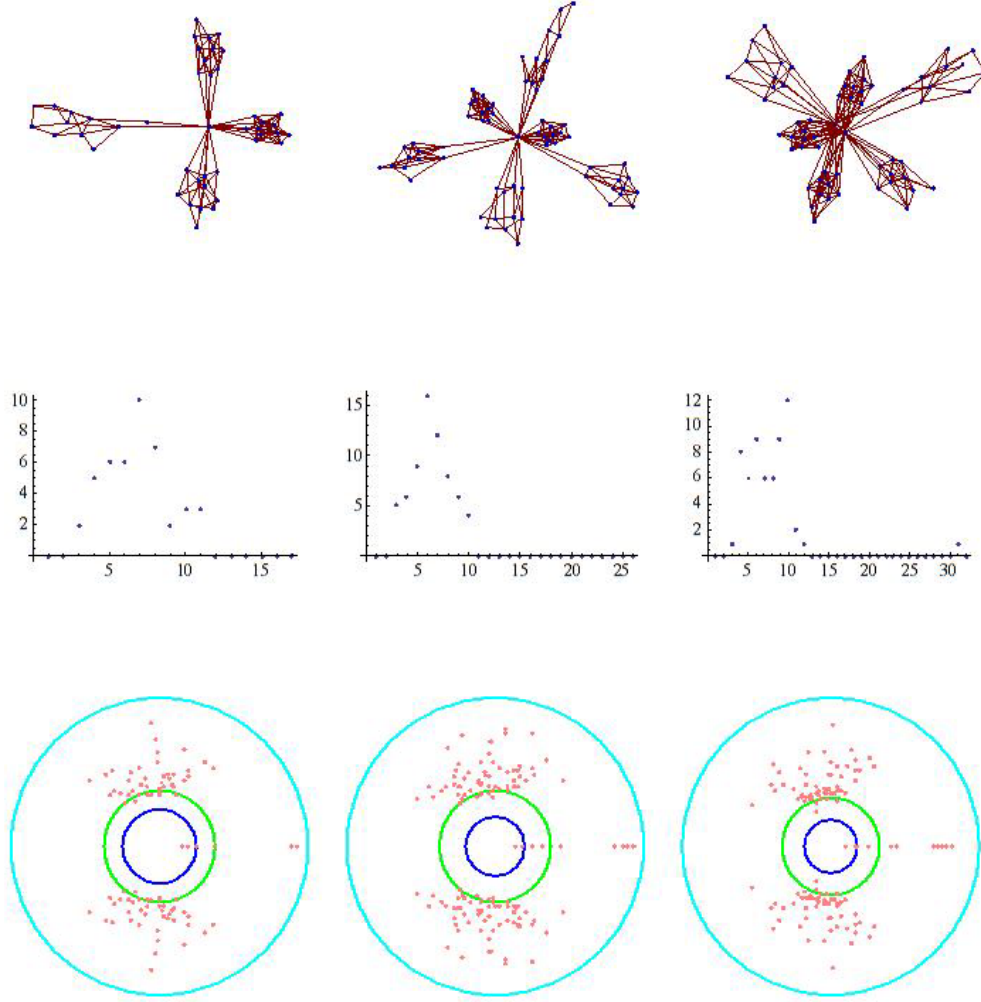
Figure 12: A Mathematica experiment. The top row show the graphs. The second row shows the histogram of degrees. In the last row the pink points are the poles of the Ihara zetas of the graphs on the top, respectively. The middle green circle is the Riemann hypothesis circle with radius $\sqrt{R}$, R=closest pole to 0. The inner circle has radius $\frac{1}{\sqrt{q}}$, where $q+1$ is the maximum degree and the outer circle has radius $\frac{1}{\sqrt{p}}$, where $p+1$ is the maximum degree. Many poles are inside the green (middle) circle and thus violate the Riemann hypothesis.
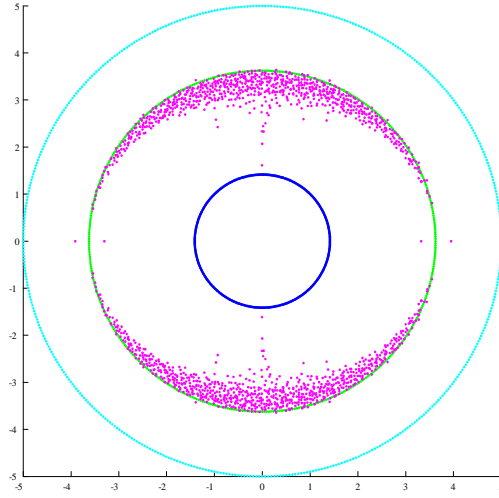
Figure 13: A Matlab Experiment. The eigenvalues of the edge adjacency matrix W_1 for a random graph are the pink points. The inner circle has radius $\sqrt{p}$. the middle green circle has radius $1/\sqrt{R}$. The outer circle has radius $\sqrt{q}$. The green (middle) circle is the Riemann hypothesis circle. Because the eigenvalues of W_1 are reciprocals of the poles of zeta, now the RH says the spectrum should be inside the green circle. The Riemann hypothesis looks approximately true. The graph has 800 vertices, mean degree $\cong 13.125$, edge probability $\cong .0164$.
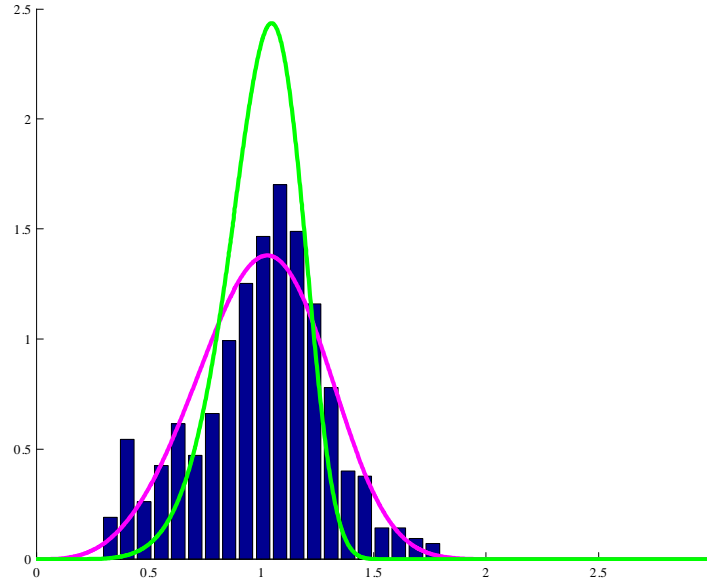

