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SPECTRA OF HEISENBERG GRAPHS OVER FINITE RINGS

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Abstract. We investigate spectra of Cayley graphs for the Heisenberg group over finite rings $\mathbb{Z}/p^n\mathbb{Z}$, where p is a prime. Emphasis is on graphs of degree four. We show that for odd p there is only one such connected graph up to isomorphism. When $p = 2$, there are at most two isomorphism classes. We study the spectra using representations of the Heisenberg group. This allows us to produce histograms and butterfly diagrams of the spectra.

1. **Introduction.** The aim of this paper is to study spectra of Cayley graphs attached to finite Heisenberg groups using histograms and Hofstadter butterfly figures. The **Heisenberg group** $H(R)$ over a ring R consists of upper triangular 3×3 matrices with entries in R and ones on the diagonal. When R is the field of real numbers $\mathbb{R}$, the group makes its presence known in quantum physics, in particular, when considering the uncertainty principle. It also is important in the theory of radar cross-ambiguity functions. See W. Schempp [19]. When the ring R is $\mathbb{Z}$, the ring of integers, there are degree 4 and 6 Cayley graphs (see the next paragraph) associated to $H(\mathbb{Z})$ whose spectra (i.e., eigenvalues of the adjacency matrix) have been much studied starting with D. R. Hofstadter's work on energy levels of Bloch electrons [7] which includes a picture of the Hofstadter butterfly. This subject also goes under the name of the spectrum of the almost Mathieu operator or the Harper operator. Or one can just look at the finite difference equation corresponding to Mathieu's equation $y'' - 2\theta\cos(2x)y = -ay$. M. P. Lamoureux's web site (http://www.math.ucalgary.ca/~mikel/mathieu.html) has a picture and references. For results concerning the Cantor-set structure of the spectra, see M. D. Choi, G. A. Elliott and Noriko Yui [2]. Other references are C. Béguin, A. Valette and A. Zuk [1] as well as M. Kotani and T. Sunada [9].

If S is a subset of a finite group G, the **Cayley graph** $X(G, S)$ has as its vertex set the set G. Edges connect vertices $g \in G$ and gs, for all $s \in S$. Usually we will assume that $s \in S$ implies $s^{-1} \in S$ so that the graph is undirected. And we will normally assume that S is a set of generators of G so that the graph will be connected. It is not hard to see that the spectrum of the adjacency matrix of $X(G, S)$ is contained in the interval $[-k, k]$, if $k = |S|$.

Heisenberg groups over finite fields have provided a tool in the search for random number generators (see Maria Zack [28]) as well as the search for Ramanujan graphs (see Perla Myers [17]). **Ramanujan graphs** were defined by A. Lubotzky, R. Phillips and P. Sarnak [13] to be finite connected k-regular graphs such that the eigenvalues λ of the adjacency matrix satisfy $|\lambda| \leq 2\sqrt{k-1}$. Other references are

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P. Diaconis and L. Saloff-Coste [5] and Terras [24]. As shown in the last reference, the size of the second largest (in absolute value) eigenvalue of the adjacency matrix governs the speed of convergence to uniform for the standard random walk on a connected regular graph. Ramanujan graphs have the best possible eigenvalue bound for connected regular graphs of fixed degree in an infinite sequence of graphs with number of vertices going to infinity. For such graphs, the random walker gets lost as quickly as possible. Equivalently, this says that such graphs can be used to build efficient communication networks.

There are more reasons to study the Heisenberg group. First, as a nilpotent group (see A. Terras [24] for the definition), it may be viewed as the closest to abelian. Second, it is an important subgroup of $GL(3, R)$ (the general linear group of matrices x such that x and x^{-1} have entries in the ring R) for those interested in creating a finite model of the symmetric space of the real $GL(3, \mathbb{R})$ analogous to the finite upper half plane model of the Poincaré upper half plane.

Of course there is more to the spectrum than the second largest eigenvalue in absolute value. Quantum physicists have long been interested in the distribution of eigenvalues or energy levels of Schrödinger operators as well as finite matrix approximations to Schrödinger operators. For example, in the 1950s E. Wigner considered spectra of real symmetric $n \times n$ matrices whose entries are independent Gaussian random variables [27]. He found that the histogram of the eigenvalues looks like a semi-circle. A histogram for a representative of our finite Heisenberg graphs can be found in Figure 1. It is definitely not a semi-circle or even a semi-ellipse.

