1. Introduction

Not written as of yet. Topics to mention.

(1) A better and more general integral.
   (a) Convergence Theorems
   (b) Integration over diverse collection of sets. (See probability theory.)
   (c) Integration relative to different weights or densities including singular weights.
   (d) Characterization of dual spaces.
   (e) Completeness.
(2) Infinite dimensional Linear algebra.
(3) ODE and PDE.
(4) Harmonic and Fourier Analysis.
(5) Probability Theory

2. Limits, sums, and other basics

2.1. **Set Operations.** Suppose that X is a set. Let $\mathcal{P}(X)$ or 2^X denote the power set of X, that is elements of $\mathcal{P}(X) = 2^X$ are subsets of A. For $A \in 2^X$ let

$$A^c = X \setminus A = \{x \in X : x \notin A\}$$

and more generally if $A, B \subset X$ let

$$B \setminus A = \{x \in B : x \notin A\}.$$

We also define the symmetric difference of A and B by

$$A \triangle B = (B \setminus A) \cup (A \setminus B).$$

As usual if $\{A_\alpha\}_{\alpha\in I}$ is an indexed collection of subsets of X we define the union and the intersection of this collection by

$$\cup_{\alpha\in I} A_\alpha := \{x \in X : \exists\ \alpha \in I\ \ni\ x \in A_\alpha\} \text{ and}$$
$$\cap_{\alpha\in I} A_\alpha := \{x \in X : x \in A_\alpha\ \forall\ \alpha \in I\ \}.$$

Notation 2.1. We will also write $\coprod_{\alpha\in I} A_\alpha$ for $\cup_{\alpha\in I} A_\alpha$ in the case that $\{A_\alpha\}_{\alpha\in I}$ are pairwise disjoint, i.e. $A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$.

Notice that $\cup$ is closely related to $\exists$ and $\cap$ is closely related to $\forall$. For example let $\{A_n\}_{n=1}^\infty$ be a sequence of subsets from X and define

$$\{A_n \text{ i.o.}\} := \{x \in X : \#\{n : x \in A_n\} = \infty\} \text{ and}$$
$$\{A_n \text{ a.a.}\} := \{x \in X : x \in A_n \text{ for all } n \text{ sufficiently large}\}.$$

(One should read $\{A_n \text{ i.o.}\}$ as A_n infinitely often and $\{A_n \text{ a.a.}\}$ as A_n almost always.) Then $x \in \{A_n \text{ i.o.}\}$ iff $\forall N \in \mathbb{N}\ \exists n \geq N \ni x \in A_n$ which may be written as

$$\{A_n \text{ i.o.}\} = \cap_{N=1}^\infty \cup_{n\geq N} A_n.$$

Similarly, $x \in \{A_n \text{ a.a.}\}$ iff $\exists\ N \in \mathbb{N} \ni \forall\ n \geq N,\ x \in A_n$ which may be written as

$$\{A_n \text{ a.a.}\} = \cup_{N=1}^\infty \cap_{n\geq N} A_n.$$

2.2. **Limits, Limsups, and Liminfs.**

Notation 2.2. The Extended real numbers is the set $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, i.e. it is $\mathbb{R}$ with two new points called ∞ and $-\infty$. We use the following conventions, $\pm\infty \cdot 0 = 0$, $\pm\infty + a = \pm\infty$ for any $a \in \mathbb{R}$, $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$ while $\infty - \infty$ is not defined.

If $\Lambda \subset \bar{\mathbb{R}}$ we will let $\sup \Lambda$ and $\inf \Lambda$ denote the least upper bound and greatest lower bound of Λ respectively. We will also use the following convention, if $\Lambda = \emptyset$, then $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

Notation 2.3. Suppose that $\{x_n\}_{n=1}^{\infty} \subset \bar{\mathbb{R}}$ is a sequence of numbers. Then

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} \text{ and} \tag{2.1}$$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\}. \tag{2.2}$$

We will also write $\underline{\lim}$ for $\liminf$ and $\overline{\lim}$ for $\limsup$.

Remark 2.4. Notice that if $a_k := \inf\{x_k : k \geq n\}$ and $b_k := \sup\{x_k : k \geq n\}$,then $\{a_k\}$ is an increasing sequence while $\{b_k\}$ is a decreasing sequence. Therefore the limits in Eq. (2.1) and Eq. (2.2) always exist and

$$\liminf_{n\to\infty} x_n = \sup_n \inf\{x_k : k \geq n\} \text{ and}$$
$$\limsup_{n\to\infty} x_n = \inf_n \sup\{x_k : k \geq n\}.$$

The following proposition contains some basic properties of liminfs and limsups.

Proposition 2.5. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers. Then*

(1) $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} a_n$ *exists in* $\bar{\mathbb{R}}$ *iff* $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n \in \bar{\mathbb{R}}$.
(2) *There is a subsequence* $\{a_{n_k}\}_{k=1}^{\infty}$ *of* $\{a_n\}_{n=1}^{\infty}$ *such that* $\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n$.
(3)
$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \tag{2.3}$$
whenever the right side of this equation is not of the form $\infty - \infty$.
(4) *If* $a_n \geq 0$ *and* $b_n \geq 0$ *for all* $n \in \mathbb{N}$, *then*
$$\limsup_{n\to\infty} (a_n b_n) \leq \limsup_{n\to\infty} a_n \cdot \limsup_{n\to\infty} b_n, \tag{2.4}$$
provided the right hand side of (2.4) is not of the form $0 \cdot \infty$ *or* $\infty \cdot 0$.

Proof. We will only prove part 1. and leave the rest as an exercise to the reader. We begin by noticing that

$$\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\} \; \forall n$$

so that

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

Now suppose that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a \in \mathbb{R}$. Then for all $\epsilon > 0$, there is an integer N such that

$$a - \epsilon \leq \inf\{a_k : k \geq N\} \leq \sup\{a_k : k \geq N\} \leq a + \epsilon,$$

i.e.

$$a-\epsilon \le a_k \le a+\epsilon \text{ for all } k \ge N.$$

Hence by the definition of the limit, $\lim_{k\to\infty} a_k = a$.

If $\liminf_{n\to\infty} a_n = \infty$, then we know for all $M \in (0,\infty)$ there is an integer N such that

$$M \le \inf\{a_k : k \ge N\}$$

and hence $\lim_{n\to\infty} a_n = \infty$. The case where $\limsup_{n\to\infty} a_n = -\infty$ is handled similarly.

Conversely, suppose that $\lim_{n\to\infty} a_n = A \in \bar{\mathbb{R}}$ exists. If $A \in \mathbb{R}$, then for every $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that $|A - a_n| \le \epsilon$ for all $n \ge N(\epsilon)$, i.e.

$$A-\epsilon \le a_n \le A+\epsilon \text{ for all } n \ge N(\epsilon).$$

From this we learn that

$$A-\epsilon \le \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n \le A+\epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$A \le \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n \le A,$$

i.e. that $A = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

If $A = \infty$, then for all $M > 0$ there exist $N(M)$ such that $a_n \ge M$ for all $n \ge N(M)$. This show that

$$\liminf_{n\to\infty} a_n \ge M$$

and since M is arbitrary it follows that

$$\infty \le \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n.$$

The proof is similar if $A = -\infty$ as well. ■

2.3. **Sums of positive functions.** In this and the next few sections, let X and Y be two sets. We will write $\alpha \subset\subset X$ to denote that α is a **finite** subset of X.

Definition 2.6. Suppose that $a : X \to [0,\infty]$ is a function and $F \subset X$ is a subset, then

$$\sum_F a = \sum_{x\in F} a(x) = \sup\left\{\sum_{x\in\alpha} a(x) : \alpha \subset\subset F\right\}.$$

Remark 2.7. Suppose that $X = \mathbb{N} = \{1,2,3,\dots\}$, then

$$\sum_{\mathbb{N}} a = \sum_{n=1}^{\infty} a(n) := \lim_{N\to\infty} \sum_{n=1}^{N} a(n).$$

Indeed for all N, $\sum_{n=1}^N a(n) \le \sum_{\mathbb{N}} a$, and thus passing to the limit we learn that

$$\sum_{n=1}^{\infty} a(n) \le \sum_{\mathbb{N}} a.$$

Conversely, if $\alpha \subset\subset \mathbb{N}$, then for all N large enough so that $\alpha \subset \{1,2,\dots,N\}$, we have $\sum_\alpha a \le \sum_{n=1}^N a(n)$ which upon passing to the limit implies that

$$\sum_\alpha a \le \sum_{n=1}^{\infty} a(n)$$

and hence by taking the supremum over α we learn that

$$\sum_{\mathbb{N}} a \leq \sum_{n=1}^{\infty} a(n).$$

Remark 2.8. Suppose that $\sum_X a < \infty$, then $\{x \in X : a(x) > 0\}$ is at most countable. To see this first notice that for any $\epsilon > 0$, the set $\{x : a(x) \geq \epsilon\}$ must be finite for otherwise $\sum_X a = \infty$. Thus

$$\{x \in X : a(x) > 0\} = \bigcup\nolimits_{k=1}^{\infty} \{x : a(x) \geq 1/k\}$$

which shows that $\{x \in X : a(x) > 0\}$ is a countable union of finite sets and thus countable.

Lemma 2.9. *Suppose that $a, b : X \to [0, \infty]$ are two functions, then*

$$\sum_X (a + b) = \sum_X a + \sum_X b \text{ and}$$
$$\sum_X \lambda a = \lambda \sum_X a$$

for all $\lambda \geq 0$.

I will only prove the first assertion, the second being easy. Let $\alpha \subset\subset X$ be a finite set, then

$$\sum_\alpha (a + b) = \sum_\alpha a + \sum_\alpha b \leq \sum_X a + \sum_X b$$

which after taking sups over α shows that

$$\sum_X (a + b) \leq \sum_X a + \sum_X b.$$

Similarly, if $\alpha, \beta \subset\subset X$, then

$$\sum_\alpha a + \sum_\beta b \leq \sum_{\alpha \cup \beta} a + \sum_{\alpha \cup \beta} b = \sum_{\alpha \cup \beta} (a + b) \leq \sum_X (a + b).$$

Taking sups over α and β then shows that

$$\sum_X a + \sum_X b \leq \sum_X (a + b).$$

Lemma 2.10. *Let X and Y be sets, $R \subset X \times Y$ and suppose that $a : R \to \bar{\mathbb{R}}$ is a function. Let ${}_xR := \{y \in Y : (x, y) \in R\}$ and $R_y := \{x \in X : (x, y) \in R\}$. Then*

$$\sup_{(x,y) \in R} a(x, y) = \sup_{x \in X} \sup_{y \in {}_xR} a(x, y) = \sup_{y \in Y} \sup_{x \in R_y} a(x, y) \text{ and}$$
$$\inf_{(x,y) \in R} a(x, y) = \inf_{x \in X} \inf_{y \in {}_xR} a(x, y) = \inf_{y \in Y} \inf_{x \in R_y} a(x, y).$$

(Recall the conventions: $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.)

Proof. Let $M = \sup_{(x,y) \in R} a(x, y)$, $N_x := \sup_{y \in {}_xR} a(x, y)$. Then $a(x, y) \leq M$ for all $(x, y) \in R$ implies $N_x = \sup_{y \in {}_xR} a(x, y) \leq M$ and therefore that

$$\sup_{x \in X} \sup_{y \in {}_xR} a(x, y) = \sup_{x \in X} N_x \leq M. \tag{2.5}$$

Similarly for any $(x, y) \in R$,

$$a(x, y) \leq N_x \leq \sup_{x \in X} N_x = \sup_{x \in X} \sup_{y \in {}_xR} a(x, y)$$

and therefore

$$\sup_{(x,y)\in R} a(x,y) \le \sup_{x\in X} \sup_{y\in_x R} a(x,y) = M \tag{2.6}$$

Equations (2.5) and (2.6) show that

$$\sup_{(x,y)\in R} a(x,y) = \sup_{x\in X} \sup_{y\in_x R} a(x,y).$$

The assertions involving infinums are proved analogously or follow from what we have just proved applied to the function $-a$. ■

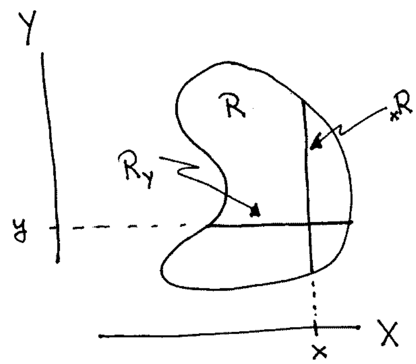

FIGURE 1. The x and y – slices of a set $R \subset X \times Y$.

Theorem 2.11 (Monotone Convergence Theorem for Sums). *Suppose that $f_n : X \to [0,\infty]$ is an increasing sequence of functions and*

$$f(x) := \lim_{n\to\infty} f_n(x) = \sup_n f_n(x).$$

Then

$$\lim_{n\to\infty} \sum_X f_n = \sum_X f$$

Proof. We will give two proves. For the first proof, let $\mathcal{P}_f(X) = \{A \subset X : A \subset\subset X\}$. Then

$$\begin{aligned}\lim_{n\to\infty} \sum_X f_n &= \sup_n \sum_X f_n = \sup_n \sup_{\alpha\in\mathcal{P}_f(X)} \sum_\alpha f_n = \sup_{\alpha\in\mathcal{P}_f(X)} \sup_n \sum_\alpha f_n \\ &= \sup_{\alpha\in\mathcal{P}_f(X)} \lim_{n\to\infty} \sum_\alpha f_n = \sup_{\alpha\in\mathcal{P}_f(X)} \sum_\alpha \lim_{n\to\infty} f_n = \sup_{\alpha\in\mathcal{P}_f(X)} \sum_\alpha f = \sum_X f.\end{aligned}$$

(Second Proof.) Let $S_n = \sum_X f_n$ and $S = \sum_X f$. Since $f_n \le f_m \le f$ for all $n \le m$, it follows that

$$S_n \le S_m \le S$$

which shows that $\lim_{n\to\infty} S_n$ exists and is less that S, i.e.

$$A := \lim_{n\to\infty} \sum_X f_n \le \sum_X f. \tag{2.7}$$

Noting that $\sum_\alpha f_n \le \sum_X f_n = S_n \le A$ for all $\alpha \subset\subset X$ and in particular,

$$\sum_\alpha f_n \le A \text{ for all } n \text{ and } \alpha \subset\subset X.$$

Letting n tend to infinity in this equation shows that

$$\sum_\alpha f \le A \text{ for all } \alpha \subset\subset X$$

and then taking the sup over all $\alpha \subset\subset X$ gives

$$\sum_X f \le A = \lim_{n\to\infty} \sum_X f_n \tag{2.8}$$

which combined with Eq. (2.7) proves the theorem. ■

Lemma 2.12 (Fatou's Lemma for Sums)**.** *Suppose that $f_n : X \to [0,\infty]$ is a sequence of functions, then*

$$\sum_X \lim\inf_{n\to\infty} f_n \le \lim\inf_{n\to\infty} \sum_X f_n.$$

Proof. Define $g_k \equiv \inf_{n\ge k} f_n$ so that $g_k \uparrow \lim\inf_{n\to\infty} f_n$ as $k\to\infty$. Since $g_k \le f_n$ for all $k \le n$,

$$\sum_X g_k \le \sum_X f_n \text{ for all } n \ge k$$

and therefore

$$\sum_X g_k \le \lim\inf_{n\to\infty} \sum_X f_n \text{ for all } k.$$

We may now use the monotone convergence theorem to let $k\to\infty$ to find

$$\sum_X \lim\inf_{n\to\infty} f_n = \sum_X \lim_{k\to\infty} g_k \overset{\text{MCT}}{=} \lim_{k\to\infty} \sum_X g_k \le \lim\inf_{n\to\infty} \sum_X f_n.$$

■

Remark 2.13. If $A = \sum_X a < \infty$, then for all $\epsilon > 0$ there exists $\alpha_\epsilon \subset\subset X$ such that

$$A \ge \sum_\alpha a \ge A - \epsilon$$

for all $\alpha \subset\subset X$ containing α_ϵ or equivalently,

$$\left| A - \sum_\alpha a \right| \le \epsilon \tag{2.9}$$

for all $\alpha \subset\subset X$ containing α_ϵ. Indeed, choose α_ϵ so that $\sum_{\alpha_\epsilon} a \ge A - \epsilon$.

2.4. **Sums of complex functions.**

Definition 2.14. Suppose that $a : X \to \mathbb{C}$ is a function, we say that

$$\sum_X a = \sum_{x\in X} a(x)$$

exists and is equal to $A \in \mathbb{C}$, if for all $\epsilon > 0$ there is a finite subset $\alpha_\epsilon \subset X$ such that for all $\alpha \subset\subset X$ containing α_ϵ we have

$$\left| A - \sum_\alpha a \right| \le \epsilon.$$

The following lemma is left as an exercise to the reader.

Lemma 2.15. *Suppose that $a, b : X \to \mathbb{C}$ are two functions such that $\sum_X a$ and $\sum_X b$ exist, then $\sum_X (a + \lambda b)$ exists for all $\lambda \in \mathbb{C}$ and*

$$\sum_X (a + \lambda b) = \sum_X a + \lambda \sum_X b.$$

Definition 2.16 (Summable)**.** We call a function $a : X \to \mathbb{C}$ **summable** if

$$\sum_X |a| < \infty.$$

Proposition 2.17. *Let $a : X \to \mathbb{C}$ be a function, then $\sum_X a$ exists iff $\sum_X |a| < \infty$, i.e. iff a is summable.*

Proof. If $\sum_X |a| < \infty$, then $\sum_X (\operatorname{Re} a)^{\pm} < \infty$ and $\sum_X (\operatorname{Im} a)^{\pm} < \infty$ and hence by Remark 2.13 these sums exists in the sense of Definition 2.14. Therefore by Lemma 2.15, $\sum_X a$ exists and

$$\sum_X a = \sum_X (\operatorname{Re} a)^+ - \sum_X (\operatorname{Re} a)^- + i \left(\sum_X (\operatorname{Im} a)^+ - \sum_X (\operatorname{Im} a)^- \right).$$

Conversely, if $\sum_X |a| = \infty$ then, because $|a| \le |\operatorname{Re} a| + |\operatorname{Im} a|$, we must have

$$\sum_X |\operatorname{Re} a| = \infty \text{ or } \sum_X |\operatorname{Im} a| = \infty.$$

Thus it suffices to consider the case where $a : X \to \mathbb{R}$ is a real function. Write $a = a^+ - a^-$ where

$$a^+(x) = \max(a(x), 0) \text{ and } a^-(x) = \max(-a(x), 0). \tag{2.10}$$

Then $|a| = a^+ + a^-$ and

$$\infty = \sum_X |a| = \sum_X a^+ + \sum_X a^-$$

which shows that either $\sum_X a^+ = \infty$ or $\sum_X a^- = \infty$. Suppose, with out loss of generality, that $\sum_X a^+ = \infty$. Let $X' := \{x \in X : a(x) \ge 0\}$, then we know that $\sum_{X'} a = \infty$ which means there are finite subsets $\alpha_n \subset X' \subset X$ such that $\sum_{\alpha_n} a \ge n$ for all n. Thus if $\alpha \subset\subset X$ is any finite set, it follows that $\lim_{n\to\infty} \sum_{\alpha_n \cup \alpha} a = \infty$, and therefore $\sum_X a$ can not exist as a number in $\mathbb{R}$. ■

Remark 2.18. Suppose that $X = \mathbb{N}$ and $a : \mathbb{N} \to \mathbb{C}$ is a sequence, then it is not necessarily true that

$$\sum_{n=1}^{\infty} a(n) = \sum_{n \in \mathbb{N}} a(n). \tag{2.11}$$

This is because

$$\sum_{n=1}^{\infty} a(n) = \lim_{N\to\infty} \sum_{n=1}^{N} a(n)$$

depends on the ordering of the sequence a where as $\sum_{n\in\mathbb{N}} a(n)$ does not. For example, take $a(n) = (-1)^n/n$ then $\sum_{n\in\mathbb{N}} |a(n)| = \infty$ i.e. $\sum_{n\in\mathbb{N}} a(n)$ does **not** exist while $\sum_{n=1}^{\infty} a(n)$ does exist. On the other hand, if

$$\sum_{n\in\mathbb{N}} |a(n)| = \sum_{n=1}^{\infty} |a(n)| < \infty$$

then Eq. (2.11) is valid.