Figure 14: The histogram of the nearest neighbor spacings of the spectrum of the random graph from the preceding figure versus the modified Wigner surmise from formula (26) with $\omega = 3$ and 6.

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Figure 12 gives the results of some Mathematica experiments on the distribution of the poles of zeta for various graphs constructed using the RealizeDegreeSequence command to create a few graphs with various degree sequences and then contracting vertices to join these graphs together. The top row show the graphs. The second row shows the histogram of degrees. The last row shows the poles of the Ihara zetas of the graphs on the top, respectively. Many poles appear to be inside the green circle rather than outside as the RH would say.

Figure 13 shows a Matlab experiment giving the spectrum of the edge adjacency matrix W_1 for a "random graph". The inner circle has radius $\sqrt{p}$. the green circle has radius $1/\sqrt{R}$. The outer circle has radius $\sqrt{q}$. The middle green circle is the Riemann hypothesis circle. Because the eigenvalues of W_1 are reciprocals of the poles of zeta, now the RH says the spectrum should be inside the middle circle. The RH looks approximately true.

Figure 14 shows the histogram of the nearest neighbor spacings of the spectrum of the random graph from the preceding figure versus a **modified Wigner surmise** (with $\omega = 3, 6$)

$$(\omega+1)\Gamma\left(\frac{\omega+2}{\omega+1}\right)^{\omega+1} x^\omega \exp\left(-\Gamma\left(\frac{\omega+2}{\omega+1}\right)^{\omega+1} x^{\omega+1}\right). \tag{26}$$

When $\omega = 3$, this is the original Wigner surmise.

Next we consider some experiments involving covering graphs. An example of a graph covering is the cube covering the tetrahedron. The theory mimics the theory of extensions of algebraic number fields. In particular, there is an analog of Galois theory and Artin L-functions attached to representations of the Galois group. This helps to explain the factorizations of the zeta functions in our earlier examples. See [45] and [51].

Definition 51 *If the graph has no multiple edges and loops we can say that the graph Y is an **unramified covering** of the graph X if we have a covering map $\pi : Y \to X$ which is an onto graph mapping (i.e., taking adjacent vertices to adjacent vertices) such that for every $x \in X$ and for every $y \in \pi^{-1}(x)$, the collection of points adjacent to $y \in Y$ is mapped 1-1 onto the collection of points adjacent to $x \in X$.*

The definition in the case of loops and multiple edges can be found in [45] or [51]. It requires directing edges and requiring the covering map to preserve local directed neighborhoods of a vertex.