Quantum chaoticists study histograms of various spectra. This MSRI website (http://www.msri.org/) has movies and transparencies of many talks from 1999 on the subject. See, for example, the talks of Sarnak from Spring, 1999. Other references are Sarnak [18] and Terras [25], [26]. In this context, one also studies the distribution function of the histogram of level spacings (differences of adjacent eigenvalues). See Fan Chung [3], J. D. Lafferty and D. Rockmore [10], Winnie Li [11], Lubotzky, [12], and Terras [23], [24] for more examples.

There are alternatives to histograms for the depiction of spectra of k-regular graphs, as was demonstrated by Hofstadter [7], who separates spectra into bands which for us correspond to higher dimensional irreducible representations of the Heisenberg group. These Hofstadter butterflies are quite beautiful. We will consider such depictions of the spectrum in the last part of this paper. See Figure 2.

In a subsequent paper we will consider Ihara zeta functions of these Heisenberg graphs. In an earlier version of this paper, we considered degree 6 Heisenberg graphs as well as graphs with very large degrees generalizing some of those in A. Medrano [15]. In particular, we considered $X(Heis(\mathbb{Z}/p^n\mathbb{Z}, S))$, with edge set

$$S = \left\{ (x, y, z) \,\middle|\, x, y, x \in \mathbb{Z}/p^n\mathbb{Z},\ 2 \text{ of } x, y, z \text{ are } 0,\ \text{the 3rd a unit } (\operatorname{mod} p) \right\}$$

of degree $3(p^n - p^{n-1})$. The spectrum has very few entries. As these graphs require a somewhat different analysis, we will publish these results elsewhere. Myers [17] considers Cayley graphs for $Heis(\mathbb{Z}/p\mathbb{Z})$ with large edge sets as well. Other references related to this work are the Ph.D. theses of some of the coauthor [4],[14], [15], and [16]. The authors would like to thank J. Schulte for a correction to Proposition 1.

2. **Spectra via Group Representations and Histograms.** As we have said, one method we use to study graph spectra comes from the basic fact that the

Fourier transform on the finite group $G = Heis(\mathbb{Z}/p^n\mathbb{Z})$ will block diagonalize the adjacency operator on a Cayley graph with vertex set G. See Proposition 2.

So we need to find a complete set of inequivalent irreducible unitary representations of G. One can find these representations for the case $n = 1$ in the paper of Schempp [19] or the book of Terras [24]. However the case of arbitrary n is a bit more complicated. Let us sketch the results we need - in particular, we need the matrix entries of the representations to write our programs. It is not sufficient to know the characters.

Notation.

We write $\mathbb{Z}/m\mathbb{Z}$ for the ring of integers $\bmod m$. And $(\mathbb{Z}/m\mathbb{Z})^*$ is the set of units or invertible elements for multiplication in this ring; i.e., the integers $x(\bmod m)$ which are relatively prime to m. In the rest of the paper we use the notation $p^f \mid x$, for an integer x, if p^f divides x. We write $p^f \parallel x$ if f is the highest power such that p^f divides x. When $f = 0$, this means $p \nmid x$. When $f = n$ this means $x = 0$.

Here $\widehat{G}$ denotes a complete set of **inequivalent irreducible unitary representations** of G.

Lemma 1. ***Conjugacy Classes of*** $Heis(\mathbb{Z}/p^n\mathbb{Z})$***.*** *The conjugacy classes partition themselves into $n+1$ types defined as follows for $0 \le e \le n$:*

$$C^{(e)}_{x,y,v} = C^{(e)} = \left\{(x,y,z) \,\middle|\, z \equiv up^{n-e} + v(\bmod p^n), u(\bmod p^e) \right\},$$

for $v(\bmod p^{n-e})$, $p^{n-e} \parallel g.c.d.(x,y)$. So $|C^{(e)}| = p^e$ and the number of distinct classes $C^{(e)}$ is $p^{n-e}\left(p^{2e} - p^{2(e-1)}\right)$, for $e \neq 0$. The number of distinct classes is p^n for $e = 0$.