Theorem 2.19 (Dominated Convergence Theorem for Sums)**.** *Suppose that $f_n : X \to \mathbb{C}$ is a sequence of functions on X such that $f(x) = \lim_{n\to\infty} f_n(x) \in \mathbb{C}$ exists for all $x \in X$. Further assume there is a* ***dominating function*** *$g : X \to [0,\infty)$ such that*

$$|f_n(x)| \le g(x) \text{ for all } x \in X \text{ and } n \in \mathbb{N} \tag{2.12}$$

and that g is summable. Then

$$\lim_{n\to\infty} \sum_{x\in X} f_n(x) = \sum_{x\in X} f(x). \tag{2.13}$$

Proof. Notice that $|f| = \lim |f_n| \le g$ so that f is summable. By considering the real and imaginary parts of f separately, it suffices to prove the theorem in the case where f is real. By Fatou's Lemma,

$$\begin{aligned}\sum_X (g \pm f) &= \sum_X \liminf_{n\to\infty} (g \pm f_n) \le \liminf_{n\to\infty} \sum_X (g \pm f_n) \\ &= \sum_X g + \liminf_{n\to\infty} \left(\pm \sum_X f_n\right).\end{aligned}$$

Since $\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty} a_n$, we have shown,

$$\sum_X g \pm \sum_X f \le \sum_X g + \begin{cases} \liminf_{n\to\infty} \sum_X f_n \\ -\limsup_{n\to\infty} \sum_X f_n \end{cases}$$

and therefore

$$\limsup_{n\to\infty} \sum_X f_n \le \sum_X f \le \liminf_{n\to\infty} \sum_X f_n.$$

This shows that $\lim\limits_{n\to\infty} \sum_X f_n$exists and is equal to $\sum_X f$. ■

Proof. (Second Proof.) Passing to the limit in Eq. (2.12) shows that $|f| \le g$ and in particular that f is summable. Given $\epsilon > 0$, let $\alpha \subset\subset X$ such that

$$\sum_{X\setminus\alpha} g \le \epsilon.$$

Then for $\beta \subset\subset X$ such that $\alpha \subset \beta$,

$$\begin{aligned}\left|\sum_\beta f - \sum_\beta f_n\right| &= \left|\sum_\beta (f - f_n)\right| \\ &\le \sum_\beta |f - f_n| = \sum_\alpha |f - f_n| + \sum_{\beta\setminus\alpha} |f - f_n| \\ &\le \sum_\alpha |f - f_n| + 2\sum_{\beta\setminus\alpha} g \\ &\le \sum_\alpha |f - f_n| + 2\epsilon.\end{aligned}$$

and hence that

$$\left|\sum_{\beta} f - \sum_{\beta} f_n\right| \leq \sum_{\alpha} |f - f_n| + 2\epsilon.$$

Since this last equation is true for all such $\beta \subset\subset X$, we learn that

$$\left|\sum_{X} f - \sum_{X} f_n\right| \leq \sum_{\alpha} |f - f_n| + 2\epsilon$$

which then implies that

$$\limsup_{n\to\infty} \left|\sum_{X} f - \sum_{X} f_n\right| \leq \limsup_{n\to\infty} \sum_{\alpha} |f - f_n| + 2\epsilon$$
$$= 2\epsilon.$$

Because $\epsilon > 0$ is arbitrary we conclude that

$$\limsup_{n\to\infty} \left|\sum_{X} f - \sum_{X} f_n\right| = 0.$$

which is the same as Eq. (2.13). ■

2.5. **Iterated sums.** Let X and Y be two sets. The proof of the following lemma is left to the reader.

Lemma 2.20. *Suppose that $a : X \to \mathbb{C}$ is function and $F \subset X$ is a subset such that $a(x) = 0$ for all $x \notin F$. Show that $\sum_F a$ exists iff $\sum_X a$ exists, and if the sums exist then*

$$\sum_X a = \sum_F a.$$

Theorem 2.21 (Tonelli's Theorem for Sums)**.** *Suppose that $a : X \times Y \to [0, \infty]$, then*

$$\sum_{X\times Y} a = \sum_X \sum_Y a = \sum_Y \sum_X a.$$

Proof. It suffices to show, by symmetry, that

$$\sum_{X\times Y} a = \sum_X \sum_Y a$$

Let $\Lambda \subset\subset X \times Y$. The for any $\alpha \subset\subset X$ and $\beta \subset\subset Y$ such that $\Lambda \subset \alpha \times \beta$, we have

$$\sum_{\Lambda} a \leq \sum_{\alpha\times\beta} a = \sum_{\alpha} \sum_{\beta} a \leq \sum_{\alpha} \sum_{Y} a \leq \sum_{X} \sum_{Y} a,$$

i.e. $\sum_\Lambda a \leq \sum_X \sum_Y a$. Taking the sup over Λ in this last equation shows

$$\sum_{X\times Y} a \leq \sum_X \sum_Y a.$$

We must now show the opposite inequality. If $\sum_{X\times Y} a = \infty$ we are done so we now assume that a is summable. By Remark 2.8, there is a countable set $\{(x'_n, y'_n)\}_{n=1}^{\infty} \subset X \times Y$ off of which a is identically 0.

Let $\{y_n\}_{n=1}^{\infty}$ be an enumeration of $\{y'_n\}_{n=1}^{\infty}$, then since $a(x,y) = 0$ if $y \notin \{y_n\}_{n=1}^{\infty}$, $\sum_{y\in Y} a(x,y) = \sum_{n=1}^{\infty} a(x,y_n)$ for all $x \in X$. Hence

$$\begin{aligned}\sum_{x\in X}\sum_{y\in Y} a(x,y) &= \sum_{x\in X}\sum_{n=1}^{\infty} a(x,y_n) = \sum_{x\in X}\lim_{N\to\infty}\sum_{n=1}^{N} a(x,y_n) \\ &= \lim_{N\to\infty}\sum_{x\in X}\sum_{n=1}^{N} a(x,y_n), \qquad (2.14)\end{aligned}$$

wherein the last inequality we have used the monotone convergence theorem with $F_N(x) := \sum_{n=1}^{N} a(x,y_n)$. If $\alpha \subset\subset X$, then

$$\sum_{x\in\alpha}\sum_{n=1}^{N} a(x,y_n) = \sum_{\alpha\times\{y_n\}_{n=1}^{N}} a \le \sum_{X\times Y} a$$

and therefore,

$$\lim_{N\to\infty}\sum_{x\in X}\sum_{n=1}^{N} a(x,y_n) \le \sum_{X\times Y} a. \qquad (2.15)$$

Hence it follows from Eqs. (2.14) and (2.15) that

$$\sum_{x\in X}\sum_{y\in Y} a(x,y) \le \sum_{X\times Y} a \qquad (2.16)$$

as desired.

Alternative proof of Eq. (2.16). Let $A = \{x'_n : n \in \mathbb{N}\}$ and let $\{x_n\}_{n=1}^{\infty}$ be an enumeration of A. Then for $x \notin A$, $a(x,y) = 0$ for all $y \in Y$.

Given $\epsilon > 0$, let $\delta : X \to [0,\infty)$ be the function such that $\sum_X \delta = \epsilon$ and $\delta(x) > 0$ for $x \in A$. (For example we may define δ by $\delta(x_n) = \epsilon/2^n$ for all n and $\delta(x) = 0$ if $x \notin A$.) For each $x \in X$, let $\beta_x \subset\subset X$ be a finite set such that

$$\sum_{y\in Y} a(x,y) \le \sum_{y\in\beta_x} a(x,y) + \delta(x).$$

Then

$$\begin{aligned}\sum_X\sum_Y a &\le \sum_{x\in X}\sum_{y\in\beta_x} a(x,y) + \sum_{x\in X}\delta(x) \\ &= \sum_{x\in X}\sum_{y\in\beta_x} a(x,y) + \epsilon = \sup_{\alpha\subset\subset X}\sum_{x\in\alpha}\sum_{y\in\beta_x} a(x,y) + \epsilon \\ &\le \sum_{X\times Y} a + \epsilon, \qquad (2.17)\end{aligned}$$

wherein the last inequality we have used

$$\sum_{x\in\alpha}\sum_{y\in\beta_x} a(x,y) = \sum_{\Lambda_\alpha} a \le \sum_{X\times Y} a$$

with

$$\Lambda_\alpha := \{(x,y) \in X \times Y : x \in \alpha \text{ and } y \in \beta_x\} \subset X \times Y.$$

Since $\epsilon > 0$ is arbitrary in Eq. (2.17), the proof is complete. ■

Theorem 2.22 (Fubini's Theorem for Sums)**.** *Now suppose that $a : X \times Y \to \mathbb{C}$ is a summable function, i.e. by Theorem 2.21 any one of the following equivalent conditions hold:*

(1) $\sum_{X\times Y} |a| < \infty$,
(2) $\sum_X \sum_Y |a| < \infty$ *or*
(3) $\sum_Y \sum_X |a| < \infty$.

Then

$$\sum_{X\times Y} a = \sum_X \sum_Y a = \sum_Y \sum_X a.$$

Proof. If $a : X \to \mathbb{R}$ is real valued the theorem follows by applying Theorem 2.21 to $a^{\pm}$ – the positive and negative parts of a. The general result holds for complex valued functions a by applying the real version just proved to the real and imaginary parts of a. ■

2.6. ℓ^p – **spaces, Minkowski and Holder Inequalities.** In this subsection, let $\mu : X \to (0, \infty]$ be a given function. Let $\mathbb{F}$ denote either $\mathbb{C}$ or $\mathbb{R}$. For $p \in (0, \infty)$ and $f : X \to \mathbb{F}$, let

$$\|f\|_p \equiv (\sum_{x\in X} |f(x)|^p \mu(x))^{1/p}$$

and for $p = \infty$ let

$$\|f\|_\infty = \sup \{|f(x)| : x \in X\}.$$

Also, for $p > 0$, let

$$\ell^p(\mu) = \{f : X \to \mathbb{F} : \|f\|_p < \infty\}.$$

In the case where $\mu(x) = 1$ for all $x \in X$ we will simply write $\ell^p(X)$ for $\ell^p(\mu)$.

Definition 2.23. A **norm** on a vector space L is a function $\|\cdot\| : L \to [0, \infty)$ such that

(1) (Homogeneity) $\|\lambda f\| = |\lambda| \|f\|$ for all $\lambda \in \mathbb{F}$ and $f \in L$.
(2) (Triangle inequality) $\|f + g\| \le \|f\| + \|g\|$ for all $f, g \in L$.
(3) (Positive definite) $\|f\| = 0$ implies $f = 0$.

A pair $(L, \|\cdot\|)$ where L is a vector space and $\|\cdot\|$ is a norm on L is called a **normed vector space.**

The rest of this section is devoted to the proof of the following theorem.

Theorem 2.24. *For $p \in [1, \infty]$, $(\ell^p(\mu), \|\cdot\|_p)$ is a normed vector space.*

Proof. The only difficulty is the proof of the triangle inequality which is the content of Minkowski's Inequality proved in Theorem 2.30 below. ■

2.6.1. *Some inequalities.*

Proposition 2.25. *Let $f : [0, \infty) \to [0, \infty)$ be a continuous strictly increasing function such that $f(0) = 0$ (for simplicity) and $\lim_{s\to\infty} f(s) = \infty$. Let $g = f^{-1}$ and for $s, t \ge 0$ let*

$$F(s) = \int_0^s f(s')ds' \text{ and } G(t) = \int_0^t g(t')dt'.$$

Then for all $s, t \ge 0$,

$$st \le F(s) + G(t)$$

and equality holds iff $t = f(s)$.

Proof. Let

$$A_s := \{(\sigma,\tau) : 0 \le \tau \le f(\sigma) \text{ for } 0 \le \sigma \le s\} \text{ and}$$
$$B_t := \{(\sigma,\tau) : 0 \le \sigma \le g(\tau) \text{ for } 0 \le \tau \le t\}$$

then as one sees from Figure 2, $[0,s]\times[0,t] \subset A_s \cup B_t$. (In the figure: $s = 3$, $t = 1$, A_3 is the region under $t = f(s)$ for $0 \le s \le 3$ and B_1 is the region to the left of the curve $s = g(t)$ for $0 \le t \le 1$.) Hence if m denotes the area of a region in the plane, then

$$st = m([0,s]\times[0,t]) \le m(A_s) + m(B_t) = F(s) + G(t).$$

As it stands, this proof is a bit on the intuitive side. However, it will become rigorous if one takes m to be Lebesgue measure on the plane which will be introduced later.

We can also give a calculus proof of this theorem under the additional assumption that f is C^1. (This restricted version of the theorem is all we need in this section.) To do this fix $t \ge 0$ and let

$$h(s) = st - F(s) = \int_0^s (t - f(\sigma))d\sigma.$$

If $\sigma > g(t) = f^{-1}(t)$, then $t - f(\sigma) < 0$ and hence if $s > g(t)$, we have

$$\begin{aligned} h(s) &= \int_0^s (t - f(\sigma))d\sigma = \int_0^{g(t)} (t - f(\sigma))d\sigma + \int_{g(t)}^s (t - f(\sigma))d\sigma \\ &\le \int_0^{g(t)} (t - f(\sigma))d\sigma = h(g(t)). \end{aligned}$$

Combining this with $h(0) = 0$ we see that $h(s)$ takes its maximum at some point $s \in (0,t]$ and hence at a point where $0 = h'(s) = t - f(s)$. The only solution to this equation is $s = g(t)$ and we have thus shown

$$st - F(s) = h(s) \le \int_0^{g(t)} (t - f(\sigma))d\sigma = h(g(t))$$

with equality when $s = g(t)$. To finish the proof we must show $\int_0^{g(t)} (t - f(\sigma))d\sigma = G(t)$. This is verified by making the change of variables $\sigma = g(\tau)$ and then integrating by parts as follows:

$$\begin{aligned} \int_0^{g(t)} (t - f(\sigma))d\sigma &= \int_0^t (t - f(g(\tau)))g'(\tau)d\tau = \int_0^t (t - \tau)g'(\tau)d\tau \\ &= \int_0^t g(\tau)d\tau = G(t). \end{aligned}$$

■

Definition 2.26. The conjugate exponent $q \in [1,\infty]$ to $p \in [1,\infty]$ is $q := \frac{p}{p-1}$ with the convention that $q = \infty$ if $p = 1$. Notice that q is characterized by any of the following identities:

$$\frac{1}{p} + \frac{1}{q} = 1,\ 1 + \frac{q}{p} = q,\ p - \frac{p}{q} = 1 \text{ and } q(p-1) = p. \tag{2.18}$$

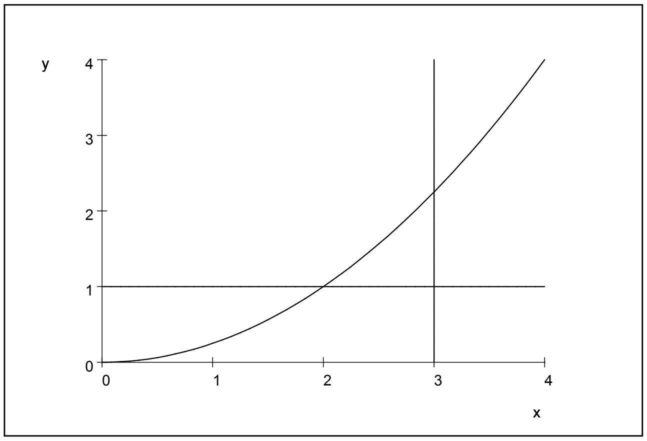

FIGURE 2. A picture proof of Proposition 2.25.

Lemma 2.27. *Let $p \in (1, \infty)$ and $q := \frac{p}{p-1} \in (1, \infty)$ be the conjugate exponent. Then*

$$st \leq \frac{s^q}{q} + \frac{t^p}{p} \text{ for all } s, t \geq 0$$

with equality if and only if $s^q = t^p$.

Proof. Let $F(s) = \frac{s^p}{p}$ for $p > 1$. Then $f(s) = s^{p-1} = t$ and $g(t) = t^{\frac{1}{p-1}} = t^{q-1}$, wherein we have used $q - 1 = p/(p-1) - 1 = 1/(p-1)$. Therefore $G(t) = t^q/q$ and hence by Proposition 2.25,

$$st \leq \frac{s^p}{p} + \frac{t^q}{q}$$

with equality iff $t = s^{p-1}$. ■

Theorem 2.28 (Hölder's inequality)**.** *Let $p, q \in [1, \infty]$ be conjugate exponents. For all $f, g : X \to \mathbb{F}$,*

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q. \tag{2.19}$$

If $p \in (1, \infty)$, then equality holds in Eq. (2.19) iff

$$(\frac{|f|}{\|f\|_p})^p = (\frac{|g|}{\|g\|_q})^q.$$

Proof. The proof of Eq. (2.19) for $p \in \{1, \infty\}$ is easy and will be left to the reader. The cases where $\|f\|_q = 0$ or ∞ or $\|g\|_p = 0$ or ∞ are easily dealt with and are also left to the reader. So we will assume that $p \in (1, \infty)$ and $0 < \|f\|_q, \|g\|_p < \infty$. Letting $s = |f|/\|f\|_p$ and $t = |g|/\|g\|_q$ in Lemma 2.27 implies

$$\frac{|fg|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{|f|^p}{\|f\|_p} + \frac{1}{q} \frac{|g|^q}{\|g\|^q}.$$

Multiplying this equation by μ and then summing gives

$$\frac{\|fg\|_1}{\|f\|_p \|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1$$

with equality iff

$$\frac{|g|}{\|g\|_q} = \frac{|f|^{p-1}}{\|f\|_p^{(p-1)}} \iff \frac{|g|}{\|g\|_q} = \frac{|f|^{p/q}}{\|f\|_p^{p/q}} \iff |g|^q \|f\|_p^p = \|g\|_q^q |f|^p.$$

■

Definition 2.29. For a complex number $\lambda \in \mathbb{C}$, let

$$\operatorname{sgn}(\lambda) = \begin{cases} \frac{\lambda}{|\lambda|} & \text{if} \quad \lambda \neq 0 \\ 0 & \text{if} \quad \lambda = 0. \end{cases}$$

Theorem 2.30 (Minkowski's Inequality)**.** *If $1 \le p \le \infty$ and $f, g \in \ell^p(\mu)$ then*

$$\|f+g\|_p \le \|f\|_p + \|g\|_p,$$

with equality iff

$$\operatorname{sgn}(f) = \operatorname{sgn}(g) \text{ when } p = 1 \text{ and}$$
$$f = cg \text{ for some } c > 0 \text{ when } p \in (1, \infty).$$

Proof. For $p = 1$,

$$\|f+g\|_1 = \sum_X |f+g|\mu \le \sum_X (|f|\mu + |g|\mu) = \sum_X |f|\mu + \sum_X |g|\mu$$

with equality iff

$$|f| + |g| = |f+g| \iff \operatorname{sgn}(f) = \operatorname{sgn}(g).$$

For $p = \infty$,

$$\|f+g\|_\infty = \sup_X |f+g| \le \sup_X (|f| + |g|)$$
$$\le \sup_X |f| + \sup_X |g| = \|f\|_\infty + \|g\|_\infty.$$

Now assume that $p \in (1, \infty)$. Since

$$|f+g|^p \le (2\max(|f|, |g|))^p = 2^p \max(|f|^p, |g|^p) \le 2^p (|f|^p + |g|^p)$$

it follows that

$$\|f+g\|_p^p \le 2^p \left(\|f\|_p^p + \|g\|_p^p\right) < \infty.$$

The theorem is easily verified if $\|f+g\|_p = 0$, so we may assume $\|f+g\|_p > 0$. Now

$$|f+g|^p = |f+g||f+g|^{p-1} \le (|f| + |g|)|f+g|^{p-1} \tag{2.20}$$

with equality iff $\operatorname{sgn}(f) = \operatorname{sgn}(g)$. Multiplying Eq. (2.20) by μ and then summing and applying Holder's inequality gives

$$\sum_X |f+g|^p \mu \le \sum_X |f| \; |f+g|^{p-1}\mu + \sum_X |g| \; |f+g|^{p-1}\mu$$
$$\le (\|f\|_p + \|g\|_p) \, \| \, |f+g|^{p-1} \, \|_q \tag{2.21}$$

with equality iff

$$\left(\frac{|f|}{\|f\|_p}\right)^p = \left(\frac{|f+g|^{p-1}}{\||f+g|^{p-1}\|_q}\right)^q = \left(\frac{|g|}{\|g\|_p}\right)^p$$
and $\operatorname{sgn}(f) = \operatorname{sgn}(g)$.

