4. The Riemann Integral

In this short chapter, the Riemann integral for Banach space valued functions is defined and developed. Our exposition will be brief, since the Lebesgue integral and the Bochner Lebesgue integral will subsume the content of this chapter. The following simple "Bounded Linear Transformation" theorem will often be used here and in the sequel to define linear transformations.

Theorem 4.1 (B. L. T. Theorem). *Suppose that Z is a normed space, X is a Banach space, and $\mathcal{S} \subset Z$ is a dense linear subspace of Z. If $T : \mathcal{S} \to X$ is a bounded linear transformation (i.e. there exists $C < \infty$ such that $\|Tz\| \le C \|z\|$ for all $z \in \mathcal{S}$), then T has a unique extension to an element $\bar{T} \in L(Z, X)$ and this extension still satisfies*

$$\left\|\bar{T}z\right\| \le C \|z\| \text{ for all } z \in \bar{\mathcal{S}}.$$

Exercise 4.1. Prove Theorem 4.1.

For the remainder of the chapter, let $[a, b]$ be a fixed compact interval and X be a Banach space. The collection $\mathcal{S} = \mathcal{S}([a, b], X)$ of **step functions**, $f : [a, b] \to X$, consists of those functions f which may be written in the form

$$f(t) = x_0 1_{[a,t_1]}(t) + \sum_{i=1}^{n-1} x_i 1_{(t_i,t_{i+1}]}(t), \tag{4.1}$$

where $\pi \equiv \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ and $x_i \in X$. For f as in Eq. (4.1), let

$$I(f) \equiv \sum_{i=0}^{n-1} (t_{i+1} - t_i) x_i \in X. \tag{4.2}$$

Exercise 4.2. Show that $I(f)$ is well defined, independent of how f is represented as a step function. (**Hint:** show that adding a point to a partition π of $[a, b]$ does not change the right side of Eq. (4.2).) Also verify that $I : \mathcal{S} \to X$ is a linear operator.

Proposition 4.2 (Riemann Integral). *The linear function $I : \mathcal{S} \to X$ extends uniquely to a continuous linear operator $\bar{I}$ from $\bar{\mathcal{S}}$ (the closure of the step functions inside of $\ell^\infty([a, b], X)$) to X and this operator satisfies,*

$$\|\bar{I}(f)\| \le (b - a) \|f\|_\infty \text{ for all } f \in \bar{\mathcal{S}}. \tag{4.3}$$

Furthermore, $C([a, b], X) \subset \bar{\mathcal{S}} \subset \ell^\infty([a, b], X)$ and for $f \in$, $\bar{I}(f)$ may be computed as

$$\bar{I}(f) = \lim_{|\pi| \to 0} \sum_{i=0}^{n-1} f(c_i^\pi)(t_{i+1} - t_i) \tag{4.4}$$

where $\pi \equiv \{a = t_0 < t_1 < \cdots < t_n = b\}$ denotes a partition of $[a, b]$, $|\pi| = \max\{|t_{i+1} - t_i| : i = 0, \ldots, n - 1\}$ is the mesh size of π and c_i^π may be chosen arbitrarily inside $[t_i, t_{i+1}]$.

Proof. Taking the norm of Eq. (4.2) and using the triangle inequality shows,

$$\|I(f)\| \le \sum_{i=0}^{n-1} (t_{i+1} - t_i) \|x_i\| \le \sum_{i=0}^{n-1} (t_{i+1} - t_i) \|f\|_\infty \le (b - a) \|f\|_\infty. \tag{4.5}$$

The existence of $\bar{I}$ satisfying Eq. (4.3) is a consequence of Theorem 4.1.