Proof. Note that

$$(a,b,c)(x,y,z)(a,b,c)^{-1} = (x,y,z + (ay - bx)).$$

This means that if $p^{n-e} \parallel g.c.d.(x,y)$, we can change z by adding multiples of p^{n-e} without changing the conjugacy class. So we obtain the formula above for the conjugacy class of (x,y,z). Next we need to count the number of such classes. When $e = n$, we must count the $(x,y) mod p^n$ such that $p \nmid g.c.d.(x,y)$. If $p \nmid x$, then y is arbitrary and there are $(p^n - p^{n-1})p^n$ such pairs. If $p \mid x$ then $p \nmid y$ and there are $p^{n-1}\left(p^n - p^{n-1}\right)$ such pairs. The total number is $p^{2n} - p^{2(n-1)}$. Suppose $e \neq 0$ or n. To see the formula for the number of classes, set $f = n - e$ and note that $p^f \parallel g.c.d.(x,y)$ iff $x = p^f u$ and $y = p^f v$ with $(u,v) mod p^e$ and $p \nmid g.c.d.(u,v)$. Thus the number of (x,y) with $p^f \parallel g.c.d.(x,y)$ is $p^{2e} - p^{2(e-1)}$. Multiply this by p^{n-e} to get the number of classes of type $C^{(e)}$. □

Now we use the little group method of Mackey and Wigner to find the **irreducible unitary representations** of $Heis(\mathbb{Z}/p^n\mathbb{Z})$. See Serre [20]. We have a semi-direct product

$$G = (\mathbb{Z}/p^n\mathbb{Z}) = A \cdot H, \text{ where } A = \left\{ (0,y,z) \in G \right\} \text{ and } H = \left\{ (x,0,0) \in G \right\}.$$

We find that $(x,0,0)^{-1}(0,y,z)(x,0,0) = (0,y,z-xy)$. The characters of A have the form $\lambda_{r,s}(0,y,z) = \exp\left(2\pi i(ry+sz)/p^n \right)$ in $\widehat{A}$. The action of $h_x = (x,0,0) \in H$ on $\lambda_{r,s}$ is given by $h_x\lambda_{r,s}(0,y,z) = \lambda_{r,s}(0,y,z-xy) = \lambda_{r-sx,s}(0,y,z)$. It follows that

$$\widehat{A}/H = \left\{ \lambda_{0,s} \mid p \nmid s \right\} \cup \bigcup_{f=1}^{n-1} \left\{ \lambda_{r,s} \,\middle|\, p^f \parallel s, \ r \bmod p^f \right\} \cup \left\{ \lambda_{r,0} \mid r \in \mathbb{Z}/p^n\mathbb{Z} \right\}.$$

Define $H_{r,s} = \left\{ \ (x,0,0) \ \mid \ \lambda_{r,s} = \lambda_{r-sx,s} \ \right\}$. Thus the representations split into types we will call $\Theta^{(f)}_{r,s,t}$, where $0 \le f \le n$.

Look at case f. Here $p^f \parallel s$, $0 \le f \le n$. Then set

$$H^{(f)}_{r,s} = \left\{ \ (x,0,0) \ \middle| \ p^{n-f} \text{ divides } x \ \right\}.$$

This group has order p^f. Define $G^{(f)}_{r,s} = A \cdot H^{(f)}_{r,s}$. Extend $\lambda_{r,s}$ to $G^{(f)}_{r,s}$ by making it constant on $H^{(f)}_{r,s}$. Let ρ_t be a character on $H^{(f)}_{r,s}$ and extend it to $G^{(f)}_{r,s}$ by making it constant on A. Here $\rho_t(x,0,0) = \exp(2\pi i t x/p^n)$, where p^{n-f} divides x. Then $\Theta^{(f)}_{r,s,t}$ is an induced representation defined by inducing the representation $\lambda_{r,s} \otimes \rho_t$ on $G^{(f)}_{r,s}$ up to the full Heisenberg group G; i.e. $\Theta^{(f)}_{r,s,t} = \mathrm{Ind}^{G}_{G^{(f)}_{r,s}}(\lambda_{r,s} \otimes \rho_t)$. Here we take r and $t \mod p^f$, $s = p^f s_1$, where p does not divide s_1 and $s_1 (\mathrm{mod}\, p^{n-f})$. The degree of $\Theta^{(f)}_{r,s,t}$ is p^{n-f}.