By Eq. (2.18), $q(p-1)=p$, and hence

$$\| |f+g|^{p-1} \|_q^q = \sum_X (|f+g|^{p-1})^q \mu = \sum_X |f+g|^p \mu. \tag{2.22}$$

Combining Eqs. (2.21) and (2.22) implies

$$\|f+g\|_p^p \le \|f\|_p \|f+g\|_p^{p/q} + \|g\|_p \|f+g\|_p^{p/q} \tag{2.23}$$

with equality iff

$$\operatorname{sgn}(f) = \operatorname{sgn}(g) \text{ and}$$
$$\left(\frac{|f|}{\|f\|_p}\right)^p = \frac{|f+g|^p}{\|f+g\|_p^p} = \left(\frac{|g|}{\|g\|_p}\right)^p. \tag{2.24}$$

Solving for $\|f+g\|_p$ in Eq. (2.23) with the aid of Eq. (2.18) shows that $\|f+g\|_p \le \|f\|_p + \|g\|_p$ with equality iff Eq. (2.24) holds which happens iff $f = cg$ with $c > 0$. ■

2.7. **Exercises .**

2.7.1. *Set Theory.* Let $f : X \to Y$ be a function and $\{A_i\}_{i\in I}$ be an indexed family of subsets of Y, verify the following assertions.

Exercise 2.1. $(\cap_{i\in I} A_i)^c = \cup_{i\in I} A_i^c$.

Exercise 2.2. Suppose that $B \subset Y$, show that $B \setminus (\cup_{i\in I} A_i) = \cap_{i\in I}(B \setminus A_i)$.

Exercise 2.3. $f^{-1}(\cup_{i\in I} A_i) = \cup_{i\in I} f^{-1}(A_i)$.

Exercise 2.4. $f^{-1}(\cap_{i\in I} A_i) = \cap_{i\in I} f^{-1}(A_i)$.

Exercise 2.5. Find a counter example which shows that $f(C \cap D) = f(C) \cap f(D)$ need not hold.

Exercise 2.6. Now suppose for each $n \in \mathbb{N} \equiv \{1, 2, \ldots\}$ that $f_n : X \to \mathbb{R}$ is a function. Let

$$D \equiv \{x \in X : \lim_{n\to\infty} f_n(x) = +\infty\}$$

show that

$$D = \cap_{M=1}^\infty \cup_{N=1}^\infty \cap_{n\ge N} \{x \in X : f_n(x) \ge M\}. \tag{2.25}$$

Exercise 2.7. Let $f_n : X \to \mathbb{R}$ be as in the last problem. Let

$$C \equiv \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\}.$$

Find an expression for C similar to the expression for D in (2.25). (Hint: use the Cauchy criteria for convergence.)

2.7.2. *Limit Problems.*

Exercise 2.8. Prove Lemma 2.15.

Exercise 2.9. Prove Lemma 2.20.

Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be two sequences of real numbers.

Exercise 2.10. Show $\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty} a_n$.

Exercise 2.11. Suppose that $\limsup_{n\to\infty} a_n = M \in \bar{\mathbb{R}}$, show that there is a subsequence $\{a_{n_k}\}_{k=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ such that $\lim_{k\to\infty} a_{n_k} = M$.

Exercise 2.12. Show that

$$\limsup_{n\to\infty}(a_n+b_n)\le\limsup_{n\to\infty}a_n+\limsup_{n\to\infty}b_n \tag{2.26}$$

provided that the right side of Eq. (2.26) is well defined, i.e. no $\infty-\infty$ or $-\infty+\infty$ type expressions. (It is OK to have $\infty+\infty=\infty$ or $-\infty-\infty=-\infty$, etc.)

Exercise 2.13. Suppose that $a_n\ge 0$ and $b_n\ge 0$ for all $n\in\mathbb{N}$. Show

$$\limsup_{n\to\infty}(a_nb_n)\le\limsup_{n\to\infty}a_n\cdot\limsup_{n\to\infty}b_n, \tag{2.27}$$

provided the right hand side of (2.27) is not of the form $0\cdot\infty$ or $\infty\cdot 0$.

2.7.3. *Dominated Convergence Theorem Problems.*

Notation 2.31. For $u_0\in\mathbb{R}^n$ and $\delta>0$, let $B_{u_0}(\delta):=\{x\in\mathbb{R}^n:|x-u_0|<\delta\}$ be the ball in $\mathbb{R}^n$ centered at u_0 with radius δ.

Exercise 2.14. Suppose $U\subset\mathbb{R}^n$ is a set and $u_0\in U$ is a point such that $U\cap(B_{u_0}(\delta)\setminus\{u_0\})\ne\emptyset$ for all $\delta>0$. Let $G:U\setminus\{u_0\}\to\mathbb{C}$ be a function on $U\setminus\{u_0\}$. Show that $\lim_{u\to u_0}G(u)$ exists and is equal to $\lambda\in\mathbb{C}$,[1] iff for all sequences $\{u_n\}_{n=1}^\infty\subset U\setminus\{u_0\}$ which converge to u_0 (i.e. $\lim_{n\to\infty}u_n=u_0$) we have $\lim_{n\to\infty}G(u_n)=\lambda$.

Exercise 2.15. Suppose that Y is a set, $U\subset\mathbb{R}^n$ is a set, and $f:U\times Y\to\mathbb{C}$ is a function satisfying:

1. For each $y\in Y$, the function $u\in U\to f(u,y)$ is continuous on U.[2]
2. There is a summable function $g:Y\to[0,\infty)$ such that
$$|f(u,y)|\le g(y)\text{ for all }y\in Y\text{ and }u\in U.$$
Show that
$$F(u):=\sum_{y\in Y}f(u,y) \tag{2.28}$$
is a continuous function for $u\in U$.

Exercise 2.16. Suppose that Y is a set, $J=(a,b)\subset\mathbb{R}$ is an interval, and $f:J\times Y\to\mathbb{C}$ is a function satisfying:

1. For each $y\in Y$, the function $u\to f(u,y)$ is differentiable on J,
2. There is a summable function $g:Y\to[0,\infty)$ such that
$$\left|\frac{\partial}{\partial u}f(u,y)\right|\le g(y)\text{ for all }y\in Y.$$
3. There is a $u_0\in J$ such that $\sum_{y\in Y}|f(u_0,y)|<\infty$.

Show:

a) for all $u\in J$ that $\sum_{y\in Y}|f(u,y)|<\infty$.

---

[1] More explicitly, $\lim_{u\to u_0}G(u)=\lambda$ means for every every $\epsilon>0$ there exists a $\delta>0$ such that
$$|G(u)-\lambda|<\epsilon\text{ whenerver }u\in U\cap(B_{u_0}(\delta)\setminus\{u_0\}).$$

[2] To say $g:=f(\cdot,y)$ is continuous on U means that $g:U\to\mathbb{C}$ is continuous relative to the metric on $\mathbb{R}^n$ restricted to U.

b) Let $F(u) := \sum_{y \in Y} f(u, y)$, show F is differentiable on J and that

$$\dot{F}(u) = \sum_{y \in Y} \frac{\partial}{\partial u} f(u, y).$$

(Hint: Use the mean value theorem.)

Exercise 2.17 (Differentiation of Power Series)**.** Suppose $R > 0$ and $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers such that $\sum_{n=0}^{\infty} |a_n| r^n < \infty$ for all $r \in (0, R)$. Show, using Exercise 2.16, $f(x) := \sum_{n=0}^{\infty} a_n x^n$ is continuously differentiable for $x \in (-R, R)$ and

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Exercise 2.18. Let $\{a_n\}_{n=-\infty}^{\infty}$ be a summable sequence of complex numbers, i.e. $\sum_{n=-\infty}^{\infty} |a_n| < \infty$. For $t \geq 0$ and $x \in \mathbb{R}$, define

$$F(t, x) = \sum_{n=-\infty}^{\infty} a_n e^{-tn^2} e^{inx},$$

where as usual $e^{ix} = \cos(x) + i \sin(x)$. Prove the following facts about F :

(1) $F(t, x)$ is continuous for $(t, x) \in [0, \infty) \times \mathbb{R}$. **Hint:** Let $Y = \mathbb{Z}$ and $u = (t, x)$ and use Exercise 2.15.
(2) $\partial F(t, x)/\partial t$, $\partial F(t, x)/\partial x$ and $\partial^2 F(t, x)/\partial x^2$ exist for $t > 0$ and $x \in \mathbb{R}$. **Hint:** Let $Y = \mathbb{Z}$ and $u = t$ for computing $\partial F(t, x)/\partial t$ and $u = x$ for computing $\partial F(t, x)/\partial x$ and $\partial^2 F(t, x)/\partial x^2$. See Exercise 2.16.
(3) F satisfies the heat equation, namely

$$\partial F(t, x)/\partial t = \partial^2 F(t, x)/\partial x^2 \text{ for } t > 0 \text{ and } x \in \mathbb{R}.$$

2.7.4. *Inequalities.*

Exercise 2.19. Generalize Proposition 2.25 as follows. Let $a \in [-\infty, 0]$ and $f : \mathbb{R} \cap [a, \infty) \to [0, \infty)$ be a continuous strictly increasing function such that $\lim_{s \to \infty} f(s) = \infty$, $f(a) = 0$ if $a > -\infty$ or $\lim_{s \to -\infty} f(s) = 0$ if $a = -\infty$. Also let $g = f^{-1}$, $b = f(0) \geq 0$,

$$F(s) = \int_0^s f(s') ds' \text{ and } G(t) = \int_0^t g(t') dt'.$$

Then for all $s, t \geq 0$,

$$st \leq F(s) + G(t \vee b) \leq F(s) + G(t)$$

and equality holds iff $t = f(s)$. In particular, taking $f(s) = e^s$, prove Young's inequality stating

$$st \leq e^s + (t \vee 1) \ln (t \vee 1) - (t \vee 1) \leq e^s + t \ln t - t.$$

Hint: Refer to the following pictures.

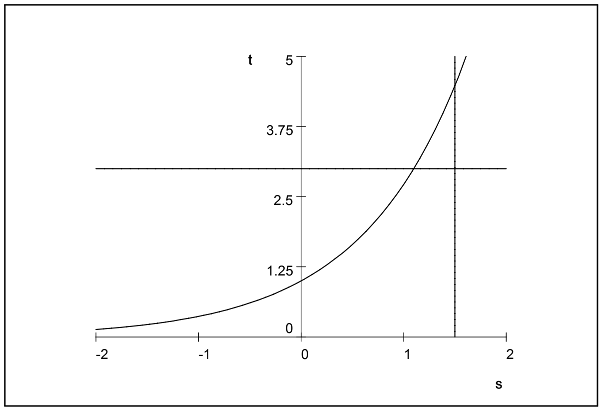

FIGURE 3. Comparing areas when $t \geq b$ goes the same way as in the text.

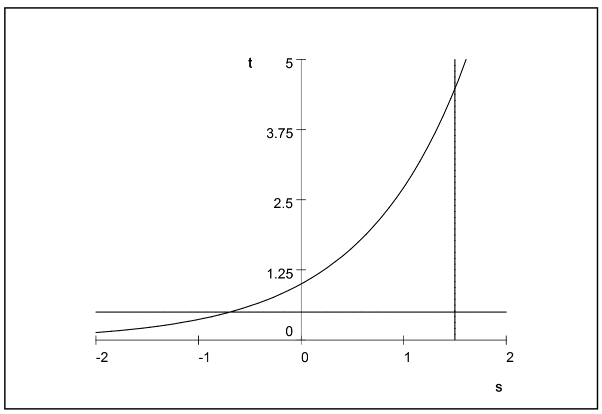

FIGURE 4. When $t \leq b$, notice that $g(t) \leq 0$ but $G(t) \geq 0$. Also notice that $G(t)$ is no longer needed to estimate st.

3. Metric, Banach and Topological Spaces

3.1. Basic metric space notions.

Definition 3.1. A function $d : X \times X \to [0, \infty)$ is called a metric if

(1) (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$
(2) (Non-degenerate) $d(x, y) = 0$ if and only if $x = y \in X$
(3) (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

As primary examples, any normed space $(X, \|\cdot\|)$ is a metric space with $d(x, y) := \|x - y\|$. Thus the space $\ell^p(\mu)$ is a metric space for all $p \in [1, \infty]$. Also any subset of a metric space is a metric space. For example a surface Σ in $\mathbb{R}^3$ is a metric space with the distance between two points on Σ being the usual distance in $\mathbb{R}^3$.

Definition 3.2. Let (X, d) be a metric space. The **open ball** $B(x, \delta) \subset X$ centered at $x \in X$ with radius $\delta > 0$ is the set

$$B(x, \delta) := \{y \in X : d(x, y) < \delta\}.$$

We will often also write $B(x,\delta)$ as $B_x(\delta)$. We also define the **closed ball** centered at $x \in X$ with radius $\delta > 0$ as the set $C_x(\delta) := \{y \in X : d(x,y) \le \delta\}$.

Definition 3.3. A sequence $\{x_n\}_{n=1}^\infty$ in a metric space (X,d) is said to be convergent if there exists a point $x \in X$ such that $\lim_{n\to\infty} d(x,x_n) = 0$. In this case we write $\lim_{n\to\infty} x_n = x$ of $x_n \to x$ as $n \to \infty$.

Exercise 3.1. Show that x in Definition 3.3 is necessarily unique.

Definition 3.4. A set $F \subset X$ is closed iff every convergent sequence $\{x_n\}_{n=1}^\infty$ which is contained in F has its limit back in F. A set $V \subset X$ is open iff V^c is closed. We will write $F \sqsubset X$ to indicate the F is a closed subset of X and $V \subset_o X$ to indicate the V is an open subset of X. We also let τ_d denote the collection of open subsets of X relative to the metric d.

Exercise 3.2. Let $\mathcal{F}$ be a collection of closed subsets of X, show $\cap\mathcal{F} := \cap_{F\in\mathcal{F}} F$ is closed. Also show that finite unions of closed sets are closed, i.e. if $\{F_k\}_{k=1}^n$ are closed sets then $\cup_{k=1}^n F_k$ is closed. (By taking complements, this shows that the collection of open sets, τ_d, is closed under finite intersections and arbitrary unions.)

The following "continuity" facts of the metric d will be used frequently in the remainder of this book.

Lemma 3.5. *For any non empty subset $A \subset X$, let $d_A(x) \equiv \inf\{d(x,a)|a \in A\}$, then*

$$|d_A(x) - d_A(y)| \le d(x,y)\ \forall\, x,y \in X. \tag{3.1}$$

Moreover the set $F_\epsilon \equiv \{x \in X | d_A(x) \ge \epsilon\}$ is closed in X.

Proof. Let $a \in A$ and $x,y \in X$, then

$$d(x,a) \le d(x,y) + d(y,a).$$

Take the inf over a in the above equation shows that

$$d_A(x) \le d(x,y) + d_A(y) \quad \forall x,y \in X.$$

Therefore, $d_A(x) - d_A(y) \le d(x,y)$ and by interchanging x and y we also have that $d_A(y) - d_A(x) \le d(x,y)$ which implies Eq. (3.1). Now suppose that $\{x_n\}_{n=1}^\infty \subset F_\epsilon$ is a convergent sequence and $x = \lim_{n\to\infty} x_n \in X$. By Eq. (3.1),

$$\epsilon - d_A(x) \le d_A(x_n) - d_A(x) \le d(x,x_n) \to 0 \text{ as } n \to \infty,$$

so that $\epsilon \le d_A(x)$. This shows that $x \in F_\epsilon$ and hence F_ϵ is closed. ■

Corollary 3.6. *The function d satisfies,*

$$|d(x,y) - d(x',y')| \le d(y,y') + d(x,x')$$

and in particular $d : X \times X \to [0,\infty)$ is continuous.

Proof. By Lemma 3.5 for single point sets and the triangle inequality for the absolute value of real numbers,

$$\begin{aligned} |d(x,y) - d(x',y')| &\le |d(x,y) - d(x,y')| + |d(x,y') - d(x',y')| \\ &\le d(y,y') + d(x,x'). \end{aligned}$$

■

Exercise 3.3. Show that $V \subset X$ is open iff for every $x \in V$ there is a $\delta > 0$ such that $B_x(\delta) \subset V$. In particular show $B_x(\delta)$ is open for all $x \in X$ and $\delta > 0$.

Lemma 3.7. *Let A be a closed subset of X and $F_\epsilon \sqsubset X$ be as defined as in Lemma 3.5. Then $F_\epsilon \uparrow A^c$ as $\epsilon \downarrow 0$.*

Proof. It is clear that $d_A(x) = 0$ for $x \in A$ so that $F_\epsilon \subset A^c$ for each $\epsilon > 0$ and hence $\cup_{\epsilon>0} F_\epsilon \subset A^c$. Now suppose that $x \in A^c \subset_o X$. By Exercise 3.3 there exists an $\epsilon > 0$ such that $B_x(\epsilon) \subset A^c$, i.e. $d(x, y) \geq \epsilon$ for all $y \in A$. Hence $x \in F_\epsilon$ and we have shown that $A^c \subset \cup_{\epsilon>0} F_\epsilon$. Finally it is clear that $F_\epsilon \subset F_{\epsilon'}$ whenever $\epsilon' \leq \epsilon$. ■

Definition 3.8. Given a set A contained a metric space X, let $\bar{A} \subset X$ be the **closure of** A defined by

$$\bar{A} := \{x \in X : \exists \, \{x_n\} \subset A \ni x = \lim_{n\to\infty} x_n\}.$$

That is to say $\bar{A}$ contains all **limit points** of A.

Exercise 3.4. Given $A \subset X$, show $\bar{A}$ is a closed set and in fact

$$\bar{A} = \cap\{F : A \subset F \subset X \text{ with } F \text{ closed}\}. \tag{3.2}$$

That is to say $\bar{A}$ is the smallest closed set containing A.

3.2. **Continuity.** Suppose that (X, d) and (Y, ρ) are two metric spaces and $f : X \to Y$ is a function.

Definition 3.9. A function $f : X \to Y$ is continuous at $x \in X$ if for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$d(f(x), f(x')) < \epsilon \text{ provided that } \rho(x, x') < \delta.$$

The function f is said to be continuous if f is continuous at all points $x \in X$.

The following lemma gives three other ways to characterize continuous functions.

Lemma 3.10 (Continuity Lemma)**.** *Suppose that (X, ρ) and (Y, d) are two metric spaces and $f : X \to Y$ is a function. Then the following are equivalent:*

1. *f is continuous.*
2. *$f^{-1}(V) \in \tau_\rho$ for all $V \in \tau_d$, i.e. $f^{-1}(V)$ is open in X if V is open in Y.*
3. *$f^{-1}(C)$ is closed in X if C is closed in Y.*
4. *For all convergent sequences $\{x_n\} \subset X$, $\{f(x_n)\}$ is convergent in Y and*

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

Proof. 1. $\Rightarrow$ 2. For all $x \in X$ and $\epsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ if $\rho(x, x') < \delta$. i.e.

$$B_x(\delta) \subset f^{-1}(B_{f(x)}(\epsilon))$$

So if $V \subset_o Y$ and $x \in f^{-1}(V)$ we may choose $\epsilon > 0$ such that $B_{f(x)}(\epsilon) \subset V$ then

$$B_x(\delta) \subset f^{-1}(B_{f(x)}(\epsilon)) \subset f^{-1}(V)$$

showing that $f^{-1}(V)$ is open.

