PDE LECTURE NOTES, MATH 237A-B

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ABSTRACT. These are lecture notes from Math 237A-B. See
C:\driverdat\Bruce\CLASSFIL\257AF94\course.tex
for notes on contraction semi-groups. Need to add examples of using the Hille Yoshida theorem in PDE.
See
See "C:\driverdat\Bruce\DATA\MATHFILE\qft-notes\co-area.tex" for co-area material and applications to Sobolev inequalities.

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1. Some Examples

Example 1.1 (Traffic Equation)**.** Consider cars travelling on a straight road, i.e. $\mathbb{R}$ and let $u(t,x)$ denote the density of cars on the road at time t and space x and $v(t,x)$ be the velocity of the cars at (t,x). Then for $J=[a,b]\subset\mathbb{R}$, $N_J(t):=\int_a^b u(t,x)dx$ is the number of cars in the set J at time t. We must have

$$\begin{aligned}\int_a^b \dot{u}(t,x)dx = \dot{N}_J(t) &= u(t,a)v(t,a)-u(t,b)v(t,b)\\ &= -\int_a^b \frac{\partial}{\partial x}\left[u(t,x)v(t,x)\right]dx.\end{aligned}$$

Since this holds for all intervals $[a,b]$, we must have

$$\dot{u}(t,x)dx = -\frac{\partial}{\partial x}\left[u(t,x)v(t,x)\right].$$

To make life more interesting, we may imagine that $v(t,x)=-F(u(t,x),u_x(t,x))$, in which case we get an equation of the form

$$\frac{\partial}{\partial t}u = \frac{\partial}{\partial x}G(u,u_x) \text{ where } G(u,u_x) = -u(t,x)F(u(t,x),u_x(t,x)).$$

A simple model might be that there is a constant maximum speed, v_m and maximum density u_m, and the traffic interpolates linearly between 0 (when $u=u_m$) to v_m when $(u=0)$, i.e. $v=v_m(1-u/u_m)$ in which case we get

$$\frac{\partial}{\partial t}u = -v_m\frac{\partial}{\partial x}\left(u(1-u/u_m)\right).$$

Example 1.2 (Burger's Equation)**.** Suppose we have a stream of particles travelling on $\mathbb{R}$, each of which has its own constant velocity and let $u(t,x)$ denote the velocity of the particle at x at time t. Let $x(t)$ denote the trajectory of the particle which is at x_0 at time t_0. We have $C=\dot{x}(t)=u(t,x(t))$. Differentiating this equation in t at $t=t_0$ implies

$$0=\left[u_t(t,x(t))+u_x(t,x(t))\dot{x}(t)\right]|_{t=t_0} = u_t(t_0,x_0)+u_x(t_0,x_0)u(t_0,x_0)$$

which leads to Burger's equation

$$0=u_t+u\,u_x.$$

Example 1.3 (Minimal surface Equation)**.** (Review Dominated convergence theorem and differentiation under the integral sign.) Let $D\subset\mathbb{R}^2$ be a bounded region with reasonable boundary, $u_0:\partial D\to\mathbb{R}$ be a given function. We wish to find the function $u:D\to\mathbb{R}$ such that $u=u_0$ on ∂D and the graph of u, $\Gamma(u)$ has least area. Recall that the area of $\Gamma(u)$ is given by

$$A(u)=Area(\Gamma(u))=\int_D\sqrt{1+|\nabla u|^2}dx.$$

Assuming u is a minimizer, let $v \in C^1(D)$ such that $v = 0$ on ∂D, then

$$\begin{aligned}
0 &= \frac{d}{ds}|_0 A(u+sv) = \frac{d}{ds}|_0 \int_D \sqrt{1+|\nabla(u+sv)|^2}dx \\
&= \int_D \frac{d}{ds}|_0 \sqrt{1+|\nabla(u+sv)|^2}dx \\
&= \int_D \frac{1}{\sqrt{1+|\nabla u|^2}} \nabla u \cdot \nabla v \ dx \\
&= -\int_D \nabla \cdot \left(\frac{1}{\sqrt{1+|\nabla u|^2}} \nabla u \right) v \ dx
\end{aligned}$$

from which it follows that

$$\nabla \cdot \left(\frac{1}{\sqrt{1+|\nabla u|^2}} \nabla u \right) = 0.$$

Example 1.4 (Heat or Diffusion Equation)**.** Suppose that $\Omega \subset \mathbb{R}^n$ is a region of space filled with a material, $\rho(x)$ is the density of the material at $x \in \Omega$ and $c(x)$ is the heat capacity. Let $u(x,t)$ denote the temperature at time $t \in [0,\infty)$ at the spatial point $x \in \Omega$. Now suppose that $B \subset \mathbb{R}^n$ is a "little" volume in $\mathbb{R}^n$, ∂B is the boundary of B, and $E_B(t)$ is the heat energy contained in the volume B at time t.