Next let us compute the character of $\Theta^{(f)}_{r,s,t}$ using the Frobenius formula for the character of an induced representation. See Terras [24], p. 271. The formula says, if $\psi = \lambda_{r,s} \otimes \rho_t$:

$$\chi_{\Theta^{(f)}_{r,s,t}}(x,y,z) = \sum_{a \in \mathbb{Z}/p^{n-f}\mathbb{Z}} \widetilde{\psi}\left(\ (a,0,0)(x,y,z)(-a,0,0) \ \right).$$

The elements $(a,0,0)$ which are summed over are representatives of the quotient $G/G^{(f)}_{r,s}$. The tilde on the character means that the function is 0 when the argument does not lie in the subgroup $G^{(f)}_{r,s}$. The argument is $(a,0,0)(x,y,z)(-a,0,0) = (x,y,z+ay)$.This is in $G^{(f)}_{r,s}$ when p^{n-f} divides x. So

$$\widetilde{\psi}(x,y,z+ay) = \begin{cases} 0, & \text{if } \ p^{n-f} \nmid x, \\ \exp\left(\ 2\pi i(ry + s(z+ay) + tx)/p^n \right) \ , & \text{if } \ p^{n-f} \mid x. \end{cases}$$

It follows upon summing over $a (\mathrm{mod}\, p^{n-f})$ that for $r, t (\mathrm{mod}\, p^f), s = p^f s_1$, where $p \nmid s_1$, we have

$$\chi_{\Theta^{(f)}_{r,s,t}}(x,y,z) = \begin{cases} 0, & \text{if } \ p^{n-f} \nmid x \ \text{ or } \ p^{n-f} \nmid y, \\ p^{n-f} \exp\left(\ 2\pi i(sz + ry + tx)/p^n \right) \ , & \text{if } \ p^{n-f} \mid x \text{ and } y. \end{cases}$$

We can also compute the matrix entries of the representations as in Terras [24], p. 270. The matrices are $p^{n-f} \times p^{n-f}$, indexed by elements $a, b \in \mathbb{Z}/p^{n-f}\mathbb{Z}$. We find the matrix entries

$$\left(\ \Theta^{(f)}_{r,s,t} \ \right)_{a,b}(x,y,z) = \begin{cases} 0, & \text{if } \ p^{n-f} \nmid (a-b+x), \\ \exp\left(\ 2\pi i \frac{t(a-b+x)+ry+s(z+ay)}{p^n} \ \right), & \text{if } p^{n-f} \mid (a-b+x). \end{cases}$$

In particular, we obtain the following result.

Proposition 1. *When $G = \mathrm{Heis}(\mathbb{Z}/p^n\mathbb{Z})$ the set $\widehat{G}$ of inequivalent irreducible unitary representations of G consists of degree p^{n-f} representations $\Theta^{(f)}_{r,s,t}$ with $0 \le f \le n$, $\quad r, t \in \mathbb{Z}/p^f\mathbb{Z}$, $\quad s = p^f s_1$, $\quad s_1 \in (\mathbb{Z}/p^{n-f}\mathbb{Z})^*$ defined by*

$$\Theta^{(f)}_{r,s,t}(x,y,z) = \exp\left(\frac{2\pi i r y}{p^n}\right) \exp\left(\frac{2\pi i s_1 z}{p^{n-f}}\right) D_f^{sy} W_f(x).$$

Here we need two $p^{n-f} \times p^{n-f}$ matrices: the diagonal matrix D_f involving powers of $w = \exp(2\pi i/p^{n-f})$ given by

$$D_f = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & w & 0 & \cdots & 0 & 0 \\ 0 & 0 & w^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & w^{p^{n-f}-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & w^{p^{n-f}-1} \end{pmatrix}$$

and the matrix $W_f(x)$ whose a, b entry is

$$(W_f(x))_{a,b} = \begin{cases} 0, & \text{if} \quad p^{n-f} \nmid (a-b+x), \\ \exp\left(2\pi i \frac{t(a-b+x)}{p^n} \right), & \text{if} \quad p^{n-f} \mid (a-b+x). \end{cases}$$

Note that when $f = 0$ (in the case of the highest dimensional representation), $W_f(x) = W^x$, where W is the $n \times n$ shift matrix

$$W_f = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

So we find the character table of $G = \mathrm{Heis}(\mathbb{Z}/p^n\mathbb{Z})$ to be that of Table 1 below.