For $f \in C([a,b],X)$, $\pi \equiv \{a = t_0 < t_1 < \cdots < t_n = b\}$ a partition of $[a,b]$, and $c_i^\pi \in [t_i, t_{i+1}]$ for $i = 0,1,2\ldots,n-1$, let

$$f_\pi(t) \equiv f(c_0)_0 1_{[t_0,t_1]}(t) + \sum_{i=1}^{n-1} f(c_i^\pi) 1_{(t_i,t_{i+1}]}(t).$$

Then $I(f_\pi) = \sum_{i=0}^{n-1} f(c_i^\pi)(t_{i+1} - t_i)$ so to finish the proof of Eq. (4.4) and that $C([a,b],X) \subset \bar{\mathcal{S}}$, it suffices to observe that $\lim_{|\pi|\to 0} \|f - f_\pi\|_\infty = 0$ because f is uniformly continuous on $[a,b]$. ■

If $f_n \in \mathcal{S}$ and $f \in \bar{\mathcal{S}}$ such that $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$, then for $a \le \alpha < \beta \le b$, then $1_{[\alpha,\beta]} f_n \in \mathcal{S}$ and $\lim_{n\to\infty} \left\|1_{[\alpha,\beta]} f - 1_{[\alpha,\beta]} f_n\right\|_\infty = 0$. This shows $1_{[\alpha,\beta]} f \in \bar{\mathcal{S}}$ whenever $f \in \bar{\mathcal{S}}$.

Notation 4.3. For $f \in \bar{\mathcal{S}}$ and $a \le \alpha \le \beta \le b$ we will write denote $\bar{I}(1_{[\alpha,\beta]} f)$ by $\int_\alpha^\beta f(t)\,dt$ or $\int_{[\alpha,\beta]} f(t)dt$. Also following the usual convention, if $a \le \beta \le \alpha \le b$, we will let

$$\int_\alpha^\beta f(t)\,dt = -\bar{I}(1_{[\beta,\alpha]} f) = -\int_\beta^\alpha f(t)\,dt.$$

The next Lemma, whose proof is left to the reader (Exercise 4.4) contains some of the many familiar properties of the Riemann integral.

Lemma 4.4. *For $f \in \bar{\mathcal{S}}([a,b],X)$ and $\alpha,\beta,\gamma \in [a,b]$, the Riemann integral satisfies:*

1. $\left\|\int_\alpha^\beta f(t)\,dt\right\|_\infty \le (\beta - \alpha) \sup\{\|f(t)\| : \alpha \le t \le \beta\}$.
2. $\int_\alpha^\gamma f(t)\,dt = \int_\alpha^\beta f(t)\,dt + \int_\beta^\gamma f(t)\,dt$.
3. *The function $G(t) := \int_a^t f(\tau) d\tau$ is continuous on $[a,b]$.*
4. *If Y is another Banach space and $T \in L(X,Y)$, then $Tf \in \bar{\mathcal{S}}([a,b],Y)$ and*
$$T\left(\int_\alpha^\beta f(t)dt\right) = \int_\alpha^\beta Tf(t)dt.$$
5. *The function $t \to \|f(t)\|_X$ is in $\bar{\mathcal{S}}([a,b],\mathbb{R})$ and*
$$\left\|\int_a^b f(t)\,dt\right\| \le \int_a^b \|f(t)\|\,dt.$$
6. *If $f,g \in \bar{\mathcal{S}}([a,b],\mathbb{R})$ and $f \le g$, then*
$$\int_a^b f(t)dt \le \int_a^b g(t)dt.$$

Theorem 4.5 (Baby Fubini Theorem)**.** *Let $a,b,c,d \in \mathbb{R}$ and $f(s,t) \in X$ be a continuous function of (s,t) for s between a and b and t between c and d. Then the maps $t \to \int_a^b f(s,t)ds \in X$ and $s \to \int_c^d f(s,t)dt$ are continuous and*

$$\int_c^d \left[\int_a^b f(s,t)ds\right] dt = \int_a^b \left[\int_c^d f(s,t)dt\right] ds. \tag{4.6}$$

Proof. With out loss of generality we may assume $a < b$ and $c < d$. By uniform continuity of f, Exercise 3.15,

$$\sup_{c\le t\le d} \|f(s,t) - f(s_0,t)\| \to 0 \text{ as } s \to s_0$$

and so by Lemma 4.4

$$\int_c^d f(s,t)dt \to \int_c^d f(s_0,t)dt \text{ as } s \to s_0$$

showing the continuity of $s \to \int_c^d f(s,t)dt$. The other continuity assertion is proved similarly.