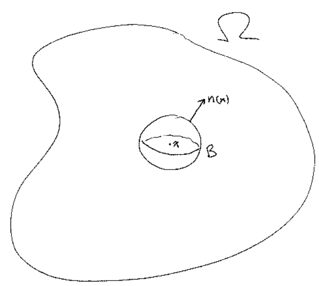


FIGURE 1. A test volume B in Ω.

Then

$$E_B(t) = \int_B \rho(x)c(x)u(t,x)dx.$$

So on one hand,

$$\dot{E}_B(t) = \int_B \rho(x)c(x)\dot{u}(t,x)dx \tag{1.1}$$

while on the other hand,

$$\dot{E}_B(t) = \int_{\partial B} \langle G(x)\nabla u(t,x), n(x)\rangle d\sigma(x), \tag{1.2}$$

where $G(x)$ is a $n \times n$–positive definite matrix representing the conduction properties of the material, $n(x)$ is the outward pointing normal to B at $x \in \partial B$, and $d\sigma$ denotes surface measure on ∂B. (We are using $\langle \cdot, \cdot \rangle$ to denote the standard dot product on $\mathbb{R}^n$.)

In order to see that we have the sign correct in (1.2), suppose that $x \in \partial B$ and $\nabla u(x) \cdot n(x) > 0$, then the temperature for points near x outside of B are hotter than those points near x inside of B and hence contribute to a increase in the heat energy inside of B. (If we get the wrong sign, then the resulting equation will have the property that heat flows from cold to hot!)

Comparing Eqs. (1.1) to (1.2) after an application of the divergence theorem shows that

$$\int_B \rho(x)c(x)\dot{u}(t,x)dx = \int_B \nabla \cdot (G(\cdot)\nabla u(t,\cdot))(x)\,dx. \tag{1.3}$$

Since this holds for all volumes $B \subset \Omega$, we conclude that the temperature functions should satisfy the following partial differential equation.

$$\rho(x)c(x)\dot{u}(t,x) = \nabla \cdot (G(\cdot)\nabla u(t,\cdot))(x)\,. \tag{1.4}$$

or equivalently that

$$\dot{u}(t,x) = \frac{1}{\rho(x)c(x)}\nabla \cdot (G(x)\nabla u(t,x)). \tag{1.5}$$

Setting $g^{ij}(x) := G_{ij}(x)/(\rho(x)c(x))$ and

$$z^j(x) := \sum_{i=1}^n \partial(G_{ij}(x)/(\rho(x)c(x)))/\partial x^i$$

the above equation may be written as:

$$\dot{u}(t,x) = Lu(t,x), \tag{1.6}$$

where

$$(Lf)(x) = \sum_{i,j} g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} f(x) + \sum_j z^j(x)\frac{\partial}{\partial x^j} f(x). \tag{1.7}$$

The operator L is a prototypical example of a second order "elliptic" differential operator.

Example 1.5 (Laplace and Poisson Equations)**.** Laplaces Equation is of the form $Lu = 0$ and solutions may represent the steady state temperature distribution for the heat equation. Equations like $\Delta u = -\rho$ appear in electrostatics for example, where u is the electric potential and ρ is the charge distribution.

Example 1.6 (Shrodinger Equation and Quantum Mechanics)**.**

$$i\frac{\partial}{\partial t}\psi(t,x) = -\frac{\Delta}{2}\psi(t,x) + V(x)\psi(t,x) \text{ with } \|\psi(\cdot,0)\|_2 = 1.$$

Interpretation,

$$\int_A |\psi(t,x)|^2\,dt = \text{ the probability of finding the particle in } A \text{ at time } t.$$

(Notice similarities to the heat equation.)

Example 1.7 (Wave Equation)**.** Suppose that we have a stretched string supported at $x = 0$ and $x = L$ and $y = 0$. Suppose that the string only undergoes vertical motion (pretty bad assumption). Let $u(t,x)$ and $T(t,x)$ denote the height and tension of the string at (t,x), $\rho_0(x)$ denote the density in equilibrium and T_0 be the equilibrium string tension. Let $J = [x, x + \Delta x] \subset [0, L]$, then

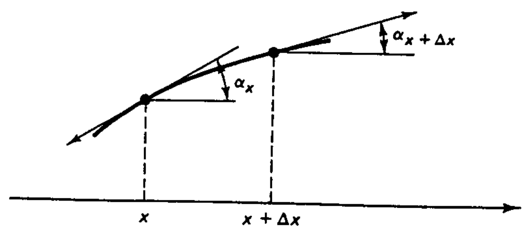


FIGURE 2. A piece of displace string

$$M_J(t) := \int_J u_t(t,x)\rho_0(x)dx$$

is the momentum of the piece of string above J. (Notice that $\rho_0(x)dx$ is the weight of the string above x.) Newton's equations state

$$\frac{dM_J(t)}{dt} = \int_J u_{t\ t}(t,x)\rho_0(x)dx = \text{Force on String.}$$