rep\class	$C^{(0)}$	$\cdots$	$C^{(e)}$	$\cdots$	$C^{(n)}$
|class|	1	$\cdots$	p^e	$\cdots$	p^n
# classes	p^n	$\cdots$	$p^{n+e} - p^{n+e-2}$	$\cdots$	$p^{2n} - p^{2n-2}$
$\Theta^{(n)}$	1	$\cdots$	$\Psi(tx+ry)$	$\cdots$	$\Psi(tx+ry)$
$\Theta^{(n-1)}$	$p\Psi(sz)$	$\cdots$	$p\Psi(tx+ry+sz)$	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\Theta^{(e)}$	$p^{n-e}\Psi(sz)$	$\cdots$	$p^{n-e}\Psi(tx+ry+sz)$	$\cdots$	0
$\Theta^{(e+1)}$	$p^{n-e-1}\Psi(sz)$	$\cdots$	0	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\Theta^{(0)}$	$p^n\Psi(sz)$	$\cdots$	0	$\cdots$	0

Table 1. Character Table of $G = \mathrm{Heis}(\mathbb{Z}/p^n\mathbb{Z})$. The conjugacy classes of type $C^{(e)} = C^{(e)}_{x,y,v}$ are as in Lemma 1 with $p^{n-e} \parallel \text{g.c.d.}(x, y)$. The representations $\Theta^{(f)} = \Theta^{(f)}_{r,s,t}$ are as in Proposition 1, with $f = 0, 1, \ldots, n$, both r and t in $\mathbb{Z}/p^f\mathbb{Z}$, $s = p^f s_1$, and $s_1 \in (\mathbb{Z}/p^{n-f}\mathbb{Z})^*$. Write $\Psi(x) = \exp(2\pi i x/p^n)$, for $x \in \mathbb{Z}/p^n\mathbb{Z}$.

We will consider Cayley graphs with vertex set $Heis(\mathbb{Z}/p^n\mathbb{Z})$ and edge set $S = \{ X^{\pm 1}, A^{\pm 1} \}$, where $X = (x, y, z)$ and $A = (a, b, c)$. We want connected graphs. Thus we need the following theorem.

Theorem 1. *Suppose that $S = \{ X^{\pm 1}, A^{\pm 1} \}$, where $X = (x, y, z)$ and $A = (a, b, c)$ are in $H = Heis(\mathbb{Z}/p^n\mathbb{Z})$. If $ay \not\equiv bx \pmod p$ then the set S generates H.*

Proof. Let G be the subgroup of $H = Heis(\mathbb{Z}/p^n\mathbb{Z})$ generated by S. Note that

$$(a,b,c)(x,y,z) = (a+x, b+y, c+z+ay)$$

and $(a,b,c)^{-1} = (-a,-b,-c+ab)$. Also we have $(a,b,c)^e(x,y,z)^f = (ae+fx, be+fy, *)$. Then for every $\begin{pmatrix} u \\ v \end{pmatrix} \in (\mathbb{Z}/p^n\mathbb{Z})^2$, the matrix equation

$$\begin{pmatrix} ae+xf \\ be+yf \end{pmatrix} = \begin{pmatrix} a & x \\ b & y \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

is solvable for some $\begin{pmatrix} e \\ f \end{pmatrix} \in (\mathbb{Z}/p^n\mathbb{Z})^2$, as the determinant of the 2×2 matrix, $ay - bx$, is a unit. So we know that G contains elements $g = (1,0,r)$ and $h = (0,1,s)$, for some r, s in $\mathbb{Z}/p^n\mathbb{Z}$. Now

$$gh = (1,1,r+s+1), \quad (gh)^{-1} = (-1,-1,-r-s), \quad hg = (1,1,r+s).$$

It follows that $(gh)^{-1}hg = (0,0,-1)$. Thus G contains $(0,0,w)$, for all $w \in \mathbb{Z}/p^n\mathbb{Z}$. Now for every (u,v,w) in H there is a w_0 such that $(u,v,w_0) \in G$. Then $(u,v,w) = (u,v,w_0)(0,0,w-w_0)$ which puts $(u,v,w) \in G$. This completes the proof that $G = H = Heis(\mathbb{Z}/p^n\mathbb{Z})$. □

Next we consider the Cayley graphs with vertex set $Heis(\mathbb{Z}/p^n\mathbb{Z})$ as in Theorem 1. We will show that for odd p they are all isomorphic. In order to do this, we need a Lemma.