Now let

$$\pi = \{a \le s_0 < s_1 < \cdots < s_m = b\} \text{ and } \pi' = \{c \le t_0 < t_1 < \cdots < t_n = d\}$$

be partitions of $[a,b]$ and $[c,d]$ respectively. For $s \in [a,b]$ let $s_\pi = s_i$ if $s \in (s_i, s_{i+1}]$ and $i \ge 1$ and $s_\pi = s_0 = a$ if $s \in [s_0, s_1]$. Define $t_{\pi'}$ for $t \in [c,d]$ analogously. Then

$$\begin{aligned}\int_a^b \left[\int_c^d f(s,t)dt\right] ds &= \int_a^b \left[\int_c^d f(s,t_{\pi'})dt\right] ds + \int_a^b \epsilon_{\pi'}(s)ds \\ &= \int_a^b \left[\int_c^d f(s_\pi,t_{\pi'})dt\right] ds + \delta_{\pi,\pi'} + \int_a^b \epsilon_{\pi'}(s)ds\end{aligned}$$

where

$$\epsilon_{\pi'}(s) = \int_c^d f(s,t)dt - \int_c^d f(s,t_{\pi'})dt$$

and

$$\delta_{\pi,\pi'} = \int_a^b \left[\int_c^d \{f(s,t_{\pi'}) - f(s_\pi,t_{\pi'})\}\, dt\right] ds.$$

The uniform continuity of f and the estimates

$$\begin{aligned}\sup_{s\in[a,b]} \|\epsilon_{\pi'}(s)\| &\le \sup_{s\in[a,b]} \int_c^d \|f(s,t) - f(s,t_{\pi'})\|\, dt \\ &\le (d-c) \sup\{\|f(s,t) - f(s,t_{\pi'})\| : (s,t) \in Q\}\end{aligned}$$

and

$$\begin{aligned}\|\delta_{\pi,\pi'}\| &\le \int_a^b \left[\int_c^d \|f(s,t_{\pi'}) - f(s_\pi,t_{\pi'})\|\, dt\right] ds \\ &\le (b-a)(d-c) \sup\{\|f(s,t) - f(s,t_{\pi'})\| : (s,t) \in Q\}\end{aligned}$$

allow us to conclude that

$$\int_a^b \left[\int_c^d f(s,t)dt\right] ds - \int_a^b \left[\int_c^d f(s_\pi,t_{\pi'})dt\right] ds \to 0 \text{ as } |\pi| + |\pi'| \to 0.$$

By symmetry (or an analogous argument),

$$\int_c^d \left[\int_a^b f(s,t)ds\right] dt - \int_c^d \left[\int_a^b f(s_\pi,t_{\pi'})ds\right] dt \to 0 \text{ as } |\pi| + |\pi'| \to 0.$$

This completes the proof since

$$\int_a^b \left[\int_c^d f(s_\pi, t_{\pi'}) dt \right] ds = \sum_{0 \le i < m, 0 \le j < n} f(s_i, t_j)(s_{i+1} - s_i)(t_{j+1} - t_j)$$
$$= \int_c^d \left[\int_a^b f(s_\pi, t_{\pi'}) ds \right] dt.$$

■

4.1. **The Fundamental Theorem of Calculus.** Our next goal is to show that our Riemann integral interacts well with differentiation, namely the fundamental theorem of calculus holds. Before doing this we will need a couple of basic definitions and results.