2. $\Rightarrow$ 1. Let $\epsilon > 0$ and $x \in X$, then, since $f^{-1}(B_{f(x)}(\epsilon)) \subset_o X$, there exists $\delta > 0$ such that $B_x(\delta) \subset f^{-1}(B_{f(x)}(\epsilon))$ i.e. if $\rho(x, x') < \delta$ then $d(f(x'), f(x)) < \epsilon$.

2. $\iff$ 3. If C is closed in Y, then $C^c \subset_o Y$ and hence $f^{-1}(C^c) \subset_o X$. Since $f^{-1}(C^c) = \left(f^{-1}(C)\right)^c$, this shows that $f^{-1}(C)$ is the complement of an open set and hence closed. Similarly one shows that 3. $\Rightarrow$ 2.

1. $\Rightarrow$ 4. If f is continuous and $x_n \to x$ in X, let $\epsilon > 0$ and choose $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ when $\rho(x, x') < \delta$. There exists an $N > 0$ such that $\rho(x, x_n) < \delta$ for all $n \geq N$ and therefore $d(f(x), f(x_n)) < \epsilon$ for all $n \geq N$. That is to say $\lim_{n\to\infty} f(x_n) = f(x)$ as $n \to \infty$.

4. $\Rightarrow$ 1. We will show that not 1. $\Rightarrow$ not 4. Not 1 implies there exists $\epsilon > 0$, a point $x \in X$ and a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $d(f(x), f(x_n)) \geq \epsilon$ while $\rho(x, x_n) < \frac{1}{n}$. Clearly this sequence $\{x_n\}$ violates 4. ■

There is of course a local version of this lemma. To state this lemma, we will use the following terminology.

Definition 3.11. Let X be metric space and $x \in X$. A subset $A \subset X$ is a **neighborhood** of x if there exists an open set $V \subset_o X$ such that $x \in V \subset A$. We will say that $A \subset X$ is an **open neighborhood** of x if A is open and $x \in A$.

Lemma 3.12 (Local Continuity Lemma)**.** *Suppose that (X, ρ) and (Y, d) are two metric spaces and $f : X \to Y$ is a function. Then following are equivalent:*

(1) *f is continuous as $x \in X$.*
(2) *For all neighborhoods $A \subset Y$ of $f(x)$, $f^{-1}(A)$ is a neighborhood of $x \in X$.*
(3) *For all sequences $\{x_n\} \subset X$ such that $x = \lim_{n\to\infty} x_n$, $\{f(x_n)\}$ is convergent in Y and*

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

The proof of this lemma is similar to Lemma 3.10 and so will be omitted.

Example 3.13. The function d_A defined in Lemma 3.5 is continuous for each $A \subset X$. In particular, if $A = \{x\}$, it follows that $y \in X \to d(y, x)$ is continuous for each $x \in X$.

Exercise 3.5. Show the closed ball $C_x(\delta) := \{y \in X : d(x, y) \leq \delta\}$ is a closed subset of X.

3.3. **Basic Topological Notions.** Using the metric space results above as motivation we will axiomatize the notion of being an open set to more general settings.

Definition 3.14. A collection of subsets τ of X is a **topology** if

(1) $\emptyset, X \in \tau$
(2) τ is closed under arbitrary unions, i.e. if $V_\alpha \in \tau$, for $\alpha \in I$ then $\bigcup_{\alpha\in I} V_\alpha \in \tau$.
(3) τ is closed under finite intersections, i.e. if $V_1, \dots, V_n \in \tau$ then $V_1 \cap \dots \cap V_n \in \tau$.

A pair (X, τ) where τ is a topology on X will be called a **topological space**.

Notation 3.15. The subsets $V \subset X$ which are in τ are called open sets and we will abbreviate this by writing $V \subset_o X$ and the those sets $F \subset X$ such that $F^c \in \tau$ are called closed sets. We will write $F \sqsubset X$ if F is a closed subset of X.

Example 3.16. (1) Let (X, d) be a metric space, we write τ_d for the collection of d – open sets in X. We have already seen that τ_d is a topology, see Exercise 3.2.

(2) Let X be any set, then $\tau = \mathcal{P}(X)$ is a topology. In this topology all subsets of X are both open and closed. At the opposite extreme we have the **trivial** topology, $\tau = \{\emptyset, X\}$. In this topology only the empty set and X are open (closed).
(3) Let $X = \{1, 2, 3\}$, then $\tau = \{\emptyset, X, \{2, 3\}\}$ is a topology on X which does not come from a metric.
(4) Again let $X = \{1, 2, 3\}$. Then $\tau = \{\{1\}, \{2, 3\}, \emptyset, X\}$. is a topology, and the sets X, $\{1\}$, $\{2, 3\}, \phi$ are open and closed. The sets $\{1, 2\}$ and $\{1, 3\}$ are neither open nor closed.

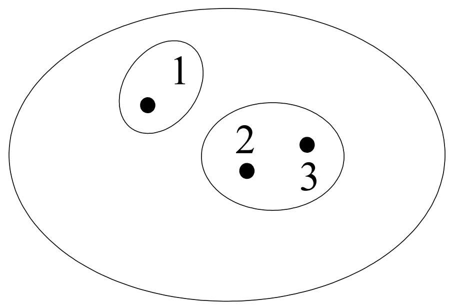

FIGURE 5. A topology.

Definition 3.17. Let (X, τ) be a topological space, $A \subset X$ and $i_A : A \to X$ be the inclusion map, i.e. $i_A(a) = a$ for all $a \in A$. Define

$$\tau_A = i_A^{-1}(\tau) = \{A \cap V : V \in \tau\},$$

the so called **relative topology** on A.

Notice that the closed sets in Y relative to τ_Y are precisely those sets of the form $C \cap Y$ where C is close in X. Indeed, $B \subset Y$ is closed iff $Y \setminus B = Y \cap V$ for some $V \in \tau$ which is equivalent to $B = Y \setminus (Y \cap V) = Y \cap V^c$ for some $V \in \tau$.

Exercise 3.6. Show the relative topology is a topology on A. Also show if (X, d) is a metric space and $\tau = \tau_d$ is the topology coming from d, then $(\tau_d)_A$ is the topology induced by making A into a metric space using the metric $d|_{A \times A}$.

Notation 3.18 (Neighborhoods of x)**.** An **open neighborhood** of a point $x \in X$ is an open set $V \subset X$ such that $x \in V$. Let $\tau_x = \{V \in \tau : x \in V\}$ denote the collection of open neighborhoods of x. A collection $\eta \subset \tau_x$ is called a **neighborhood base** at $x \in X$ if for all $V \in \tau_x$ there exists $W \in \eta$ such that $W \subset V$.

The notation τ_x should not be confused with

$$\tau_{\{x\}} := i_{\{x\}}^{-1}(\tau) = \{\{x\} \cap V : V \in \tau\} = \{\emptyset, \{x\}\}.$$

When (X, d) is a metric space, a typical example of a neighborhood base for x is $\eta = \{B_x(\epsilon) : \epsilon \in \mathbb{D}\}$ where $\mathbb{D}$ is any dense subset of $(0, 1]$.

Definition 3.19. Let (X, τ) be a topological space and A be a subset of X.

(1) The **closure** of A is the smallest closed set $\bar{A}$ containing A, i.e.
$$\bar{A} := \cap \{F : A \subset F \sqsubset X\}.$$
(Because of Exercise 3.4 this is consistent with Definition 3.8 for the closure of a set in a metric space.)
(2) The **interior** of A is the largest open set A^o contained in A, i.e.
$$A^o = \cup \{V \in \tau : V \subset A\}.$$
(3) The **accumulation points** of A is the set
$$\mathrm{acc}(A) = \{x \in X : V \cap A \setminus \{x\} \neq \emptyset \text{ for all } V \in \tau_x\}.$$
(4) The **boundary** of A is the set $\partial A := \bar{A} \setminus A^o$.
(5) A is a **neighborhood** of a point $x \in X$ if $x \in A^o$. This is equivalent to requiring there to be an open neighborhood of V of $x \in X$ such that $V \subset A$.

Remark 3.20. The relationships between the interior and the closure of a set are:
$$(A^o)^c = \bigcap \{V^c : V \in \tau \text{ and } V \subset A\} = \bigcap \{C : C \text{ is closed } C \supset A^c\} = \overline{A^c}$$
and similarly, $(\bar{A})^c = (A^c)^o$. Hence the boundary of A may be written as
$$\partial A \equiv \bar{A} \setminus A^o = \bar{A} \cap (A^o)^c = \bar{A} \cap \overline{A^c}, \tag{3.3}$$
which is to say ∂A consists of the points in both the closure of A and A^c.

Proposition 3.21. *Let $A \subset X$ and $x \in X$.*
(1) *If $V \subset_o X$ and $A \cap V = \emptyset$ then $\bar{A} \cap V = \emptyset$.*
(2) *$x \in \bar{A}$ iff $V \cap A \neq \emptyset$ for all $V \in \tau_x$.*
(3) *$x \in \partial A$ iff $V \cap A \neq \emptyset$ and $V \cap A^c \neq \emptyset$ for all $V \in \tau_x$.*
(4) *$\bar{A} = A \cup \mathrm{acc}(A)$.*

Proof. 1. Since $A \cap V = \emptyset$, $A \subset V^c$ and since V^c is closed, $\bar{A} \subset V^c$. That is to say $\bar{A} \cap V = \emptyset$.

2. By Remark 3.20[3], $\bar{A} = ((A^c)^o)^c$ so $x \in \bar{A}$ iff $x \notin (A^c)^o$ which happens iff $V \not\subset A^c$ for all $V \in \tau_x$, i.e. iff $V \cap A \neq \emptyset$ for all $V \in \tau_x$.

3. This assertion easily follows from the Item 2. and Eq. (3.3).

4. Item 4. is an easy consequence of the definition of $\mathrm{acc}(A)$ and item 2. ■

Lemma 3.22. *Let $A \subset Y \subset X$, $\bar{A}^Y$ denote the closure of A in Y with its relative topology and $\bar{A} = \bar{A}^X$ be the closure of A in X, then $\bar{A}^Y = \bar{A}^X \cap Y$.*

Proof. Using the comments after Definition 3.17,
$$\bar{A}^Y = \cap \{B \sqsubset Y : A \subset B\} = \cap \{C \cap Y : A \subset C \sqsubset X\}$$
$$= Y \cap (\cap \{C : A \subset C \sqsubset X\}) = Y \cap \bar{A}^X.$$

Alternative proof. Let $x \in Y$ then $x \in \bar{A}^Y$ iff for all $V \in \tau_x^Y$, $V \cap A \neq \emptyset$. This happens iff for all $U \in \tau_x^X$, $U \cap Y \cap A = U \cap A \neq \emptyset$ which happens iff $x \in \bar{A}^X$. That is to say $\bar{A}^Y = \bar{A}^X \cap Y$. ■

[3]Here is another direct proof of item 2. which goes by showing $x \notin \bar{A}$ iff there exists $V \in \tau_x$ such that $V \cap A = \emptyset$. If $x \notin \bar{A}$ then $V = \overline{A^c} \in \tau_x$ and $V \cap A \subset V \cap \bar{A} = \emptyset$. Conversely if there exists $V \in \tau_x$ such that $V \cap A = \emptyset$ then by Item 1. $\bar{A} \cap V = \emptyset$.

Definition 3.23. Let (X,τ) be a topological space and $A\subset X$. We say a subset $\mathcal{U}\subset\tau$ is an **open cover** of A if $A\subset\cup\mathcal{U}$. The set A is said to be **compact** if every open cover of A has finite a sub-cover, i.e. if $\mathcal{U}$ is an open cover of A there exists $\mathcal{U}_0\subset\subset\mathcal{U}$ such that $\mathcal{U}_0$ is a cover of A. (We will write $A\sqsubset\sqsubset X$ to denote that $A\subset X$ and A is compact.) A subset $A\subset X$ is **precompact** if $\bar{A}$ is compact.

Proposition 3.24. *Suppose that $K\subset X$ is a compact set and $F\subset K$ is a closed subset. Then F is compact. If $\{K_i\}_{i=1}^n$ is a finite collections of compact subsets of X then $K=\cup_{i=1}^n K_i$ is also a compact subset of X.*

Proof. Let $\mathcal{U}\subset\tau$ is an open cover of F, then $\mathcal{U}\cup\{F^c\}$ is an open cover of K. The cover $\mathcal{U}\cup\{F^c\}$ of K has a finite subcover which we denote by $\mathcal{U}_0\cup\{F^c\}$ where $\mathcal{U}_0\subset\subset\mathcal{U}$. Since $F\cap F^c=\emptyset$, it follows that $\mathcal{U}_0$ is the desired subcover of F.

For the second assertion suppose $\mathcal{U}\subset\tau$ is an open cover of K. Then $\mathcal{U}$ covers each compact set K_i and therefore there exists a finite subset $\mathcal{U}_i\subset\subset\mathcal{U}$ for each i such that $K_i\subset\cup\mathcal{U}_i$. Then $\mathcal{U}_0:=\cup_{i=1}^n\mathcal{U}_i$ is a finite cover of K. ■

Definition 3.25. We say a collection $\mathcal{F}$ of closed subsets of a topological space (X,τ) has the **finite intersection property if** $\cap\mathcal{F}_0\neq\emptyset$ for all $\mathcal{F}_0\subset\subset\mathcal{F}$.

The notion of compactness may be expressed in terms of closed sets as follows.

Proposition 3.26. *A topological space X is compact iff every family of closed sets $\mathcal{F}\subset\mathcal{P}(X)$ with the **finite intersection property** satisfies $\bigcap\mathcal{F}\neq\emptyset$.*

Proof. $(\Rightarrow)$ Suppose that X is compact and $\mathcal{F}\subset\mathcal{P}(X)$ is a collection of closed sets such that $\bigcap\mathcal{F}=\emptyset$. Let

$$\mathcal{U}=\mathcal{F}^c:=\{C^c:C\in\mathcal{F}\}\subset\tau,$$

then $\mathcal{U}$ is a cover of X and hence has a finite subcover, $\mathcal{U}_0$. Let $\mathcal{F}_0=\mathcal{U}_0^c\subset\subset\mathcal{F}$, then $\cap\mathcal{F}_0=\emptyset$ so that $\mathcal{F}$ does not have the finite intersection property.

$(\Leftarrow)$ If X is not compact, there exists an open cover $\mathcal{U}$ of X with no finite subcover. Let $\mathcal{F}=\mathcal{U}^c$, then $\mathcal{F}$ is a collection of closed sets with the finite intersection property while $\bigcap\mathcal{F}=\emptyset$. ■

Exercise 3.7. Let (X,τ) be a topological space. Show that $A\subset X$ is compact iff (A,τ_A) is a compact topological space.

Definition 3.27. Let (X,τ) be a topological space. A sequence $\{x_n\}_{n=1}^\infty\subset X$ **converges** to a point $x\in X$ if for all $V\in\tau_x$, $x_n\in V$ almost always (abbreviated a.a.), i.e. $\#(\{n:x_n\notin V\})<\infty$. We will write $x_n\to x$ as $n\to\infty$ or $\lim_{n\to\infty}x_n=x$ when x_n converges to x.

Example 3.28. Let $Y=\{1,2,3\}$ and $\tau=\{Y,\emptyset,\{1,2\},\{2,3\},\{2\}\}$ and $y_n=2$ for all n. Then $y_n\to y$ for every $y\in Y$. So limits need not be unique!

Definition 3.29. Let (X,τ_X) and (Y,τ_Y) be topological spaces. A function $f:X\to Y$ is **continuous** if $f^{-1}(\tau_Y)\subset\tau_X$. We will also say that f is τ_X/τ_Y – continuous or (τ_X,τ_Y) – continuous. We also say that f is continuous at a point $x\in X$ if for every open neighborhood V of $f(x)$ there is an open neighborhood U of x such that $U\subset f^{-1}(V)$. See Figure 6.

Definition 3.30. A map $f:X\to Y$ between topological spaces is called a **homeomorphism** provided that f is bijective, f is continuous and $f^{-1}:Y\to X$ is continuous. If there exists $f:X\to Y$ which is a homeomorphism, we say that

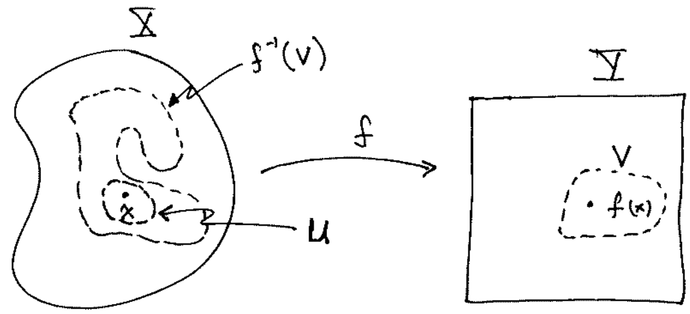

FIGURE 6. Checking that a function is continuous at $x \in X$.

X and Y are homeomorphic. (As topological spaces X and Y are essentially the same.)

Exercise 3.8. Show $f : X \to Y$ is continuous iff f is continuous at all points $x \in X$.

Exercise 3.9. Show $f : X \to Y$ is continuous iff $f^{-1}(C)$ is closed in X for all closed subsets C of Y.

Exercise 3.10. Suppose $f : X \to Y$ is continuous and $K \subset X$ is compact, then $f(K)$ is a compact subset of Y.

Exercise 3.11 (Dini's Theorem)**.** Let X be a compact topological space and $f_n : X \to [0, \infty)$ be a sequence of continuous functions such that $f_n(x) \downarrow 0$ as $n \to \infty$ for each $x \in X$. Show that in fact $f_n \downarrow 0$ uniformly in x, i.e. $\sup_{x\in X} f_n(x) \downarrow 0$ as $n \to \infty$. **Hint:** Given $\epsilon > 0$, consider the open sets $V_n := \{x \in X : f_n(x) < \epsilon\}$.

Definition 3.31 (First Countable)**.** A topological space, (X, τ), is **first countable** iff every point $x \in X$ has a countable neighborhood base. (All metric space are first countable.)

When τ is first countable, we may formulate many topological notions in terms of sequences.

Proposition 3.32. *If $f : X \to Y$ is continuous at $x \in X$ and $\lim_{n\to\infty} x_n = x \in X$, then $\lim_{n\to\infty} f(x_n) = f(x) \in Y$. Moreover, if there exists a countable neighborhood base η of $x \in X$, then f is continuous at x iff $\lim\limits_{n\to\infty} f(x_n) = f(x)$ for all sequences $\{x_n\}_{n=1}^\infty \subset X$ such that $x_n \to x$ as $n \to \infty$.*

Proof. If $f : X \to Y$ is continuous and $W \in \tau_Y$ is a neighborhood of $f(x) \in Y$, then there exists a neighborhood V of $x \in X$ such that $f(V) \subset W$. Since $x_n \to x$, $x_n \in V$ a.a. and therefore $f(x_n) \in f(V) \subset W$ a.a., i.e. $f(x_n) \to f(x)$ as $n \to \infty$.