Lemma 2. *Consider the element $(x,y,z) \in Heis(\mathbb{Z}/p^n\mathbb{Z})$. Then, for all integers e, $(x,y,z)^e = \left(ex, ey, ez + \frac{e(e-1)xy}{2} \right)$.*

Proof. This follows by induction on e. □

Corollary 1. *Let $(x,y,z) \in Heis(\mathbb{Z}/p^n\mathbb{Z})$. Then we have the following facts.*
Case 1) *Assume p is an odd prime.*
a) $(x,y,z)^{p^n} = (0,0,0)$.
b) If x or y is not divisible by p, then the order of (x,y,z) is p^n.
Case 2) *Let $p = 2$.*
a) $(x,y,z)^{2^{n+1}} = (0,0,0)$.
b) If both x and y are odd, the order of (x,y,z) is 2^{n+1}.
c) If one of x and y is odd and the other is even, then the order of (x,y,z) is 2^n.

Theorem 2. *Suppose the prime $p \neq 2$. Then the Cayley graphs $X(G,S)$ with vertex set $G = Heis(\mathbb{Z}/p^n\mathbb{Z})$ and edge set $S = \{ X^{\pm 1}, A^{\pm 1} \}$, where $X = (x,y,z)$ and $A = (a,b,c)$ with $ay \not\equiv bx (\mathrm{mod}\, p)$, are all isomorphic. When $p = 2$, there are at most 2 isomorphism classes of graphs.*

Proof. First assume that p is not 2. Note that the center of $Heis(\mathbb{Z}/p^n\mathbb{Z})$ consists of elements $(0,0,z)$, $z \in \mathbb{Z}/p^n\mathbb{Z}$. The order of both A and X is p^n. And $AX \neq XA$ implies A and X are not in the center. Look at $C = XAX^{-1}A^{-1}$. Then we have the relations.

$$A^{p^n} = X^{p^n} = C^{p^n} = I, \quad XC = CX, \quad XA = AXC. \tag{1}$$

It follows that

$$A^i X^j C^k A^{i'} X^{j'} C^{k'} = A^{i+i'} X^{j+j'} C^{k+k'+i'j}. \tag{2}$$

Note that $(i', j', k')(i, j, k) = (i+i', j+j', k+k'+i'j)$. Define the map $F(A^i X^j C^k) = (i, j, k)^{-1} \in Heis(\mathbb{Z}/p^n\mathbb{Z})$, for all $i, j, k \in \mathbb{Z}$. Then F gives a graph isomorphism $F : X(G, S) \to X(G, S_0)$, where

$$S_0 = \left\{ \ (\pm 1, 0, 0), (0, \pm 1, 0) \ \right\}.$$

To see this, you need to know that every element of $G = Heis(\mathbb{Z}/p^n\mathbb{Z})$ can be written uniquely in the form $A^i X^j C^k$, where $0 \leq i, j, k < p^n$. Since A and X generate G, the relations (1) above show that every element of G can be written in this form and since there are exactly p^{3n} elements in G, the result is unique. The proof for $p = 2$ and $n \geq 2$ is similar except that we have two cases. In the case that exactly two of a, b, x, y are odd, we obtain an isomorphism from $X(G, S)$ to $X(G, S_0)$, where $S_0 = \left\{ \ (\pm 1, 0, 0), (0, \pm 1, 0) \ \right\}$. In the case that $p = 2$, $n \geq 2$, and exactly three of a, b, x, y are odd, we obtain an isomorphism from $X(G, S)$ to $X(G, S_0)$ with $S_0 = \left\{ \ (\pm 1, 0, 0), (1, 1, 0)^{\pm 1} \ \right\}$. Without loss of generality, we can assume a and b are both odd and thus that the order of A is 2^{n+1}. Set $C = XAX^{-1}A^{-1} = (0, 0, bx - ay)$. Since $bx - ay$ is odd, the order of C is 2^n. Then we have $A^{2^n} = C^{2^{n-1}}$. In this case it can occur that $i + i' \geq 2^n$. If so, we can find $m \in \{0, 1\}$ so that $i + i' - 2^n m \leq 2^n$ and replace equation (2) with

$$A^i X^j C^k A^{i'} X^{j'} C^{k'} = A^{i+i'-2^n m} X^{j+j'} C^{k+k'+i'j+2^{n-1}m}. \tag{3}$$