Definition 4.6. Let $(a, b) \subset \mathbb{R}$. A function $f : (a, b) \to X$ is differentiable at $t \in (a, b)$ iff $L := \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$ exists in X. The limit L, if it exists, will be denoted by $\dot{f}(t)$ or $\frac{df}{dt}(t)$. We also say that $f \in C^1((a, b) \to X)$ if f is differentiable at all points $t \in (a, b)$ and $\dot{f} \in C((a, b) \to X)$.

Proposition 4.7. *Suppose that $f : [a, b] \to X$ is a continuous function such that $\dot{f}(t)$ exists and is equal to zero for $t \in (a, b)$. Then f is constant.*

Proof. Let $\epsilon > 0$ and $\alpha \in (a, b)$ be given. (We will later let $\epsilon \downarrow 0$ and $\alpha \downarrow a$.) By the definition of the derivative, for all $\tau \in (a, b)$ there exists $\delta_\tau > 0$ such that

$$\|f(t) - f(\tau)\| = \left\| f(t) - f(\tau) - \dot{f}(\tau)(t - \tau) \right\| \le \epsilon |t - \tau| \text{ if } |t - \tau| < \delta_\tau. \tag{4.7}$$

Let

$$A = \{t \in [\alpha, b] : \|f(t) - f(\alpha)\| \le \epsilon(t - \alpha)\} \tag{4.8}$$

and t_0 be the least upper bound for A. Eq. (4.7) with $\tau = \alpha$ shows $t_0 > \alpha$ and a simple continuity argument shows $t_0 \in A$, i.e.

$$\|f(t_0) - f(\alpha)\| \le \epsilon(t_0 - \alpha) \tag{4.9}$$

For the sake of contradiction, suppose that $t_0 < b$. By Eqs. (4.7) and (4.9),

$$\|f(t) - f(\alpha)\| \le \|f(t) - f(t_0)\| + \|f(t_0) - f(\alpha)\| \le \epsilon(t_0 - \alpha) + \epsilon(t - t_0) = \epsilon(t - \alpha)$$

for $0 \le t - t_0 < \delta_{t_0}$ which violates the definition of t_0 being an upper bound. Thus we have shown Eq. (4.8) holds for all $t \in [\alpha, b]$. Since $\epsilon > 0$ and $\alpha > a$ were arbitrary we may conclude, using the continuity of f, that $\|f(t) - f(a)\| = 0$ for all $t \in [a, b]$. ■

Remark 4.8. The usual real variable proof of Proposition 4.7 makes use Rolle's theorem which in turn uses the extreme value theorem. This latter theorem is not available to vector valued functions. However with the aid of the Hahn Banach Theorem 18.16 and Lemma 4.4, it is possible to reduce the proof of Proposition 4.7 and the proof of the Fundamental Theorem of Calculus 4.9 to the real valued case, see Exercise 18.12.

Theorem 4.9 (Fundamental Theorem of Calculus)**.** *Suppose that $f \in C([a, b], X)$, Then*

(1) $\frac{d}{dt} \int_a^t f(\tau)\, d\tau = f(t)$ *for all* $t \in (a, b)$.

(2) *Now assume that $F \in C([a,b],X)$, F is continuously differentiable on (a,b), and $\dot{F}$ extends to a continuous function on $[a,b]$ which is still denoted by $\dot{F}$. Then*

$$\int_a^b \dot{F}(t)\,dt = F(b) - F(a).$$

Proof. Let $h > 0$ be a small number and consider

$$\begin{aligned}\|\int_a^{t+h} f(\tau)d\tau - \int_a^t f(\tau)d\tau - f(t)h\| &= \|\int_t^{t+h} (f(\tau) - f(t))\,d\tau\| \\ &\le \int_t^{t+h} \|(f(\tau) - f(t))\|\,d\tau \\ &\le h\epsilon(h),\end{aligned}$$