Conversely suppose that $\eta \equiv \{W_n\}_{n=1}^\infty$ is a countable neighborhood base at x and $\lim\limits_{n\to\infty} f(x_n) = f(x)$ for all sequences $\{x_n\}_{n=1}^\infty \subset X$ such that $x_n \to x$. By replacing W_n by $W_1 \cap \cdots \cap W_n$ if necessary, we may assume that $\{W_n\}_{n=1}^\infty$ is a decreasing sequence of sets. If f were **not** continuous at x then there exists $V \in \tau_{f(x)}$ such that $x \notin f^{-1}(V)^0$. Therefore, W_n is not a subset of $f^{-1}(V)$ for all n. Hence for each n, we may choose $x_n \in W_n \setminus f^{-1}(V)$. This sequence then has the property that $x_n \to x$ as $n \to \infty$ while $f(x_n) \notin V$ for all n and hence $\lim_{n\to\infty} f(x_n) \neq f(x)$. ■

Lemma 3.33. *Suppose there exists $\{x_n\}_{n=1}^\infty \subset A$ such that $x_n \to x$, then $x \in \bar{A}$. Conversely if (X,τ) is a first countable space (like a metric space) then if $x \in \bar{A}$ there exists $\{x_n\}_{n=1}^\infty \subset A$ such that $x_n \to x$.*

Proof. Suppose $\{x_n\}_{n=1}^\infty \subset A$ and $x_n \to x \in X$. Since $\bar{A}^c$ is an open set, if $x \in \bar{A}^c$ then $x_n \in \bar{A}^c \subset A^c$ a.a. contradicting the assumption that $\{x_n\}_{n=1}^\infty \subset A$. Hence $x \in \bar{A}$.

For the converse we now assume that (X,τ) is first countable and that $\{V_n\}_{n=1}^\infty$ is a countable neighborhood base at x such that $V_1 \supset V_2 \supset V_3 \supset \dots$. By Proposition 3.21, $x \in \bar{A}$ iff $V \cap A \neq \emptyset$ for all $V \in \tau_x$. Hence $x \in \bar{A}$ implies there exists $x_n \in V_n \cap A$ for all n. It is now easily seen that $x_n \to x$ as $n \to \infty$. ■

Definition 3.34 (Support)**.** Let $f : X \to Y$ be a function from a topological space (X, τ_X) to a vector space Y. Then we define the support of f by

$$\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}},$$

a closed subset of X.

Example 3.35. For example, let $f(x) = \sin(x)1_{[0,4\pi]}(x) \in \mathbb{R}$, then

$$\{f \neq 0\} = (0, 4\pi) \setminus \{\pi, 2\pi, 3\pi\}$$

and therefore $\operatorname{supp}(f) = [0, 4\pi]$.

Notation 3.36. If X and Y are two topological spaces, let $C(X,Y)$ denote the continuous functions from X to Y. If Y is a Banach space, let

$$BC(X,Y) := \{f \in C(X,Y) : \sup_{x\in X} \|f(x)\|_Y < \infty\}$$

and

$$C_c(X,Y) := \{f \in C(X,Y) : \operatorname{supp}(f) \text{ is compact}\}.$$

If $Y = \mathbb{R}$ or $\mathbb{C}$ we will simply write $C(X)$, $BC(X)$ and $C_c(X)$ for $C(X,Y)$, $BC(X,Y)$ and $C_c(X,Y)$ respectively.

The next result is included for completeness but will not be used in the sequel so may be omitted.

Lemma 3.37. *Suppose that $f : X \to Y$ is a map between topological spaces. Then the following are equivalent:*

1. *f is continuous.*
2. *$f(\bar{A}) \subset \overline{f(A)}$ for all $A \subset X$*
3. *$\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ for all $B \sqsubset X$.*

Proof. If f is continuous, then $f^{-1}\left(\overline{f(A)}\right)$ is closed and since $A \subset f^{-1}(f(A)) \subset f^{-1}\left(\overline{f(A)}\right)$ it follows that $\bar{A} \subset f^{-1}\left(\overline{f(A)}\right)$. From this equation we learn that $f(\bar{A}) \subset \overline{f(A)}$ so that (1) implies (2) Now assume (2), then for $B \subset Y$ (taking $A = f^{-1}(\bar{B})$) we have

$$f(\overline{f^{-1}(B)}) \subset f(\overline{f^{-1}(\bar{B})}) \subset \overline{f(f^{-1}(\bar{B}))} \subset \bar{B}$$

and therefore

$$\overline{f^{-1}(B)} \subset f^{-1}(\bar{B}). \tag{3.4}$$

This shows that (2) implies (3) Finally if Eq. (3.4) holds for all B, then when B is closed this shows that

$$\overline{f^{-1}(B)} \subset f^{-1}(\bar{B}) = f^{-1}(B) \subset \overline{f^{-1}(B)}$$

which shows that

$$f^{-1}(B) = \overline{f^{-1}(B)}.$$

Therefore $f^{-1}(B)$ is closed whenever B is closed which implies that f is continuous. ■

3.4. **Completeness.**

Definition 3.38 (Cauchy sequences)**.** A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is **Cauchy** provided that

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0.$$

Exercise 3.12. Show that convergent sequences are always Cauchy sequences. The converse is not always true. For example, let $X = \mathbb{Q}$ be the set of rational numbers and $d(x, y) = |x - y|$. Choose a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ which converges to $\sqrt{2} \in \mathbb{R}$, then $\{x_n\}_{n=1}^{\infty}$ is $(\mathbb{Q}, d)$ – Cauchy but not $(\mathbb{Q}, d)$ – convergent. The sequence does converge in $\mathbb{R}$ however.

Definition 3.39. A metric space (X, d) is **complete** if all Cauchy sequences are convergent sequences.

Exercise 3.13. Let (X, d) be a complete metric space. Let $A \subset X$ be a subset of X viewed as a metric space using $d|_{A\times A}$. Show that $(A, d|_{A\times A})$ is complete iff A is a closed subset of X.

Definition 3.40. If $(X, \|\cdot\|)$ is a normed vector space, then we say $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence if $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$. The normed vector space is a **Banach space** if it is complete, i.e. if every $\{x_n\}_{n=1}^{\infty} \subset X$ which is Cauchy is convergent where $\{x_n\}_{n=1}^{\infty} \subset X$ is convergent iff there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. As usual we will abbreviate this last statement by writing $\lim_{n\to\infty} x_n = x$.

Lemma 3.41. *Suppose that X is a set then the bounded functions $\ell^{\infty}(X)$ on X is a Banach space with the norm*

$$\|f\| = \|f\|_{\infty} = \sup_{x\in X} |f(x)|.$$

Moreover if X is a topological space the set $BC(X) \subset \ell^{\infty}(X) = B(X)$ is closed subspace of $\ell^{\infty}(X)$ and hence is also a Banach space.

Proof. Let $\{f_n\}_{n=1}^{\infty} \subset \ell^{\infty}(X)$ be a Cauchy sequence. Since for any $x \in X$, we have

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_{\infty} \tag{3.5}$$

which shows that $\{f_n(x)\}_{n=1}^{\infty} \subset \mathbb{F}$ is a Cauchy sequence of numbers. Because $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is complete, $f(x) := \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. Passing to the limit $n \to \infty$ in Eq. (3.5) implies

$$|f(x) - f_m(x)| \le \limsup_{n\to\infty} \|f_n - f_m\|_{\infty}$$

and taking the supremum over $x \in X$ of this inequality implies

$$\|f - f_m\|_\infty \leq \limsup_{n\to\infty} \|f_n - f_m\|_\infty \to 0 \text{ as } m \to \infty$$

showing $f_m \to f$ in $\ell^\infty(X)$.

For the second assertion, suppose that $\{f_n\}_{n=1}^\infty \subset BC(X) \subset \ell^\infty(X)$ and $f_n \to f \in \ell^\infty(X)$. We must show that $f \in BC(X)$, i.e. that f is continuous. To this end let $x, y \in X$, then

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq 2\|f - f_n\|_\infty + |f_n(x) - f_n(y)|. \end{aligned}$$

Thus if $\epsilon > 0$, we may choose n large so that $2\|f - f_n\|_\infty < \epsilon/2$ and then for this n there exists an open neighborhood V_x of $x \in X$ such that $|f_n(x) - f_n(y)| < \epsilon/2$ for $y \in V_x$. Thus $|f(x) - f(y)| < \epsilon$ for $y \in V_x$ showing the limiting function f is continuous. ∎

Remark 3.42. Let X be a set, Y be a Banach space and $\ell^\infty(X, Y)$ denote the bounded functions $f : X \to Y$ equipped with the norm $\|f\| = \|f\|_\infty = \sup_{x\in X} \|f(x)\|_Y$. If X is a topological space, let $BC(X, Y)$ denote those $f \in \ell^\infty(X, Y)$ which are continuous. The same proof used in Lemma 3.41 shows that $\ell^\infty(X, Y)$ is a Banach space and that $BC(X, Y)$ is a closed subspace of $\ell^\infty(X, Y)$.

Theorem 3.43 (Completeness of $\ell^p(\mu)$)**.** *Let X be a set and $\mu : X \to (0, \infty]$ be a given function. Then for any $p \in [1, \infty]$, $(\ell^p(\mu), \|\cdot\|_p)$ is a Banach space.*

Proof. We have already proved this for $p = \infty$ in Lemma 3.41 so we now assume that $p \in [1, \infty)$. Let $\{f_n\}_{n=1}^\infty \subset \ell^p(\mu)$ be a Cauchy sequence. Since for any $x \in X$,

$$|f_n(x) - f_m(x)| \leq \frac{1}{\mu(x)} \|f_n - f_m\|_p \to 0 \text{ as } m, n \to \infty$$

it follows that $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence of numbers and $f(x) := \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. By Fatou's Lemma,

$$\begin{aligned} \|f_n - f\|_p^p &= \sum_X \mu \cdot \liminf_{m\to\infty} |f_n - f_m|^p \leq \liminf_{m\to\infty} \sum_X \mu \cdot |f_n - f_m|^p \\ &= \liminf_{m\to\infty} \|f_n - f_m\|_p^p \to 0 \text{ as } n \to \infty. \end{aligned}$$

This then shows that $f = (f - f_n) + f_n \in \ell^p(\mu)$ (being the sum of two ℓ^p – functions) and that $f_n \xrightarrow{\ell^p} f$. ∎

Example 3.44. Here are a couple of examples of complete metric spaces.

(1) $X = \mathbb{R}$ and $d(x, y) = |x - y|$.
(2) $X = \mathbb{R}^n$ and $d(x, y) = \|x - y\|_2 = \sum_{i=1}^n (x_i - y_i)^2$.
(3) $X = \ell^p(\mu)$ for $p \in [1, \infty]$ and any weight function μ.
(4) $X = C([0, 1], \mathbb{R})$ – the space of continuous functions from $[0, 1]$ to $\mathbb{R}$ and $d(f, g) := \max_{t\in[0,1]} |f(t) - g(t)|$. This is a special case of Lemma 3.41.
(5) Here is a typical example of a non-complete metric space. Let $X = C([0, 1], \mathbb{R})$ and

$$d(f, g) := \int_0^1 |f(t) - g(t)|\, dt.$$

3.5. **Compactness in Metric Spaces.** Let (X,ρ) be a metric space and let $B_x'(\epsilon) = B_x(\epsilon) \setminus \{x\}$.

Definition 3.45. A point $x \in X$ is an accumulation point of a subset $E \subset X$ if $\emptyset \neq E \cap V \setminus \{x\}$ for all $V \subset_o X$ containing x.

Let us start with the following elementary lemma which is left as an exercise to the reader.

Lemma 3.46. *Let $E \subset X$ be a subset of a metric space (X,ρ) . Then the following are equivalent:*

1. *$x \in X$ is an accumulation point of E.*
2. *$B_x'(\epsilon) \cap E \neq \emptyset$ for all $\epsilon > 0$.*
3. *$B_x(\epsilon) \cap E$ is an infinite set for all $\epsilon > 0$.*
4. *There exists $\{x_n\}_{n=1}^\infty \subset E \setminus \{x\}$ with $\lim_{n\to\infty} x_n = x$.*

Definition 3.47. A metric space (X,ρ) is said to be $\epsilon -$ **bounded** $(\epsilon > 0)$ provided there exists a finite cover of X by balls of radius ϵ. The metric space is **totally bounded** if it is $\epsilon -$ bounded for all $\epsilon > 0$.

Theorem 3.48. *Let X be a metric space. The following are equivalent.*

(a) *X is compact.*
(b) *Every infinite subset of X has an accumulation point.*
(c) *X is totally bounded and complete.*

Proof. The proof will consist of showing that $a \Rightarrow b \Rightarrow c \Rightarrow a$.

$(a \Rightarrow b)$ We will show that **not** $b \Rightarrow$ **not** a. Suppose there exists $E \subset X$, such that $\#(E) = \infty$ and E has no accumulation points. Then for all $x \in X$ there exists $\delta_x > 0$ such that $V_x := B_x(\delta_x)$ satisfies $(V_x \setminus \{x\}) \cap E = \emptyset$. Clearly $\mathcal{V} = \{V_x\}_{x\in X}$ is a cover of X, yet $\mathcal{V}$ has no finite sub cover. Indeed, for each $x \in X$, $V_x \cap E$ consists of at most one point, therefore if $\Lambda \subset\subset X$, $\cup_{x\in\Lambda} V_x$ can only contain a finite number of points from E, in particular $X \neq \cup_{x\in\Lambda} V_x$. (See Figure 7.)

Figure 7. The construction of an open cover with no finite sub-cover.

$(b \Rightarrow c)$ To show X is complete, let $\{x_n\}_{n=1}^\infty \subset X$ be a sequence and $E := \{x_n : n \in \mathbb{N}\}$. If $\#(E) < \infty$, then $\{x_n\}_{n=1}^\infty$ has a subsequence $\{x_{n_k}\}$ which is constant and hence convergent. If E is an infinite set it has an accumulation point by assumption and hence Lemma 3.46 implies that $\{x_n\}$ has a convergence subsequence.

We now show that X is totally bounded. Let $\epsilon > 0$ be given and choose $x_1 \in X$. If possible choose $x_2 \in X$ such that $d(x_2, x_1) \geq \epsilon$, then if possible choose $x_3 \in X$ such that $d(x_3, \{x_1, x_2\}) \geq \epsilon$ and continue inductively choosing points $\{x_j\}_{j=1}^n \subset X$ such that $d(x_n, \{x_1, \ldots, x_{n-1}\}) \geq \epsilon$. This process must terminate, for otherwise we could choose $E = \{x_j\}_{j=1}^\infty$ and infinite number of distinct points such that $d(x_j, \{x_1, \ldots, x_{j-1}\}) \geq \epsilon$ for all $j = 2, 3, 4, \ldots$. Since for all $x \in X$ the $B_x(\epsilon/3) \cap E$ can contain at most one point, no point $x \in X$ is an accumulation point of E. (See Figure 8.)

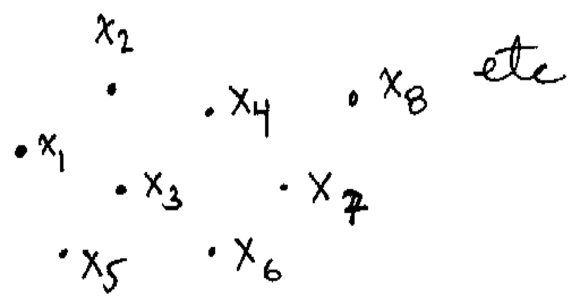

FIGURE 8. Constructing a set with out an accumulation point.

$(c \Rightarrow a)$ For sake of contradiction, assume there exists a cover an open cover $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ of X with no finite subcover. Since X is totally bounded for each $n \in \mathbb{N}$ there exists $\Lambda_n \subset\subset X$ such that

$$X = \bigcup_{x \in \Lambda_n} B_x(1/n) \subset \bigcup_{x \in \Lambda_n} C_x(1/n).$$

Choose $x_1 \in \Lambda_1$ such that no finite subset of $\mathcal{V}$ covers $K_1 := C_{x_1}(1)$. Since $K_1 = \cup_{x \in \Lambda_2} K_1 \cap C_x(1/2)$, there exists $x_2 \in \Lambda_2$ such that $K_2 := K_1 \cap C_{x_2}(1/2)$ can not be covered by a finite subset of $\mathcal{V}$. Continuing this way inductively, we construct sets $K_n = K_{n-1} \cap C_{x_n}(1/n)$ with $x_n \in \Lambda_n$ such no K_n can be covered by a finite subset of $\mathcal{V}$. Now choose $y_n \in K_n$ for each n. Since $\{K_n\}_{n=1}^\infty$ is a decreasing sequence of closed sets such that $\operatorname{diam}(K_n) \leq 2/n$, it follows that $\{y_n\}$ is a Cauchy and hence convergent with

$$y = \lim_{n \to \infty} y_n \in \cap_{m=1}^\infty K_m.$$

Since $\mathcal{V}$ is a cover of X, there exists $V \in \mathcal{V}$ such that $x \in V$. Since $K_n \downarrow \{y\}$ and $\operatorname{diam}(K_n) \to 0$, it now follows that $K_n \subset V$ for some n large. But this violates the assertion that K_n can not be covered by a finite subset of $\mathcal{V}$.(See Figure 9.)

■

Remark 3.49. Let X be a topological space and Y be a Banach space. By combining Exercise 3.10 and Theorem 3.48 it follows that $C_c(X, Y) \subset BC(X, Y)$.

Corollary 3.50. *Let X be a metric space then X is compact iff **all** sequences $\{x_n\} \subset X$ have convergent subsequences.*

Proof. Suppose X is compact and $\{x_n\} \subset X$.

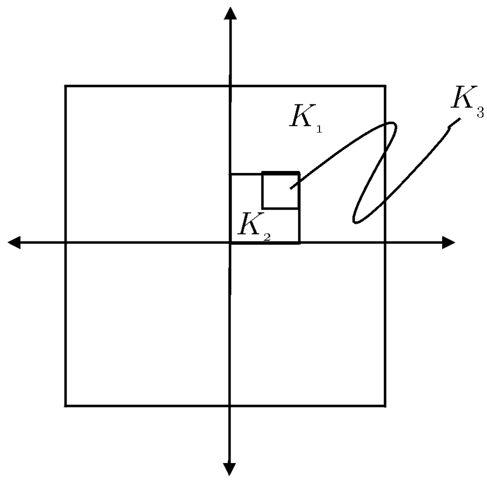

FIGURE 9. Nested Sequence of cubes.

(1) If $\#(\{x_n : n = 1, 2, \dots\}) < \infty$ then choose $x \in X$ such that $x_n = x$ i.o. and let $\{n_k\} \subset \{n\}$ such that $x_{n_k} = x$ for all k. Then $x_{n_k} \to x$
(2) If $\#(\{x_n : n = 1, 2, \dots\}) = \infty$. We know $E = \{x_n\}$ has an accumulation point $\{x\}$, hence there exists $x_{n_k} \to x$.

Conversely if E is an infinite set let $\{x_n\}_{n=1}^{\infty} \subset E$ be a sequence of distinct elements of E. We may, by passing to a subsequence, assume $x_n \to x \in X$ as $n \to \infty$. Now $x \in X$ is an accumulation point of E by Theorem 3.48 and hence X is compact. ■

Corollary 3.51. *Compact subsets of* $\mathbb{R}^n$ *are the closed and bounded sets.*

Proof. If K is closed and bounded then K is complete (being the closed subset of a complete space) and K is contained in $[-M, M]^n$ for some positive integer M. For $\delta > 0$, let

$$\Lambda_\delta = \delta\mathbb{Z}^n \cap [-M, M]^n := \{\delta x : x \in \mathbb{Z}^n \text{ and } \delta|x_i| \le M \text{ for } i = 1, 2, \dots, n\}.$$

We will show, by choosing $\delta > 0$ sufficiently small, that

$$K \subset [-M, M]^n \subset \cup_{x \in \Lambda_\delta} B(x, \epsilon) \tag{3.6}$$

which shows that K is totally bounded. Hence by Theorem 3.48, K is compact.