This same equation holds with our choice of generators in S_0 in Case 2. From here the proof proceeds as before. □

Definition 1. For all prime p, define the Cayley graph

$$\mathcal{H}(p^n) = X\left(\mathrm{Heis}(\mathbb{Z}/p^n\mathbb{Z}), \{(\pm 1, 0, 0), (0, \pm 1, 0)\}\right).$$

When $p = 2$, define a second Cayley graph

$$\mathcal{H}(2^n)' = X\left(\mathrm{Heis}(\mathbb{Z}/2^n\mathbb{Z}), \left\{(1, 1, 0)^{\pm 1}, (\pm 1, 0, 0)\right\}\right).$$

Next we study the spectra of the graphs $\mathcal{H}(p^n)$ and $\mathcal{H}(p^n)'$. Theorem 3 says that the spectra approach a continuous line segment $[-4, 4]$ as p^n approaches infinity. To prove it we need a familiar result in spectral graph theory.

Proposition 2. *Suppose $X = X(G, S)$ is a Cayley graph of a finite group G. Then the eigenvalues of the adjacency operator of X are the eigenvalues of the $d_\pi \times d_\pi$ matrices $M_\pi = \sum_{s \in S} \pi(s)$, each taken with multiplicity d_π as π runs through $\widehat{G}$, a complete set of irreducible unitary representations of G.*

Proof. See Terras [24], p. 257. □

Examples.

It is not hard to use Propositions 1, 2, and a home computer to find spectra of $\mathcal{H}(q)$ and $\mathcal{H}(q)'$ when q is less than or equal to the size of a matrix your computer can handle. Figure 1 shows a histogram for the graph $\mathcal{H}(64)$. The histogram is obtained by dividing up the interval $[-4, 4]$ into n subintervals. The height of the rectangle at the jth subinterval I_j is the number of eigenvalues of the adjacency matrix of $\mathcal{H}(64)$ in subinterval I_j. Here the degree is 4 and the shape of the histogram is quite distinctive. It is the shape we see for any of these degree 4 Heisenberg graphs. On the web site

http : //math.ucsd.edu/˜aterras/heis.pdf

one can find histograms for degree 6 Heisenberg graphs as well.

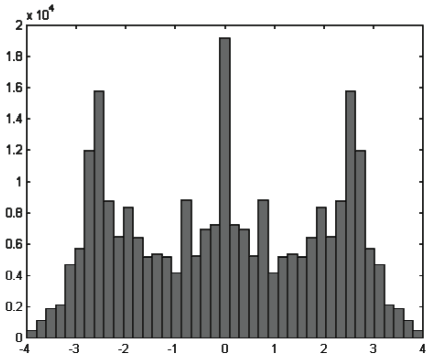

Figure 1. Histogram for spectrum of adjacency operator of $\mathcal{H}(64)$.

Theorem 3. *The spectra of the degree 4 Heisenberg graphs $\mathcal{H}(p^n)$ and $\mathcal{H}(p^n)'$ (see Definition 1) approach a continuous interval $[-4,4]$ as $p^n \to \infty$.*

Proof. Use Proposition 2 and look only at the eigenvalues corresponding to the one dimensional representations of $G = Heis(\mathbb{Z}/p^n\mathbb{Z})$. This corresponds to the top line in Table 1. Let us assume that p is odd and our set S is the set $S_0 = \{ (\pm 1, 0, 0), (0, \pm 1, 0) \}$. Then we have an eigenvalue corresponding to

$$\Theta^{(f)}_{r,0,t}(x,y,z) = \exp\left(2\pi i(tx + ry)/p^n \right)$$

where $r, t \in \mathbb{Z}/p^n\mathbb{Z}$. Then the corresponding eigenvalue of the adjacency matrix of $\mathcal{H}(p^n)$ is

$$\lambda_{r,t} = 2\cos\left(\tfrac{2\pi t}{p^n} \right) + 2\cos\left(\tfrac{2\pi r}{p^n} \right). \tag{4}$$