where $\epsilon(h) \equiv \max_{\tau\in[t,t+h]} \|(f(\tau) - f(t))\|$. Combining this with a similar computation when $h < 0$ shows, for all $h \in \mathbb{R}$ sufficiently small, that

$$\|\int_a^{t+h} f(\tau)d\tau - \int_a^t f(\tau)d\tau - f(t)h\| \le |h|\epsilon(h),$$

where now $\epsilon(h) \equiv \max_{\tau\in[t-|h|,t+|h|]} \|(f(\tau)-f(t))\|$. By continuity of f at t, $\epsilon(h) \to 0$ and hence $\frac{d}{dt}\int_a^t f(\tau)\,d\tau$ exists and is equal to $f(t)$.

For the second item, set $G(t) \equiv \int_a^t \dot{F}(\tau)\,d\tau - F(t)$. Then G is continuous by Lemma 4.4 and $\dot{G}(t) = 0$ for all $t \in (a,b)$ by item 1. An application of Proposition 4.7 shows G is a constant and in particular $G(b) = G(a)$, i.e. $\int_a^b \dot{F}(\tau)\,d\tau - F(b) = -F(a)$. ■

Corollary 4.10 (Mean Value Inequality)**.** *Suppose that $f : [a,b] \to X$ is a continuous function such that $\dot{f}(t)$ exists for $t \in (a,b)$ and $\dot{f}$ extends to a continuous function on $[a,b]$. Then*

$$\|f(b) - f(a)\| \le \int_a^b \|\dot{f}(t)\|dt \le (b-a)\cdot \left\|\dot{f}\right\|_\infty. \tag{4.10}$$

Proof. By the fundamental theorem of calculus, $f(b) - f(a) = \int_a^b \dot{f}(t)dt$ and then by Lemma 4.4,

$$\|f(b) - f(a)\| = \left\|\int_a^b \dot{f}(t)dt\right\| \le \int_a^b \|\dot{f}(t)\|dt \le \int_a^b \left\|\dot{f}\right\|_\infty dt = (b-a)\cdot\left\|\dot{f}\right\|_\infty.$$

■

Proposition 4.11 (Equality of Mixed Partial Derivatives)**.** *Let $Q = (a,b)\times(c,d)$ be an open rectangle in $\mathbb{R}^2$ and $f \in C(Q,X)$. Assume that $\frac{\partial}{\partial t}f(s,t)$, $\frac{\partial}{\partial s}f(s,t)$ and $\frac{\partial}{\partial t}\frac{\partial}{\partial s}f(s,t)$ exists and are continuous for $(s,t) \in Q$, then $\frac{\partial}{\partial s}\frac{\partial}{\partial t}f(s,t)$ exists for $(s,t) \in Q$ and*

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}f(s,t) = \frac{\partial}{\partial t}\frac{\partial}{\partial s}f(s,t) \text{ for } (s,t) \in Q. \tag{4.11}$$

Proof. Fix $(s_0,t_0)\in Q$. By two applications of Theorem 4.9,

$$\begin{aligned} f(s,t) &= f(s_{t_0},t)+\int_{s_0}^{s}\frac{\partial}{\partial\sigma}f(\sigma,t)d\sigma \\ &= f(s_0,t)+\int_{s_0}^{s}\frac{\partial}{\partial\sigma}f(\sigma,t_0)d\sigma+\int_{s_0}^{s}d\sigma\int_{t_0}^{t}d\tau\frac{\partial}{\partial\tau}\frac{\partial}{\partial\sigma}f(\sigma,\tau) \end{aligned} \tag{4.12}$$

and then by Fubini's Theorem 4.5 we learn

$$f(s,t)=f(s_0,t)+\int_{s_0}^{s}\frac{\partial}{\partial\sigma}f(\sigma,t_0)d\sigma+\int_{t_0}^{t}d\tau\int_{s_0}^{s}d\sigma\frac{\partial}{\partial\tau}\frac{\partial}{\partial\sigma}f(\sigma,\tau).$$