Suppose that $y \in [-M, M]^n$, then there exists $x \in \Lambda_\delta$ such that $|y_i - x_i| \le \delta$ for $i = 1, 2, \dots, n$. Hence

$$d^2(x, y) = \sum_{i=1}^{n} (y_i - x_i)^2 \le n\delta^2$$

which shows that $d(x, y) \le \sqrt{n}\delta$. Hence if choose $\delta < \epsilon/\sqrt{n}$ we have shows that $d(x, y) < \epsilon$, i.e. Eq. (3.6) holds. ■

Example 3.52. Let $X = \ell^p(\mathbb{N})$ with $p \in [1, \infty)$ and $\rho \in X$ such that $\rho(k) \ge 0$ for all $k \in \mathbb{N}$. The set

$$K := \{x \in X : |x(k)| \le \rho(k) \text{ for all } k \in \mathbb{N}\}$$

is compact. To prove this, let $\{x_n\}_{n=1}^{\infty} \subset K$ be a sequence. By compactness of closed bounded sets in $\mathbb{C}$, for each $k \in \mathbb{N}$ there is a subsequence of $\{x_n(k)\}_{n=1}^{\infty} \subset \mathbb{C}$ which is convergent. By Cantor's diagonalization trick, we may choose a subsequence $\{y_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $y(k) := \lim_{n\to\infty} y_n(k)$ exists for all $k \in \mathbb{N}$.[4] Since $|y_n(k)| \le \rho(k)$ for all n it follows that $|y(k)| \le \rho(k)$, i.e. $y \in K$. Finally

$$\lim_{n\to\infty} \|y - y_n\|_p^p = \lim_{n\to\infty} \sum_{k=1}^{\infty} |y(k) - y_n(k)|^p = \sum_{k=1}^{\infty} \lim_{n\to\infty} |y(k) - y_n(k)|^p = 0$$

where we have used the Dominated convergence theorem. (Note $|y(k) - y_n(k)|^p \le 2^p \rho^p(k)$ and ρ^p is summable.) Therefore $y_n \to y$ and we are done.

Alternatively, we can prove K is compact by showing that K is closed and totally bounded. It is simple to show K is closed, for if $\{x_n\}_{n=1}^{\infty} \subset K$ is a convergent sequence in X, $x := \lim_{n\to\infty} x_n$, then $|x(k)| \le \lim_{n\to\infty} |x_n(k)| \le \rho(k)$ for all $k \in \mathbb{N}$. This shows that $x \in K$ and hence K is closed. To see that K is totally bounded, let $\epsilon > 0$ and choose N such that $\left(\sum_{k=N+1}^{\infty} |\rho(k)|^p\right)^{1/p} < \epsilon$. Since $\prod_{k=1}^{N} C_{\rho(k)}(0) \subset \mathbb{C}^N$ is closed and bounded, it is compact. Therefore there exists a finite subset $\Lambda \subset \prod_{k=1}^{N} C_{\rho(k)}(0)$ such that

$$\prod_{k=1}^{N} C_{\rho(k)}(0) \subset \cup_{z\in\Lambda} B_z^N(\epsilon)$$

where $B_z^N(\epsilon)$ is the open ball centered at $z \in \mathbb{C}^N$ relative to the $\ell^p(\{1,2,3,\dots,N\})$ – norm. For each $z \in \Lambda$, let $\tilde{z} \in X$ be defined by $\tilde{z}(k) = z(k)$ if $k \le N$ and $\tilde{z}(k) = 0$ for $k \ge N+1$. I now claim that

$$K \subset \cup_{z\in\Lambda} B_{\tilde{z}}(2\epsilon) \tag{3.7}$$

which, when verified, shows K is totally bounced. To verify Eq. (3.7), let $x \in K$ and write $x = u + v$ where $u(k) = x(k)$ for $k \le N$ and $u(k) = 0$ for $k < N$. Then by construction $u \in B_{\tilde{z}}(\epsilon)$ for some $\tilde{z} \in \Lambda$ and

$$\|v\|_p \le \left(\sum_{k=N+1}^{\infty} |\rho(k)|^p\right)^{1/p} < \epsilon.$$

So we have

$$\|x - \tilde{z}\|_p = \|u + v - \tilde{z}\|_p \le \|u - \tilde{z}\|_p + \|v\|_p < 2\epsilon.$$

Exercise 3.14 (Extreme value theorem)**.** Let (X, τ) be a compact topological space and $f : X \to \mathbb{R}$ be a continuous function. Show $-\infty < \inf f \le \sup f < \infty$ and

[4] The argument is as follows. Let $\{n_j^1\}_{j=1}^{\infty}$ be a subsequence of $\mathbb{N} = \{n\}_{n=1}^{\infty}$ such that $\lim_{j\to\infty} x_{n_j^1}(1)$ exists. Now choose a subsequence $\{n_j^2\}_{j=1}^{\infty}$ of $\{n_j^1\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} x_{n_j^2}(2)$ exists and similalry $\{n_j^3\}_{j=1}^{\infty}$ of $\{n_j^2\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} x_{n_j^3}(3)$ exists. Continue on this way inductively to get

$$\{n\}_{n=1}^{\infty} \supset \{n_j^1\}_{j=1}^{\infty} \supset \{n_j^2\}_{j=1}^{\infty} \supset \{n_j^3\}_{j=1}^{\infty} \supset \dots$$

such that $\lim_{j\to\infty} x_{n_j^k}(k)$ exists for all $k \in \mathbb{N}$. Let $m_j := n_j^j$ so that eventually $\{m_j\}_{j=1}^{\infty}$ is a subsequnce of $\{n_j^k\}_{j=1}^{\infty}$ for all k. Therefore, we may take $y_j := x_{m_j}$.

there exists $a, b \in X$ such that $f(a) = \inf f$ and $f(b) = \sup f$. [5] **Hint:** use Exercise 3.10 and Corollary 3.51.

Exercise 3.15 (Uniform Continuity)**.** Let (X, d) be a compact metric space, (Y, ρ) be a metric space and $f : X \to Y$ be a continuous function. Show that f is uniformly continuous, i.e. if $\epsilon > 0$ there exists $\delta > 0$ such that $\rho(f(y), f(x)) < \epsilon$ if $x, y \in X$ with $d(x, y) < \delta$. **Hint:** I think the easiest proof is by using a sequence argument.

Definition 3.53. Let L be a vector space. We say that two norms, $|\cdot|$ and $\|\cdot\|$, on L are equivalent if there exists constants $\alpha, \beta \in (0, \infty)$ such that

$$\|f\| \leq \alpha |f| \text{ and } |f| \leq \beta \|f\| \text{ for all } f \in L.$$

Lemma 3.54. *Let L be a finite dimensional vector space. Then any two norms $|\cdot|$ and $\|\cdot\|$ on L are equivalent. (This is typically not true for norms on infinite dimensional spaces.)*

Proof. Let $\{f_i\}_{i=1}^n$ be a basis for L and define a new norm on L by

$$\left\| \sum_{i=1}^n a_i f_i \right\|_1 \equiv \sum_{i=1}^n |a_i| \text{ for } a_i \in \mathbb{F}.$$

By the triangle inequality of the norm $|\cdot|$, we find

$$\left| \sum_{i=1}^n a_i f_i \right| \leq \sum_{i=1}^n |a_i| |f_i| \leq M \sum_{i=1}^n |a_i| = M \left\| \sum_{i=1}^n a_i f_i \right\|_1$$

where $M = \max_i |f_i|$. Thus we have

$$|f| \leq M \|f\|_1$$

for all $f \in L$. This inequality shows that $|\cdot|$ is continuous relative to $\|\cdot\|_1$. Now let $S := \{f \in L : \|f\|_1 = 1\}$, a compact subset of L relative to $\|\cdot\|_1$. Therefore by Exercise 3.14 there exists $f_0 \in S$ such that

$$m = \inf \{|f| : f \in S\} = |f_0| > 0.$$

Hence given $0 \neq f \in L$, then $\frac{f}{\|f\|_1} \in S$ so that

$$m \leq \left| \frac{f}{\|f\|_1} \right| = |f| \frac{1}{\|f\|_1}$$

or equivalently

$$\|f\|_1 \leq \frac{1}{m} |f|.$$

This shows that $|\cdot|$ and $\|\cdot\|_1$ are equivalent norms. Similarly one shows that $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent and hence so are $|\cdot|$ and $\|\cdot\|$. ■

Definition 3.55. A subset D of a topological space X is **dense** if $\bar{D} = X$. A topological space is said to be **separable** if it contains a countable dense subset, D.

Example 3.56. The following are examples of countable dense sets.

[5] Here is a proof if X is a metric space. Let $\{x_n\}_{n=1}^\infty \subset X$ be a sequence such that $f(x_n) \uparrow \sup f$. By compactness of X we may assume, by passing to a subsequence if necessary that $x_n \to b \in X$ as $n \to \infty$. By continuity of f, $f(b) = \sup f$.

(1) The rational number $\mathbb{Q}$ are dense in $\mathbb{R}$ equipped with the usual topology.
(2) More generally, $\mathbb{Q}^d$ is a countable dense subset of $\mathbb{R}^d$ for any $d \in \mathbb{N}$.
(3) Even more generally, for any function $\mu : \mathbb{N} \to (0, \infty)$, $\ell^p(\mu)$ is separable for all $1 \le p < \infty$. For example, let $\Gamma \subset \mathbb{F}$ be a countable dense set, then
$$D := \{x \in \ell^p(\mu) : x_i \in \lneqq \text{ for all } i \text{ and } \#\{j : x_j \ne 0\} < \infty\}.$$
The set Γ can be taken to be $\mathbb{Q}$ if $\mathbb{F} = \mathbb{R}$ or $\mathbb{Q} + i\mathbb{Q}$ if $\mathbb{F} = \mathbb{C}$.
(4) If (X, ρ) is a metric space which is separable then every subset $Y \subset X$ is also separable in the induced topology.

To prove 4. above, let $A = \{x_n\}_{n=1}^\infty \subset X$ be a countable dense subset of X. Let $\rho(x, Y) = \inf\{\rho(x, y) : y \in Y\}$ be the distance from x to Y. Recall that $\rho(\cdot, Y) : X \to [0, \infty)$ is continuous. Let $\epsilon_n = \rho(x_n, Y) \ge 0$ and for each n let $y_n \in B_{x_n}(\frac{1}{n}) \cap Y$ if $\epsilon_n = 0$ otherwise choose $y_n \in B_{x_n}(2\epsilon_n) \cap Y$. Then if $y \in Y$ and $\epsilon > 0$ we may choose $n \in \mathbb{N}$ such that $\rho(y, x_n) \le \epsilon_n < \epsilon/3$ and $\frac{1}{n} < \epsilon/3$. If $\epsilon_n > 0$, $\rho(y_n, x_n) \le 2\epsilon_n < 2\epsilon/3$ and if $\epsilon_n = 0$, $\rho(y_n, x_n) < \epsilon/3$ and therefore
$$\rho(y, y_n) \le \rho(y, x_n) + \rho(x_n, y_n) < \epsilon.$$
This shows that $B \equiv \{y_n\}_{n=1}^\infty$ is a countable dense subset of Y.

Lemma 3.57. *Any compact metric space (X, d) is separable.*

Proof. To each integer n, there exists $\Lambda_n \subset\subset X$ such that $X = \cup_{x \in \Lambda_n} B(x, 1/n)$. Let $D := \cup_{n=1}^\infty \Lambda_n$ – a countable subset of X. Moreover, it is clear by construction that $\bar{D} = X$. ■

3.6. **Compactness in Function Spaces.** In this section, let (X, τ) be a topological space.

Definition 3.58. Let $\mathcal{F} \subset C(X)$.

(1) $\mathcal{F}$ is equicontinuous at $x \in X$ iff for all $\epsilon > 0$ there exists $U \in \tau_x$ such that $|f(y) - f(x)| < \epsilon$ for all $y \in U$ and $f \in \mathcal{F}$.
(2) $\mathcal{F}$ is equicontinuous if $\mathcal{F}$ is equicontinuous at all points $x \in X$.
(3) $\mathcal{F}$ is pointwise bounded if $\sup\{|f(x)| : |f \in \mathcal{F}\} < \infty$ for all $x \in X$.

Theorem 3.59 (Ascoli-Arzela Theorem)**.** *Let (X, τ) be a compact topological space and $\mathcal{F} \subset C(X)$. Then $\mathcal{F}$ is precompact in $C(X)$ iff $\mathcal{F}$ is equicontinuous and pointwise bounded.*

Proof. ($\Leftarrow$) Since $B(X)$ is a complete metric space, we must show $\mathcal{F}$ is totally bounded. Let $\epsilon > 0$ be given. By equicontinuity there exists $V_x \in \tau_x$ for all $x \in X$ such that $|f(y) - f(x)| < \epsilon/2$ if $y \in V_x$ and $f \in \mathcal{F}$. Since X is compact we may choose $\Lambda \subset\subset X$ such that $X = \cup_{x \in \Lambda} V_x$. We have now decomposed X into "blocks" $\{V_x\}_{x \in \Lambda}$ such that each $f \in \mathcal{F}$ is constant to within ϵ on V_x. Since $\sup\{|f(x)| : x \in \Lambda \text{ and } f \in \mathcal{F}\} < \infty$, it is now evident that
$$M \equiv \sup\{|f(x)| : x \in X \text{ and } f \in \mathcal{F}\} \le \sup\{|f(x)| : x \in \Lambda \text{ and } f \in \mathcal{F}\} + \epsilon < \infty.$$

Let $\mathbb{D} \equiv \{k\epsilon/2 : k \in \mathbb{Z}\} \cap [-M, M]$. If $f \in \mathcal{F}$ and $\phi \in \mathbb{D}^\Lambda$ (i.e. $\phi : \Lambda \to \mathbb{D}$ is a function) is chosen so that $|\phi(x) - f(x)| \le \epsilon/2$ for all $x \in \Lambda$, then
$$|f(y) - \phi(x)| \le |f(y) - f(x)| + |f(x) - \phi(x)| < \epsilon \;\forall\; x \in \Lambda \text{ and } y \in V_x.$$
From this it follows that $\mathcal{F} = \bigcup\left\{\mathcal{F}_\phi : \phi \in \mathbb{D}^\Lambda\right\}$ where, for $\phi \in \mathbb{D}^\Lambda$,
$$\mathcal{F}_\phi \equiv \{f \in \mathcal{F} : |f(y) - \phi(x)| < \epsilon \text{ for } y \in V_x \text{ and } x \in \Lambda\}.$$

Let $\Gamma := \left\{\phi \in \mathbb{D}^\Lambda : \mathcal{F}_\phi \neq \emptyset\right\}$ and for each $\phi \in \Gamma$ choose $f_\phi \in \mathcal{F}_\phi \cap \mathcal{F}$. For $f \in \mathcal{F}_\phi$, $x \in \Lambda$ and $y \in V_x$ we have

$$|f(y) - f_\phi(y)| \leq |f(y) - \phi(x))| + |\phi(x) - f_\phi(y)| < 2\epsilon.$$

So $\|f - f_\phi\| < 2\epsilon$ for all $f \in \mathcal{F}_\phi$ showing that $\mathcal{F}_\phi \subset B_{f_\phi}(2\epsilon)$. Therefore,

$$\mathcal{F} = \cup_{\phi\in\Gamma}\mathcal{F}_\phi \subset \cup_{\phi\in\Gamma}B_{f_\phi}(2\epsilon)$$

and because $\epsilon > 0$ was arbitrary we have shown that $\mathcal{F}$ is totally bounded.

($\Rightarrow$) Since $\|\cdot\| : C(X) \to [0,\infty)$ is a continuous function on $C(X)$ it is bounded on any compact subset $\mathcal{F} \subset C(X)$. This shows that $\sup\{\|f\| : f \in \mathcal{F}\} < \infty$ which clearly implies that $\mathcal{F}$ is pointwise bounded.[6] Suppose $\mathcal{F}$ were **not** equicontinuous at some point $x \in X$ that is to say there exists $\epsilon > 0$ such that for all $V \in \tau_x$, $\sup_{y\in V}\sup_{f\in\mathcal{F}}|f(y) - f(x)| > \epsilon$.[7] Equivalently said, to each $V \in \tau_x$ we may choose

$$f_V \in \mathcal{F} \text{ and } x_V \in V \text{ such that } |f_V(x) - f_V(x_V)| \geq \epsilon. \tag{3.8}$$

Set $\mathcal{C}_V = \overline{\{f_W : W \in \tau_x \text{ and } W \subset V\}}^{\|\cdot\|_\infty} \subset \mathcal{F}$ and notice for any $\mathcal{V} \subset\subset \tau_x$ that

$$\cap_{V\in\mathcal{V}}\mathcal{C}_V \supseteq \mathcal{C}_{\cap\mathcal{V}} \neq \emptyset,$$

so that $\{\mathcal{C}_V\}_V \in \tau_x \subset \mathcal{F}$ has the finite intersection property.[8] Since $\mathcal{F}$ is compact, it follows that there exists some

$$f \in \bigcap_{V\in\tau_x}\mathcal{C}_V \neq \emptyset.$$

Since f is continuous, there exists $V \in \tau_x$ such that $|f(x) - f(y)| < \epsilon/3$ for all $y \in V$. Because $f \in \mathcal{C}_V$, there exists $W \subset V$ such that $\|f - f_W\| < \epsilon/3$. We now arrive at a contradiction;

$$\begin{aligned}\epsilon &\leq |f_W(x) - f_W(x_W)| \leq |f_W(x) - f(x)| + |f(x) - f(x_W)| + |f(x_W) - f_W(x_W)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

■

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[6] One could also prove that $\mathcal{F}$ is pointwise bounded by considering the continuous evaluation maps $e_x : C(X) \to \mathbb{R}$ given by $e_x(f) = f(x)$ for all $x \in X$.

[7] If X is first countable we could finish the proof with the following argument. Let $\{V_n\}_{n=1}^\infty$ be a neighborhood base at x such that $V_1 \supset V_2 \supset V_3 \supset \dots$. By the assumption that $\mathcal{F}$ is not equicontinuous at x, there exist $f_n \in \mathcal{F}$ and $x_n \in V_n$ such that $|f_n(x) - f_n(x_n)| \geq \epsilon\ \forall\ n$. Since $\mathcal{F}$ is a compact metric space by passing to a subsequence if necessary we may assume that f_n converges uniformly to some $f \in \mathcal{F}$. Because $x_n \to x$ as $n \to \infty$ we learn that

$$\begin{aligned}\epsilon &\leq |f_n(x) - f_n(x_n)| \leq |f_n(x) - f(x)| + |f(x) - f(x_n)| + |f(x_n) - f_n(x_n)| \\ &\leq 2\|f_n - f\| + |f(x) - f(x_n)| \to 0 \text{ as } n \to \infty\end{aligned}$$

which is a contradiction.

[8] If we are willing to use Net's described in Appendix D below we could finish the proof as follows. Since $\mathcal{F}$ is compact, the net $\{f_V\}_{V\in\tau_x} \subset \mathcal{F}$ has a cluster point $f \in \mathcal{F} \subset C(X)$. Choose a subnet $\{g_\alpha\}_{\alpha\in A}$ of $\{f_V\}_{V\in\tau_X}$ such that $g_\alpha \to f$ uniformly. Then, since $x_V \to x$ implies $x_{V_\alpha} \to x$, we may conclude from Eq. (3.8) that

$$\epsilon \leq |g_\alpha(x) - g_\alpha(x_{V_\alpha})| \to |g(x) - g(x)| = 0$$

which is a contradiction.

3.7. **Bounded Linear Operators Basics.**

Definition 3.60. Let X and Y be normed spaces and $T : X \to Y$ be a linear map. Then T is said to be bounded provided there exists $C < \infty$ such that $\|T(x)\| \leq C\|x\|_X$ for all $x \in X$. We denote the best constant by $\|T\|$, i.e.

$$\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|} = \sup_{x \neq 0} \{\|T(x)\| : \|x\| = 1\}.$$

The number $\|T\|$ is called the operator norm of T.