As r varies between 0 and $p^n - 1$, the graph of $\cos(2\pi r/p^n)$ approximates a continuous line between -1 and $+1$. The result follows. In the case $p = 2$, we can make a similar argument for $\mathcal{H}(p^n)'$. □

3. **Comparisons of Spectra and Butterflies.** One should compare our previous figures with histograms of spectra of degree 6 Heisenberg graphs as well as **torus graphs**

$$\mathcal{T}^{(n)}(q) = X((\mathbb{Z}/q\mathbb{Z})^n, \{\pm e_1, \pm e_2, \cdots, \pm e_n\}),$$

where e_i denotes the vector with 1 in the ith coordinate and 0 elsewhere. Because the torus groups $(\mathbb{Z}/q\mathbb{Z})^n$ are abelian, it is relatively easy to generate these figures. In fact, the eigenvalues of the adjacency matrix of $\mathcal{T}^{(n)}(q)$ are, for $a, b \in (\mathbb{Z}/q\mathbb{Z})^n$

$$\lambda_{a,b} = 2\left(\cos\left(\tfrac{2\pi i a_1 b_1}{q} \right) + \cos\left(\tfrac{2\pi i a_2 b_2}{q} \right) + \cdots + \cos\left(\tfrac{2\pi i a_n b_n}{q} \right) \right).$$

Note that, by equation (4), the spectrum of the degree 4 Heisenberg graph $\mathcal{H}(q)$ contains the spectrum of $\mathcal{T}^{(2)}(q)$.

The histogram for $\mathcal{T}^{(1)}(q) = X(\mathbb{Z}/q\mathbb{Z}, \{\pm 1\})$ is easily analyzed and seen to approach the limiting density $f(x) = \frac{1}{\pi\sqrt{1-(x/2)^2}}$, as $q \to \infty$. If you make the substitution $u = x^2/4$, you obtain the density for the arc sine law (see Feller [6]). It follows that the limiting density for $\mathcal{T}^{(n)}(q)$ is $\underbrace{f * f * \cdots * f}_{n}$. There is a big difference

between the three peak $\mathcal{H}(q)$ histogram in Figure 1 and the one peak histogram for $\mathcal{T}^{(2)}(q)$. The histograms for degree 6 Heisenberg graphs as well as some torus graphs can be found on the website: http://math.ucsd.edu/~aterras/heis.pdf.

In Figure 2 we present a Hofstadter butterfly-type picture. To draw this picture, one must separate the part of the spectrum of the Cayley graph $\mathcal{H}(q)$ corresponding to the q-dimensional representations of $\mathrm{Heis}(\mathbb{Z}/q\mathbb{Z})$ denoted $\pi_s = \Theta^{(0)}_{0,s,0}$, $s \in (\mathbb{Z}/q\mathbb{Z})^*$. Plot the part of the spectrum corresponding to π_s by Proposition 2 as points in the plane with y-coordinate s/q and x coordinate given by the various eigenvalues λ of the matrix $M_{\pi_s} = \sum_{u \in S} \pi_s(u)$,where $S = \{(\pm 1, 0, 0), (0 \pm 1, 0)\}$ denotes the edge set for the Cayley graph $\mathcal{H}(q) = X(\mathrm{Heis}(\mathbb{Z}/q\mathbb{Z}), S)$. A Hofstadter butterfly-type picture for a degree 6 Heisenberg graph can be found on the website mentioned above.

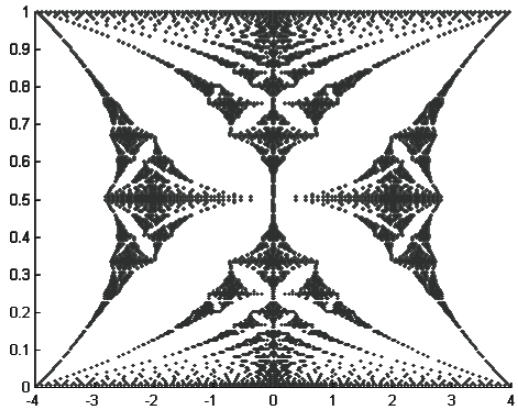

Figure 2. Hofstadter Butterfly Graph for the Spectrum of $\mathcal{H}(169)$.

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