Differentiating this equation in t and then in s (again using two more applications of Theorem 4.9) shows Eq. (4.11) holds. ■

4.2. **Exercises.**

Exercise 4.3. Let $\ell^\infty([a,b],X)\equiv\{f:[a,b]\to X:\|f\|_\infty\equiv\sup_{t\in[a,b]}\|f(t)\|<\infty\}$. Show that $(\ell^\infty([a,b],X),\|\cdot\|_\infty)$ is a complete Banach space.

Exercise 4.4. Prove Lemma 4.4.

Exercise 4.5. Using Lemma 4.4, show $f=(f_1,\dots,f_n)\in\bar{\mathcal{S}}([a,b],\mathbb{R}^n)$ iff $f_i\in\bar{\mathcal{S}}([a,b],\mathbb{R})$ for $i=1,2,\dots,n$ and

$$\int_a^b f(t)dt=\left(\int_a^b f_1(t)dt,\dots,\int_a^b f_n(t)dt\right).$$

Exercise 4.6. Give another proof of Proposition 4.11 which does not use Fubini's Theorem 4.5 as follows.

(1) By a simple translation argument we may assume $(0,0)\in Q$ and we are trying to prove Eq. (4.11) holds at $(s,t)=(0,0)$.
(2) Let $h(s,t):=\frac{\partial}{\partial t}\frac{\partial}{\partial s}f(s,t)$ and

$$G(s,t):=\int_0^s d\sigma\int_0^t d\tau h(\sigma,\tau)$$

so that Eq. (4.12) states

$$f(s,t)=f(0,t)+\int_0^s\frac{\partial}{\partial\sigma}f(\sigma,t_0)d\sigma+G(s,t)$$

and differentiating this equation at $t=0$ shows

$$\frac{\partial}{\partial t}f(s,0)=\frac{\partial}{\partial t}f(0,0)+\frac{\partial}{\partial t}G(s,0). \tag{4.13}$$

Now show using the definition of the derivative that

$$\frac{\partial}{\partial t}G(s,0)=\int_0^s d\sigma h(\sigma,0). \tag{4.14}$$

Hint: Consider

$$G(s,t)-t\int_0^s d\sigma h(\sigma,0)=\int_0^s d\sigma\int_0^t d\tau\left[h(\sigma,\tau)-h(\sigma,0)\right].$$

(3) Now differentiate Eq. (4.13) in s using Theorem 4.9 to finish the proof.

Exercise 4.7. Give another proof of Eq. (4.6) in Theorem 4.5 based on Proposition 4.11. To do this let $t_0 \in (c,d)$ and $s_0 \in (a,b)$ and define

$$G(s,t) := \int_{t_0}^{t} d\tau \int_{s_0}^{s} d\sigma f(\sigma,\tau)$$

Show G satisfies the hypothesis of Proposition 4.11 which combined with two applications of the fundamental theorem of calculus implies

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}G(s,t) = \frac{\partial}{\partial s}\frac{\partial}{\partial t}G(s,t) = f(s,t).$$

Use two more applications of the fundamental theorem of calculus along with the observation that $G = 0$ if $t = t_0$ or $s = s_0$ to conclude

$$G(s,t) = \int_{s_0}^{s} d\sigma \int_{t_0}^{t} d\tau \frac{\partial}{\partial \tau}\frac{\partial}{\partial \sigma}G(\sigma,\tau) = \int_{s_0}^{s} d\sigma \int_{t_0}^{t} d\tau \frac{\partial}{\partial \tau} f(\sigma,\tau). \tag{4.15}$$

Finally let $s = b$ and $t = d$ in Eq. (4.15) and then let $s_0 \downarrow a$ and $t_0 \downarrow c$ to prove Eq. (4.6).