Proposition 3.61. *Suppose that X and Y are normed spaces and $T : X \to Y$ is a linear map. The the following are equivalent:*

(a) *T is continuous.*
(b) *T is continuous at 0.*
(c) *T is bounded.*

Proof. (a) $\Rightarrow$ (b) trivial. (b) $\Rightarrow$ (c) If T continuous at 0 then there exist $\delta > 0$ such that $\|T(x)\| \leq 1$ if $\|x\| \leq \delta$. Therefore for any $x \in X$, $\|T(\delta x/\|x\|)\| \leq 1$ which implies that $\|T(x)\| \leq \frac{1}{\delta}\|x\|$ and hence $\|T\| \leq \frac{1}{\delta} < \infty$. (c) $\Rightarrow$ (a) Let $x \in X$ and $\epsilon > 0$ be given. Then

$$\|T(y) - T(x)\| = \|T(y - x)\| \leq \|T\| \, \|y - x\| < \epsilon$$

provided $\|y - x\| < \epsilon/\|T\| \equiv \delta$. ■

In the examples to follow all integrals are the standard Riemann integrals, see Section 4 below for the definition and the basic properties of the Riemann integral.

Example 3.62. Suppose that $K : [0,1] \times [0,1] \to \mathbb{C}$ is a continuous function. For $f \in C([0,1])$, let

$$Tf(x) = \int_0^1 K(x,y)f(y)dy.$$

Since

$$\begin{aligned} |Tf(x) - Tf(z)| &\leq \int_0^1 |K(x,y) - K(z,y)| \, |f(y)| \, dy \\ &\leq \|f\|_\infty \max_y |K(x,y) - K(z,y)| \end{aligned} \tag{3.9}$$

and the latter expression tends to 0 as $x \to z$ by uniform continuity of K. Therefore $Tf \in C([0,1])$ and by the linearity of the Riemann integral, $T : C([0,1]) \to C([0,1])$ is a linear map. Moreover,

$$|Tf(x)| \leq \int_0^1 |K(x,y)| \, |f(y)| \, dy \leq \int_0^1 |K(x,y)| \, dy \cdot \|f\|_\infty \leq A \, \|f\|_\infty$$

where

$$A := \sup_{x \in [0,1]} \int_0^1 |K(x,y)| \, dy < \infty. \tag{3.10}$$

This shows $\|T\| \leq A < \infty$ and therefore T is bounded. We may in fact show $\|T\| = A$. To do this let $x_0 \in [0,1]$ be such that

$$\sup_{x \in [0,1]} \int_0^1 |K(x,y)| \, dy = \int_0^1 |K(x_0,y)| \, dy.$$

Such an x_0 can be found since, using a similar argument to that in Eq. (3.9), $x \to \int_0^1 |K(x,y)|\, dy$ is continuous. Given $\epsilon > 0$, let

$$f_\epsilon(y) := \frac{\overline{K(x_0,y)}}{\sqrt{\epsilon + |K(x_0,y)|^2}}$$

and notice that $\lim_{\epsilon \downarrow 0} \|f_\epsilon\|_\infty = 1$ and

$$\|Tf_\epsilon\|_\infty \geq |Tf_\epsilon(x_0)| = Tf_\epsilon(x_0) = \int_0^1 \frac{|K(x_0,y)|^2}{\sqrt{\epsilon + |K(x_0,y)|^2}} dy.$$

Therefore,

$$\begin{aligned}\|T\| &\geq \lim_{\epsilon \downarrow 0} \frac{1}{\|f_\epsilon\|_\infty} \int_0^1 \frac{|K(x_0,y)|^2}{\sqrt{\epsilon + |K(x_0,y)|^2}} dy \\ &= \lim_{\epsilon \downarrow 0} \int_0^1 \frac{|K(x_0,y)|^2}{\sqrt{\epsilon + |K(x_0,y)|^2}} dy = A\end{aligned}$$

since

$$\begin{aligned}0 \leq |K(x_0,y)| - \frac{|K(x_0,y)|^2}{\sqrt{\epsilon + |K(x_0,y)|^2}} &= \frac{|K(x_0,y)|}{\sqrt{\epsilon + |K(x_0,y)|^2}} \left[\sqrt{\epsilon + |K(x_0,y)|^2} - |K(x_0,y)|\right] \\ &\leq \sqrt{\epsilon + |K(x_0,y)|^2} - |K(x_0,y)|\end{aligned}$$

and the latter expression tends to zero uniformly in y as $\epsilon \downarrow 0$.

We may also consider other norms on $C([0,1])$. Let (for now) $L^1([0,1])$ denote $C([0,1])$ with the norm

$$\|f\|_1 = \int_0^1 |f(x)|\, dx,$$

then $T : L^1([0,1], dm) \to C([0,1])$ is bounded as well. Indeed, let $M = \sup\{|K(x,y)| : x, y \in [0,1]\}$, then

$$|(Tf)(x)| \leq \int_0^1 |K(x,y)f(y)|\, dy \leq M \|f\|_1$$

which shows $\|Tf\|_\infty \leq M \|f\|_1$ and hence,

$$\|T\|_{L^1 \to C} \leq \max\{|K(x,y)| : x, y \in [0,1]\} < \infty.$$

We can in fact show that $\|T\| = M$ as follows. Let $(x_0, y_0) \in [0,1]^2$ satisfying $|K(x_0,y_0)| = M$. Then given $\epsilon > 0$, there exists a neighborhood $U = I \times J$ of (x_0,y_0) such that $|K(x,y) - K(x_0,y_0)| < \epsilon$ for all $(x,y) \in U$. Let $f \in C_c(I,[0,\infty))$ such that $\int_0^1 f(x)dx = 1$. Choose $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and $\alpha K(x_0,y_0) = M$, then

$$\begin{aligned}|(T\alpha f)(x_0)| &= \left|\int_0^1 K(x_0,y)\alpha f(y)dy\right| = \left|\int_I K(x_0,y)\alpha f(y)dy\right| \\ &\geq \operatorname{Re}\int_I \alpha K(x_0,y)f(y)dy \geq \int_I (M-\epsilon) f(y)dy = (M-\epsilon)\|\alpha f\|_{L^1}\end{aligned}$$

and hence

$$\|T\alpha f\|_C \geq (M-\epsilon)\|\alpha f\|_{L^1}$$

showing that $\|T\| \geq M - \epsilon$. Since $\epsilon > 0$ is arbitrary, we learn that $\|T\| \geq M$ and hence $\|T\| = M$.

One may also view T as a map from $T : C([0,1]) \to L^1([0,1])$ in which case one may show

$$\|T\|_{L^1 \to C} \leq \int_0^1 \max_y |K(x,y)| \, dx < \infty.$$

For the next three exercises, let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ and $T : X \to Y$ be a linear transformation so that T is given by matrix multiplication by an $m \times n$ matrix. Let us identify the linear transformation T with this matrix.

Exercise 3.16. Assume the norms on X and Y are the ℓ^1 – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\| = \sum_{j=1}^n |x_j|$. Then the operator norm of T is given by

$$\|T\| = \max_{1 \leq j \leq n} \sum_{i=1}^m |T_{ij}| .$$

Exercise 3.17. ms on X and Y are the ℓ^∞ – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\| = \max_{1 \leq j \leq n} |x_j|$. Then the operator norm of T is given by

$$\|T\| = \max_{1 \leq i \leq m} \sum_{j=1}^n |T_{ij}| .$$

Exercise 3.18. Assume the norms on X and Y are the ℓ^2 – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\|^2 = \sum_{j=1}^n x_j^2$. Show $\|T\|^2$ is the largest eigenvalue of the matrix $T^{tr}T : \mathbb{R}^n \to \mathbb{R}^n$.

Exercise 3.19. If X is finite dimensional normed space then all linear maps are bounded.

Notation 3.63. Let $L(X,Y)$ denote the bounded linear operators from X to Y. If $Y = \mathbb{F}$ we write X^* for $L(X,\mathbb{F})$ and call X^* the (continuous) **dual space** to X.

Lemma 3.64. *Let X,Y be normed spaces, then the operator norm $\|\cdot\|$ on $L(X,Y)$ is a norm. Moreover if Z is another normed space and $T : X \to Y$ and $S : Y \to Z$ are linear maps, then $\|ST\| \leq \|S\|\|T\|$, where $ST := S \circ T$.*

Proof. As usual, the main point in checking the operator norm is a norm is to verify the triangle inequality, the other axioms being easy to check. If $A, B \in L(X,Y)$ then the triangle inequality is verified as follows:

$$\begin{aligned} \|A + B\| &= \sup_{x \neq 0} \frac{\|Ax + Bx\|}{\|x\|} \leq \sup_{x \neq 0} \frac{\|Ax\| + \|Bx\|}{\|x\|} \\ &\leq \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} + \sup_{x \neq 0} \frac{\|Bx\|}{\|x\|} = \|A\| + \|B\| . \end{aligned}$$

For the second assertion, we have for $x \in X$, that

$$\|STx\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\|.$$

From this inequality and the definition of $\|ST\|$, it follows that $\|ST\| \leq \|S\|\|T\|$. ■

Proposition 3.65. *Suppose that X is a normed vector space and Y is a Banach space. Then $(L(X,Y), \|\cdot\|_{op})$ is a Banach space. In particular the dual space X^* is always a Banach space.*

We will use the following characterization of a Banach space in the proof of this proposition.

Theorem 3.66. *A normed space* $(X, \|\cdot\|)$ *is a Banach space iff for every sequence* $\{x_n\}_{n=1}^\infty$ *such that* $\sum_{n=1}^\infty \|x_n\| < \infty$ *then* $\lim_{N\to\infty} \sum_{n=1}^N x_n = S$ *exists in* X *(that is to say every absolutely convergent series is a convergent series in* X*). As usual we will denote* S *by* $\sum_{n=1}^\infty x_n$.

Proof. ($\Rightarrow$)If X is complete and $\sum_{n=1}^\infty \|x_n\| < \infty$ then sequence $S_N \equiv \sum_{n=1}^N x_n$ for $N \in \mathbb{N}$ is Cauchy because (for $N > M$)

$$\|S_N - S_M\| \le \sum_{n=M+1}^N \|x_n\| \to 0 \text{ as } M, N \to \infty.$$

Therefore $S = \sum_{n=1}^\infty x_n := \lim_{N\to\infty} \sum_{n=1}^N x_n$ exists in X.

($\Longleftarrow$) Suppose that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence and let $\{y_k = x_{n_k}\}_{k=1}^\infty$ be a subsequence of $\{x_n\}_{n=1}^\infty$ such that $\sum_{n=1}^\infty \|y_{n+1} - y_n\| < \infty$. By assumption

$$y_{N+1} - y_1 = \sum_{n=1}^N (y_{n+1} - y_n) \to S = \sum_{n=1}^\infty (y_{n+1} - y_n) \in X \text{ as } N \to \infty.$$

This shows that $\lim_{N\to\infty} y_N$ exists and is equal to $x := y_1 + S$. Since $\{x_n\}_{n=1}^\infty$ is Cauchy,

$$\|x - x_n\| \le \|x - y_k\| + \|y_k - x_n\| \to 0 \text{ as } k, n \to \infty$$

showing that $\lim_{n\to\infty} x_n$ exists and is equal to x. ∎

Proof. (Proof of Proposition 3.65.) We must show $(L(X,Y), \|\cdot\|_{op})$ is complete. Suppose that $T_n \in L(X,Y)$ is a sequence of operators such that $\sum_{n=1}^\infty \|T_n\| < \infty$. Then

$$\sum_{n=1}^\infty \|T_n x\| \le \sum_{n=1}^\infty \|T_n\| \|x\| < \infty$$

and therefore by the completeness of Y, $Sx := \sum_{n=1}^\infty T_n x = \lim_{N\to\infty} S_N x$ exists in Y, where $S_N := \sum_{n=1}^N T_n$. The reader should check that $S : X \to Y$ so defined in linear. Since,

$$\|Sx\| = \lim_{N\to\infty} \|S_N x\| \le \lim_{N\to\infty} \sum_{n=1}^N \|T_n x\| \le \sum_{n=1}^\infty \|T_n\| \|x\|,$$

S is bounded and

$$\|S\| \le \sum_{n=1}^\infty \|T_n\|. \tag{3.11}$$

Similarly,

$$\|Sx - S_M x\| = \lim_{N\to\infty} \|S_N x - S_M x\| \le \lim_{N\to\infty} \sum_{n=M+1}^{N} \|T_n\| \|x\| = \sum_{n=M+1}^{\infty} \|T_n\| \|x\|$$

and therefore,

$$\|S - S_M\| \le \sum_{n=M}^{\infty} \|T_n\| \to 0 \text{ as } M \to \infty.$$

■

Of course we did not actually need to use Theorem 3.66 in the proof. Here is another proof. Let $\{T_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $L(X,Y)$. Then for each $x \in X$,

$$\|T_n x - T_m x\| \le \|T_n - T_m\| \|x\| \to 0 \text{ as } m,n \to \infty$$

showing $\{T_n x\}_{n=1}^{\infty}$ is Cauchy in Y. Using the completeness of Y, there exists an element $Tx \in Y$ such that

$$\lim_{n\to\infty} \|T_n x - Tx\| = 0.$$

It is a simple matter to show $T : X \to Y$ is a linear map. Moreover,

$$\|Tx - T_n x\| \le \|Tx - T_m x\| + \|T_m x - T_n x\| \le \|Tx - T_m x\| + \|T_m - T_n\| \|x\|$$

and therefore

$$\|Tx - T_n x\| \le \lim\sup_{m\to\infty} (\|Tx - T_m x\| + \|T_m - T_n\| \|x\|) = \|x\| \cdot \lim\sup_{m\to\infty} \|T_m - T_n\|.$$

Hence

$$\|T - T_n\| \le \lim\sup_{m\to\infty} \|T_m - T_n\| \to 0 \text{ as } n \to \infty.$$

Thus we have shown that $T_n \to T$ in $L(X,Y)$ as desired.

3.8. **Inverting Elements in $L(X)$ and Linear ODE.**

Definition 3.67. A linear map $T : X \to Y$ is an **isometry** if $\|Tx\|_Y = \|x\|_X$ for all $x \in X$. T is said to be **invertible** if T is a bijection and T^{-1} is bounded.

Notation 3.68. We will write $GL(X,Y)$ for those $T \in L(X,Y)$ which are invertible. If $X = Y$ we simply write $L(X)$ and $GL(X)$ for $L(X,X)$ and $GL(X,X)$ respectively.

Proposition 3.69. *Suppose X is a Banach space and $\Lambda \in L(X) \equiv L(X,X)$ satisfies $\sum_{n=0}^{\infty} \|\Lambda^n\| < \infty$. Then $I - \Lambda$ is invertible and*

$$(I - \Lambda)^{-1} = \text{“}\frac{1}{I - \Lambda}\text{”} = \sum_{n=0}^{\infty} \Lambda^n \text{ and } \left\|(I - \Lambda)^{-1}\right\| \le \sum_{n=0}^{\infty} \|\Lambda^n\|.$$

In particular if $\|\Lambda\| < 1$ then the above formula holds and

$$\left\|(I - \Lambda)^{-1}\right\| \le \frac{1}{1 - \|\Lambda\|}.$$

Proof. Since $L(X)$ is a Banach space and $\sum_{n=0}^{\infty}\|\Lambda^n\|<\infty$, it follows from Theorem 3.66 that

$$S:=\lim_{N\to\infty}S_N:=\lim_{N\to\infty}\sum_{n=0}^{N}\Lambda^n$$

exists in $L(X)$. Moreover, by Exercise 3.38 below,

$$\begin{aligned}(I-\Lambda)\,S&=(I-\Lambda)\lim_{N\to\infty}S_N=\lim_{N\to\infty}(I-\Lambda)\,S_N\\&=\lim_{N\to\infty}(I-\Lambda)\sum_{n=0}^{N}\Lambda^n=\lim_{N\to\infty}(I-\Lambda^{N+1})=I\end{aligned}$$

and similarly $S\,(I-\Lambda)=I$. This shows that $(I-\Lambda)^{-1}$ exists and is equal to S. Moreover, $(I-\Lambda)^{-1}$ is bounded because

$$\left\|(I-\Lambda)^{-1}\right\|=\|S\|\le\sum_{n=0}^{\infty}\|\Lambda^n\|.$$

If we further assume $\|\Lambda\|<1$, then $\|\Lambda^n\|\le\|\Lambda\|^n$ and

$$\sum_{n=0}^{\infty}\|\Lambda^n\|\le\sum_{n=0}^{\infty}\|\Lambda\|^n\le\frac{1}{1-\|\Lambda\|}<\infty.$$

■

Corollary 3.70. *Let X and Y be Banach spaces. Then $GL(X,Y)$ is an open (possibly empty) subset of $L(X,Y)$. More specifically, if $A\in GL(X,Y)$ and $B\in L(X,Y)$ satisfies*

$$\|B-A\|<\|A^{-1}\|^{-1}\tag{3.12}$$

then $B\in GL(X,Y)$

$$B^{-1}=\sum_{n=0}^{\infty}\left[I_X-A^{-1}B\right]^n A^{-1}\in L(Y,X)\tag{3.13}$$

and

$$\left\|B^{-1}\right\|\le\|A^{-1}\|\frac{1}{1-\|A^{-1}\|\,\|A-B\|}.$$

Proof. Let A and B be as above, then

$$B=A-(A-B)=A\left[I_X-A^{-1}(A-B))\right]=A(I_X-\Lambda)$$

where $\Lambda:X\to X$ is given by

$$\Lambda:=A^{-1}(A-B)=I_X-A^{-1}B.$$

Now

$$\|\Lambda\|=\left\|A^{-1}(A-B))\right\|\le\|A^{-1}\|\,\|A-B\|<\|A^{-1}\|\|A^{-1}\|^{-1}=1.$$

Therefore $I-\Lambda$ is invertible and hence so is B (being the product of invertible elements) with

$$B^{-1}=(I-\Lambda)^{-1}A^{-1}=\left[I_X-A^{-1}(A-B))\right]^{-1}A^{-1}.$$

For the last assertion we have,

$$\left\|B^{-1}\right\|\le\left\|(I_X-\Lambda)^{-1}\right\|\|A^{-1}\|\le\|A^{-1}\|\frac{1}{1-\|\Lambda\|}\le\|A^{-1}\|\frac{1}{1-\|A^{-1}\|\,\|A-B\|}.$$

■

For an application of these results to linear ordinary differentiatl equations, see Section 5.2.

3.9. Supplement: Sums in Banach Spaces.

Definition 3.71. Suppose that X is a normed space and $\{v_\alpha \in X : \alpha \in A\}$ is a given collection of vectors in X. We say that $s = \sum_{\alpha \in A} v_\alpha \in X$ if for all $\epsilon > 0$ there exists a finite set $\Gamma_\epsilon \subset A$ such that $\left\| s - \sum_{\alpha\in\Lambda} v_\alpha \right\| < \epsilon$ for all $\Lambda \subset\subset A$ such that $\Gamma_\epsilon \subset \Lambda$. (Unlike the case of real valued sums, this does not imply that $\sum_{\alpha\in\Lambda} \|v_\alpha\| < \infty$. See Proposition 12.19 below, from which one may manufacture counter-examples to this false premise.)