Definition 52 *If Y/X is a d-sheeted covering with projection map $\pi : Y \longrightarrow X$, we say that it is a **normal covering** when there are d graph automorphisms $\sigma : Y \longrightarrow Y$ such that $\pi \circ \sigma = \pi$. The Galois group $G(Y/X)$ is the set of these maps σ.*

For covering graphs one can say more about the expected shape of the spectrum of the edge adjacency matrix or equivalently describe the region bounding the poles of the Ihara zeta. Angel, Friedman and Hoory [1] give a method to compute the region encompassing the spectrum of the analogous operator to the edge adjacency matrix W_1 on the universal cover of a graph X. In section 2 we mentioned the Alon conjecture for regular graphs. Angel, Friedman and Hoory give an **analog of the Alon conjecture for irregular graphs**. Roughly their conjecture says that the new edge adjacency spectrum of a large random covering graph is near the edge adjacency spectrum of the universal covering. Here "new" means not occurring in the spectrum of W_1 for the base graph. This conjecture can be shown to imply the approximate Riemann hypothesis for the new poles of a large random cover.

We show some examples related to this conjecture. Figure 15 shows the spectrum of the edge adjacency matrix of a random cover of the base graph consisting of 2 loops with an extra vertex on 1 loop. The inner circle has radius 1. The middle circle has radius $1/\sqrt{R}$. The outer circle has radius $\sqrt{3}$. The Riemann hypothesis is approximately true for this graph zeta.

Figure 16 shows the nearest neighbor spacings for the points in Figure 15 compared with the modified Wigner surmise in formula (26), for various small values of ω.

Figure 17 shows the spectrum of the edge adjacency matrix for a Galois $\mathbb{Z}_{163} \times \mathbb{Z}_{45}$ covering of the base graph consisting of 2 loops with an extra vertex on 1 loop. The inner circle has radius 1. The middle circle has radius $1/\sqrt{R}$, with R as in Definition 14. The outer circle has radius $\sqrt{3}$. The Riemann hypothesis looks very false.

Figure 18 shows the histogram of the nearest neighbor spacings for the spectrum of the edge adjacency matrix of the graph in the preceding figure compared with spacings of a Poisson random variable (e^{-x}) and the Wigner surmise from formula (25).
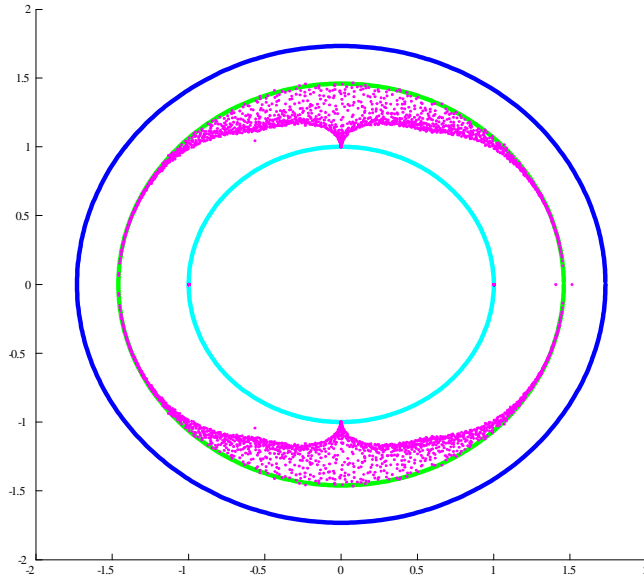
Figure 15: Matlab experiment. The pink points are the eigenvalues of the edge adjacency matrix of a random cover of the base graph consisting of 2 loops with an extra vertex on 1 loop. Thus we plot the reciprocals of the poles of zeta. The inner circle has radius 1. The middle circle has radius $1/\sqrt{R}$. The outer circle has radius $\sqrt{3}$. The Riemann hypothesis is approximately true for this graph zeta. The cover has 801 sheets(copies of a spanning tree).
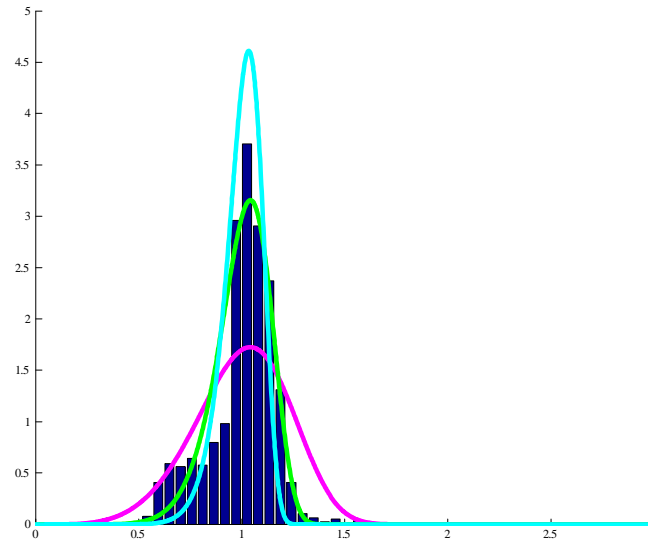