Lemma 3.72. *(1) When X is a Banach space, $\sum_{\alpha\in A} v_\alpha$ exists in X iff for all $\epsilon > 0$ there exists $\Gamma_\epsilon \subset\subset A$ such that $\left\|\sum_{\alpha\in\Lambda} v_\alpha\right\| < \epsilon$ for all $\Lambda \subset\subset A \setminus \Gamma_\epsilon$. Also if $\sum_{\alpha\in A} v_\alpha$ exists in X then $\{\alpha \in A : v_a \neq 0\}$ is at most countable. (2) If $s = \sum_{\alpha\in A} v_\alpha \in X$ exists and $T : X \to Y$ is a bounded linear map between normed spaces, then $\sum_{\alpha\in A} Tv_\alpha$ exists in Y and*

$$Ts = T\sum_{\alpha\in A} v_\alpha = \sum_{\alpha\in A} Tv_\alpha.$$

Proof. (1) Suppose that $s = \sum_{\alpha\in A} v_\alpha$ exists and $\epsilon > 0$. Let $\Gamma_\epsilon \subset\subset A$ be as in Definition 3.71. Then for $\Lambda \subset\subset A \setminus \Gamma_\epsilon$,

$$\begin{aligned}\left\|\sum_{\alpha\in\Lambda} v_\alpha\right\| &\le \left\|\sum_{\alpha\in\Lambda} v_\alpha + \sum_{\alpha\in\Gamma_\epsilon} v_\alpha - s\right\| + \left\|\sum_{\alpha\in\Gamma_\epsilon} v_\alpha - s\right\| \\ &= \left\|\sum_{\alpha\in\Gamma_\epsilon\cup\Lambda} v_\alpha - s\right\| + \epsilon < 2\epsilon.\end{aligned}$$

Conversely, suppose for all $\epsilon > 0$ there exists $\Gamma_\epsilon \subset\subset A$ such that $\left\|\sum_{\alpha\in\Lambda} v_\alpha\right\| < \epsilon$ for all $\Lambda \subset\subset A \setminus \Gamma_\epsilon$. Let $\gamma_n := \cup_{k=1}^n \Gamma_{1/k} \subset A$ and set $s_n := \sum_{\alpha\in\gamma_n} v_\alpha$. Then for $m > n$,

$$\|s_m - s_n\| = \left\|\sum_{\alpha\in\gamma_m\setminus\gamma_n} v_\alpha\right\| \le 1/n \to 0 \text{ as } m, n \to \infty.$$

Therefore $\{s_n\}_{n=1}^\infty$ is Cauchy and hence convergent in X. Let $s := \lim_{n\to\infty} s_n$, then for $\Lambda \subset\subset A$ such that $\gamma_n \subset \Lambda$, we have

$$\left\|s - \sum_{\alpha\in\Lambda} v_\alpha\right\| \le \|s - s_n\| + \left\|\sum_{\alpha\in\Lambda\setminus\gamma_n} v_\alpha\right\| \le \|s - s_n\| + \frac{1}{n}.$$

Since the right member of this equation goes to zero as $n \to \infty$, it follows that $\sum_{\alpha\in A} v_\alpha$ exists and is equal to s.

Let $\gamma := \cup_{n=1}^\infty \gamma_n$ – a countable subset of A. Then for $\alpha \notin \gamma$, $\{\alpha\} \subset A \setminus \gamma_n$ for all n and hence

$$\|v_\alpha\| = \left\|\sum_{\beta\in\{\alpha\}} v_\beta\right\| \le 1/n \to 0 \text{ as } n \to \infty.$$

Therefore $v_\alpha = 0$ for all $\alpha \in A \setminus \gamma$.

(2) Let Γ_ϵ be as in Definition 3.71 and $\Lambda \subset\subset A$ such that $\Gamma_\epsilon \subset \Lambda$. Then

$$\left\| Ts - \sum_{\alpha\in\Lambda} Tv_\alpha \right\| \le \|T\| \left\| s - \sum_{\alpha\in\Lambda} v_\alpha \right\| < \|T\| \epsilon$$

which shows that $\sum_{\alpha\in\Lambda} Tv_\alpha$ exists and is equal to Ts. ■

3.10. **Word of Caution.**

Example 3.73. Let (X, d) be a metric space. It is always true that $\overline{B_x(\epsilon)} \subset C_x(\epsilon)$ since $C_x(\epsilon)$ is a closed set containing $B_x(\epsilon)$. However, it is not always true that $\overline{B_x(\epsilon)} = C_x(\epsilon)$. For example let $X = \{1, 2\}$ and $d(1, 2) = 1$, then $B_1(1) = \{1\}$, $\overline{B_1(1)} = \{1\}$ while $C_1(1) = X$. For another counter example, take

$$X = \left\{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } x = 1\right\}$$

with the usually Euclidean metric coming from the plane. Then

$$\begin{aligned} B_{(0,0)}(1) &= \left\{(0, y) \in \mathbb{R}^2 : |y| < 1\right\}, \\ \overline{B_{(0,0)}(1)} &= \left\{(0, y) \in \mathbb{R}^2 : |y| \le 1\right\}, \text{ while} \\ C_{(0,0)}(1) &= \overline{B_{(0,0)}(1)} \cup \{(0, 1)\}. \end{aligned}$$

In spite of the above examples, Lemmas 3.74 and 3.75 below shows that for certain metric spaces of interest it is true that $\overline{B_x(\epsilon)} = C_x(\epsilon)$.

Lemma 3.74. *Suppose that $(X, |\cdot|)$ is a normed vector space and d is the metric on X defined by $d(x, y) = |x - y|$. Then*

$$\begin{aligned} \overline{B_x(\epsilon)} &= C_x(\epsilon) \text{ and} \\ \partial B_x(\epsilon) &= \{y \in X : d(x, y) = \epsilon\}. \end{aligned}$$

Proof. We must show that $C := C_x(\epsilon) \subset \overline{B_x(\epsilon)} =: \bar{B}$. For $y \in C$, let $v = y - x$, then

$$|v| = |y - x| = d(x, y) \le \epsilon.$$

Let $\alpha_n = 1 - 1/n$ so that $\alpha_n \uparrow 1$ as $n \to \infty$. Let $y_n = x + \alpha_n v$, then $d(x, y_n) = \alpha_n d(x, y) < \epsilon$, so that $y_n \in B_x(\epsilon)$ and $d(y, y_n) = 1 - \alpha_n \to 0$ as $n \to \infty$. This shows that $y_n \to y$ as $n \to \infty$ and hence that $y \in \bar{B}$. ■

3.10.1. *Riemannian Metrics.* This subsection is not completely self contained and may safely be skipped.

Lemma 3.75. *Suppose that X is a Riemannian (or sub-Riemannian) manifold and d is the metric on X defined by*

$$d(x, y) = \inf\{\ell(\sigma) : \sigma(0) = x \text{ and } \sigma(1) = y\}$$

where $\ell(\sigma)$ is the length of the curve σ. We define $\ell(\sigma) = \infty$ if σ is not piecewise smooth.

Then

$$\begin{aligned} \overline{B_x(\epsilon)} &= C_x(\epsilon) \text{ and} \\ \partial B_x(\epsilon) &= \{y \in X : d(x, y) = \epsilon\}. \end{aligned}$$

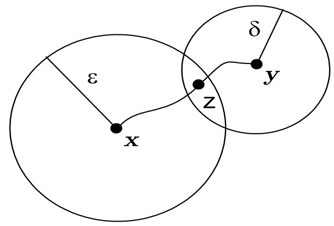

FIGURE 10. An almost length minimizing curve joining x to y.

Proof. Let $C := C_x(\epsilon) \subset \overline{B_x(\epsilon)} =: \bar{B}$. We will show that $C \subset \bar{B}$ by showing $\bar{B}^c \subset C^c$. Suppose that $y \in \bar{B}^c$ and choose $\delta > 0$ such that $B_y(\delta) \cap \bar{B} = \emptyset$. In particular this implies that

$$B_y(\delta) \cap B_x(\epsilon) = \emptyset.$$

We will finish the proof by showing that $d(x,y) \geq \epsilon + \delta > \epsilon$ and hence that $y \in C^c$. This will be accomplished by showing: if $d(x,y) < \epsilon + \delta$ then $B_y(\delta) \cap B_x(\epsilon) \neq \emptyset$.

If $d(x,y) < \max(\epsilon, \delta)$ then either $x \in B_y(\delta)$ or $y \in B_x(\epsilon)$. In either case $B_y(\delta) \cap B_x(\epsilon) \neq \emptyset$. Hence we may assume that $\max(\epsilon, \delta) \leq d(x,y) < \epsilon + \delta$. Let $\alpha > 0$ be a number such that

$$\max(\epsilon, \delta) \leq d(x,y) < \alpha < \epsilon + \delta$$

and choose a curve σ from x to y such that $\ell(\sigma) < \alpha$. Also choose $0 < \delta' < \delta$ such that $0 < \alpha - \delta' < \epsilon$ which can be done since $\alpha - \delta < \epsilon$. Let $k(t) = d(y, \sigma(t))$ a continuous function on $[0,1]$ and therefore $k([0,1]) \subset \mathbb{R}$ is a connected set which contains 0 and $d(x,y)$. Therefore there exists $t_0 \in [0,1]$ such that $d(y, \sigma(t_0)) = k(t_0) = \delta'$. Let $z = \sigma(t_0) \in B_y(\delta)$ then

$$d(x,z) \leq \ell(\sigma|_{[0,t_0]}) = \ell(\sigma) - \ell(\sigma|_{[t_0,1]}) < \alpha - d(z,y) = \alpha - \delta' < \epsilon$$

and therefore $z \in B_x(\epsilon) \cap B_x(\delta) \neq \emptyset$. ■

Remark 3.76. Suppose again that X is a Riemannian (or sub-Riemannian) manifold and

$$d(x,y) = \inf \{\ell(\sigma) : \sigma(0) = x \text{ and } \sigma(1) = y\}.$$

Let σ be a curve from x to y and let $\epsilon = \ell(\sigma) - d(x,y)$. Then for all $0 \leq u < v \leq 1$,

$$d(\sigma(u), \sigma(v)) \leq \ell(\sigma|_{[u,v]}) + \epsilon.$$

So if σ is within ϵ of a length minimizing curve from x to y that $\sigma|_{[u,v]}$ is within ϵ of a length minimizing curve from $\sigma(u)$ to $\sigma(v)$. In particular if $d(x,y) = \ell(\sigma)$ then $d(\sigma(u), \sigma(v)) = \ell(\sigma|_{[u,v]})$ for all $0 \leq u < v \leq 1$, i.e. if σ is a length minimizing curve from x to y that $\sigma|_{[u,v]}$ is a length minimizing curve from $\sigma(u)$ to $\sigma(v)$.

To prove these assertions notice that

$$\begin{aligned} d(x,y) + \epsilon = \ell(\sigma) &= \ell(\sigma|_{[0,u]}) + \ell(\sigma|_{[u,v]}) + \ell(\sigma|_{[v,1]}) \\ &\geq d(x, \sigma(u)) + \ell(\sigma|_{[u,v]}) + d(\sigma(v), y) \end{aligned}$$

and therefore

$$\begin{aligned} \ell(\sigma|_{[u,v]}) &\leq d(x,y) + \epsilon - d(x, \sigma(u)) - d(\sigma(v), y) \\ &\leq d(\sigma(u), \sigma(v)) + \epsilon. \end{aligned}$$

3.11. **Exercises.**

Exercise 3.20. Prove Lemma 3.46.

Exercise 3.21. Let $X = C([0,1],\mathbb{R})$ and for $f \in X$, let

$$\|f\|_1 := \int_0^1 |f(t)|\, dt.$$

Show that $(X, \|\cdot\|_1)$ is normed space and show by example that this space is **not** complete.

Exercise 3.22. Let (X,d) be a metric space. Suppose that $\{x_n\}_{n=1}^\infty \subset X$ is a sequence and set $\epsilon_n := d(x_n, x_{n+1})$. Show that for $m > n$ that

$$d(x_n, x_m) \le \sum_{k=n}^{m-1} \epsilon_k \le \sum_{k=n}^{\infty} \epsilon_k.$$

Conclude from this that if

$$\sum_{k=1}^\infty \epsilon_k = \sum_{n=1}^\infty d(x_n, x_{n+1}) < \infty$$

then $\{x_n\}_{n=1}^\infty$ is Cauchy. Moreover, show that if $\{x_n\}_{n=1}^\infty$ is a convergent sequence and $x = \lim_{n\to\infty} x_n$ then

$$d(x, x_n) \le \sum_{k=n}^\infty \epsilon_k.$$

Exercise 3.23. Show that (X,d) is a complete metric space iff every sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $\sum_{n=1}^\infty d(x_n, x_{n+1}) < \infty$ is a convergent sequence in X. You may find it useful to prove the following statements in the course of the proof.

1. If $\{x_n\}$ is Cauchy sequence, then there is a subsequence $y_j \equiv x_{n_j}$ such that $\sum_{j=1}^\infty d(y_{j+1}, y_j) < \infty$.
2. If $\{x_n\}_{n=1}^\infty$ is Cauchy and there exists a subsequence $y_j \equiv x_{n_j}$ of $\{x_n\}$ such that $x = \lim_{j\to\infty} y_j$ exists, then $\lim_{n\to\infty} x_n$ also exists and is equal to x.

Exercise 3.24. Suppose that $f : [0,\infty) \to [0,\infty)$ is a C^2 – function such that $f(0) = 0$, $f' > 0$ and $f'' \le 0$ and (X,ρ) is a metric space. Show that $d(x,y) = f(\rho(x,y))$ is a metric on X. In particular show that

$$d(x,y) \equiv \frac{\rho(x,y)}{1+\rho(x,y)}$$

is a metric on X. (Hint: use calculus to verify that $f(a+b) \le f(a) + f(b)$ for all $a, b \in [0,\infty)$.)

Exercise 3.25. Let $d : C(\mathbb{R}) \times C(\mathbb{R}) \to [0,\infty)$ be defined by

$$d(f,g) = \sum_{n=1}^\infty 2^{-n} \frac{\|f-g\|_n}{1+\|f-g\|_n},$$

where $\|f\|_n \equiv \sup\{|f(x)| : |x| \le n\} = \max\{|f(x)| : |x| \le n\}$.

1. Show that d is a metric on $C(\mathbb{R})$.
2. Show that a sequence $\{f_n\}_{n=1}^\infty \subset C(\mathbb{R})$ converges to $f \in C(\mathbb{R})$ as $n \to \infty$ iff f_n converges to f uniformly on compact subsets of $\mathbb{R}$.
3. Show that $(C(\mathbb{R}), d)$ is a complete metric space.

Exercise 3.26. Let $\{(X_n, d_n)\}_{n=1}^{\infty}$ be a sequence of metric spaces, $X := \prod_{n=1}^{\infty} X_n$, and for $x = (x(n))_{n=1}^{\infty}$ and $y = (y(n))_{n=1}^{\infty}$ in X let

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x(n), y(n))}{1 + d_n(x(n), y(n))}.$$

Show: 1) (X, d) is a metric space, 2) a sequence $\{x_k\}_{k=1}^{\infty} \subset X$ converges to $x \in X$ iff $x_k(n) \to x(n) \in X_n$ as $k \to \infty$ for every $n = 1, 2, \ldots$, and 3) X is complete if X_n is complete for all n.

Exercise 3.27 (Tychonoff's Theorem)**.** Let us continue the notation of the previous problem. Further assume that the spaces X_n are compact for all n. Show (X, d) is compact. **Hint:** Either use Cantor's method to show every sequence $\{x_m\}_{m=1}^{\infty} \subset X$ has a convergent subsequence or alternatively show (X, d) is complete and totally bounded.

Exercise 3.28. Let (X_i, d_i) for $i = 1, \ldots, n$ be a finite collection of metric spaces and for $1 \le p \le \infty$ and $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $X := \prod_{i=1}^{n} X_i$, let

$$\rho_p(x, y) = \begin{cases} \left(\sum_{i=1}^{n} [d_i(x_i, y_i)]^p\right)^{1/p} & \text{if} \quad p \neq \infty \\ \max_i d_i(x_i, y_i) & \text{if} \quad p = \infty \end{cases}.$$

(1) Show (X, ρ_p) is a metric space for $p \in [1, \infty]$. **Hint**: Minkowski's inequality.
(2) Show that all of the metric $\{\rho_p : 1 \le p \le \infty\}$ are equivalent, i.e. for any $p, q \in [1, \infty]$ there exists constants $c, C < \infty$ such that

$$\rho_p(x, y) \le C\rho_q(x, y) \text{ and } \rho_q(x, y) \le c\rho_p(x, y) \text{ for all } x, y \in X.$$

Hint: This can be done with explicit estimates or more simply using Lemma 3.54.
(3) Show that the topologies associated to the metrics ρ_p are the same for all $p \in [1, \infty]$.

Exercise 3.29. Let C be a closed proper subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n \setminus C$. Show there exists a $y \in C$ such that $d(x, y) = d_C(x)$.

Exercise 3.30. Let $\mathbb{F} = \mathbb{R}$ in this problem and $A \subset \ell^2(\mathbb{N})$ be defined by

$$\begin{aligned} A &= \{x \in \ell^2(\mathbb{N}) : x(n) \ge 1 + 1/n \text{ for some } n \in \mathbb{N}\} \\ &= \cup_{n=1}^{\infty} \{x \in \ell^2(\mathbb{N}) : x(n) \ge 1 + 1/n\}. \end{aligned}$$

Show A is a closed subset of $\ell^2(\mathbb{N})$ with the property that $d_A(0) = 1$ while there is no $y \in A$ such that $d_A(y) = 1$. (Remember that in general an infinite union of closed sets need not be closed.)

3.11.1. *Banach Space Problems.*

Exercise 3.31. Show that all finite dimensional normed vector spaces $(L, \|\cdot\|)$ are necessarily complete. Also show that closed and bounded sets (relative to the given norm) are compact.

Exercise 3.32. Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Show the map

$$(\lambda, x, y) \in \mathbb{F} \times X \times X \to x + \lambda y \in X$$

is continuous relative to the topology on $\mathbb{F} \times X \times X$ defined by the norm

$$\|(\lambda, x, y)\|_{\mathbb{F} \times X \times X} := |\lambda| + \|x\| + \|y\|.$$

(See Exercise 3.28 for more on the metric associated to this norm.) Also show that $\|\cdot\| : X \to [0, \infty)$ is continuous.

Exercise 3.33. Let $p \in [1, \infty]$ and X be an infinite set. Show the closed unit ball in $\ell^p(X)$ is not compact.

Exercise 3.34. Let $X = \mathbb{N}$ and for $p, q \in [1, \infty)$ let $\|\cdot\|_p$ denote the $\ell^p(\mathbb{N})$ – norm. Show $\|\cdot\|_p$ and $\|\cdot\|_q$ are inequivalent norms for $p \neq q$ by showing

$$\sup_{f \neq 0} \frac{\|f\|_p}{\|f\|_q} = \infty \text{ if } p < q.$$

Exercise 3.35. Folland Problem 5.5. Closure of subspaces are subspaces.

Exercise 3.36. Folland Problem 5.9. Showing $C^k([0, 1])$ is a Banach space.

Exercise 3.37. Folland Problem 5.11. Showing Holder spaces are Banach spaces.

Exercise 3.38. Let X, Y and Z be normed spaces. Prove the maps

$$(S, x) \in L(X, Y) \times X \longrightarrow Sx \in Y$$

and

$$(S, T) \in L(X, Y) \times L(Y, Z) \longrightarrow ST \in L(X, Z)$$

are continuous relative to the norms

$$\|(S, x)\|_{L(X,Y)\times X} := \|S\|_{L(X,Y)} + \|x\|_X \text{ and}$$
$$\|(S, T)\|_{L(X,Y)\times L(Y,Z)} := \|S\|_{L(X,Y)} + \|T\|_{L(Y,Z)}$$

on $L(X, Y) \times X$ and $L(X, Y) \times L(Y, Z)$ respectively.

3.11.2. *Ascoli-Arzela Theorem Problems.*

Exercise 3.39. Let $T \in (0, \infty)$ and $\mathcal{F} \subset C([0, T])$ be a family of functions such that:

(1) $\dot{f}(t)$ exists for all $t \in (0, T)$ and $f \in \mathcal{F}$.
(2) $\sup_{f\in\mathcal{F}} |f(0)| < \infty$ and
(3) $M := \sup_{f\in\mathcal{F}} \sup_{t\in(0,T)} \left|\dot{f}(t)\right| < \infty$.

Show $\mathcal{F}$ is precompact in the Banach space $C([0, T])$ equipped with the norm $\|f\|_\infty = \sup_{t\in[0,T]} |f(t)|$.

Exercise 3.40. Folland Problem 4.63.

Exercise 3.41. Folland Problem 4.64.

3.11.3. *General Topological Space Problems.*

Exercise 3.42. Give an example of continuous map, $f : X \to Y$, and a compact subset K of Y such that $f^{-1}(K)$ is not compact.

Exercise 3.43. Let V be an open subset of $\mathbb{R}$. Show V may be written as a disjoint union of open intervals $J_n = (a_n, b_n)$, where $a_n, b_n \in \mathbb{R}\cup \{\pm\infty\}$ for $n = 1, 2, \cdots < N$ with $N = \infty$ possible.