Figure 16: The nearest neighbor spacings for the spectrum of the edge adjacency matrix of the previous graph compared with 3 versions of the modified Wigner surmise from formula (26). Here $\omega = 3, 6, 9$.
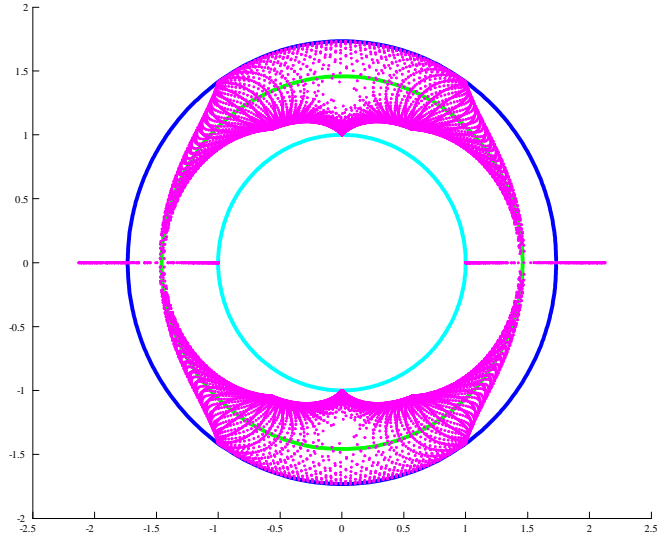
Figure 17: Matlab experiment. The pink dots show the eigenvalues of the edge adjacency matrix W_1 for a Galois $\mathbb{Z}_{163} \times \mathbb{Z}_{45}$ covering of the base graph consisting of 2 loops with an extra vertex on 1 loop. The inner circle has radius 1. The middle circle has radius $1/\sqrt{R}$, with R as in Definition 14. The outer circle has radius $\sqrt{3}$. The Riemann hypothesis is very false.
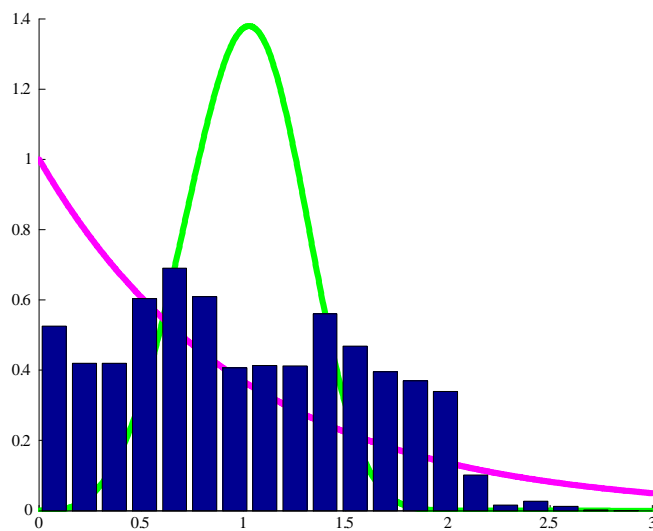
Figure 18: The histogram of the nearest neighbor spacings for the spectrum of the edge adjacency matrix W_1 of the graph in the preceding figure compared with spacings of a Poisson random variable (e^{-x}) and the Wigner surmise from formula (25).

Exercise 53 *Compute more examples of poles of zeta functions of graphs. In particular, it would be interesting to look at graphs with degrees satisfying a power law* d^{-e}, *where* $2 \le e \le 3$ *say.*

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