Since the string is to only undergo vertical motion we require

$$T(t, x+\Delta x)\cos(\alpha_{x+\Delta x}) - T(t,x)\cos(\alpha_x) = 0$$

for all Δx and therefore that $T(t,x)\cos(\alpha_x) = T_0$, i.e.

$$T(t,x) = \frac{T_0}{\cos(\alpha_x)}.$$

The vertical tension component is given by

$$\begin{aligned} T(t,x+\Delta x)\sin(\alpha_{x+\Delta x}) - T(t,x)\sin(\alpha_x) &= T_0\left[\frac{\sin(\alpha_{x+\Delta x})}{\sin(\alpha_{x+\Delta x})} - \frac{\sin(\alpha_x)}{\cos(\alpha_x)}\right] \\ &= T_0\left[u_x(t,x+\Delta x) - u_x(t,x)\right]. \end{aligned}$$

Finally there may be a component due to gravity and air resistance, say

$$\text{gravity} = -\int_J \rho_0(x)dx \text{ and resistance} = -\int_J k(x)u_t(t,x)dx.$$

So Newton's equations become

$$\begin{aligned} \int_x^{x+\Delta x} u_{tt}(t,x)\rho_0(x)dx &= T_0\left[u_x(t,x+\Delta x) - u_x(t,x)\right] \\ &\quad - \int_x^{x+\Delta x}\rho_0(x)dx - \int_x^{x+\Delta x} k(x)u_t(t,x)dx \end{aligned}$$

and differentiating this in Δx gives

$$u_{t\ t}(t,x)\rho_0(x) = u_{x\ x}(t,x) - \rho_0(x) - k(x)u_t(t,x)$$

or equivalently that

$$u_{t\ t}(t,x) = \frac{1}{\rho_0(x)}u_{x\ x}(t,x) - 1 - \frac{k(x)}{\rho_0(x)}u_t(t,x). \tag{1.8}$$

Example 1.8 (Maxwell Equations in Free Space)**.**

$$\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B}$$
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$
$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0.$$

Notice that

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\nabla \times \mathbf{E}) = \Delta \mathbf{E} - \nabla (\nabla \cdot \mathbf{E}) = \Delta \mathbf{E}$$

and similarly, $\frac{\partial^2 \mathbf{B}}{\partial t^2} = \Delta \mathbf{B}$ so that all the components of the electromagnetic fields satisfy the wave equation.

Example 1.9 (Navier – Stokes)**.** Here $u(t,x)$ denotes the velocity of a fluid ad (t,x), $p(t,x)$ is the pressure. The Navier – Stokes equations state,

$$\frac{\partial u}{\partial t} + \partial_u u = \nu \Delta u - \nabla p + f \text{ with } u(0,x) = u_0(x) \tag{1.9}$$

$$\nabla \cdot u = 0 \text{ (incompressibility)} \tag{1.10}$$

where f are the components of a given external force and u_0 is a given divergence free vector field, ν is the viscosity constant. The Euler equations are found by taking $\nu = 0$. Equation (1.9) is Newton's law of motion again, $F = ma$. See http://www.claymath.org for more information on this Million Dollar problem.

1.1. **Some More Geometric Examples.**

Example 1.10 (Einstein Equations)**.** Einstein's equations from general relativity are

$$\mathrm{Ric}_g - \frac{1}{2} g S_g = T$$

where T is the stress energy tensor.

Example 1.11 (Yamabe Problem)**.** Does there exists a metric $g_1 = u^{4/(n-2)} g_0$ in the conformal class of g_0 so that g_1 has constant scalar curvature. This is equivalent to solving

$$-\gamma \Delta_{g_0} u + S_{g_0} u = k u^{\alpha}$$

where $\gamma = 4\frac{n-1}{n-2}$, $\alpha = \frac{n+2}{n-2}$, k is a constant and S_{g_0} is the scalar curvature of g_0.

Example 1.12 (Ricci Flow)**.** Hamilton introduced the Ricci – flow,

$$\frac{\partial g}{\partial t} = \mathrm{Ric}_g,$$

as another method to create "good" metrics on manifolds. This is a possible solution to the 3 dimensional Poincaré conjecture, again go to the Clay math web site for this problem.

2. First Order Quasi-Linear Scalar PDE

2.1. **Linear Evolution Equations.** Consider the first order partial differential equation

$$\partial_t u(t,x) = \sum_{i=1}^n a_i(x)\partial_i u(t,x) \text{ with } u(0,x) = f(x) \tag{2.1}$$

where $x \in \mathbb{R}^n$ and $a_i(x)$ are smooth functions on $\mathbb{R}^n$. Let $A(x) = (a_1(x), \ldots, a_n(x))$ and for $u \in C^1(\mathbb{R}^n, \mathbb{C})$, let

$$\tilde{A}u(x) := \frac{d}{dt}|_0 u(x + tZ(x)) = \nabla u(x) \cdot A(x) = \sum_{i=1}^n a_i(x)\partial_i u(x),$$

i.e. $\tilde{A}(x)$ is the first order differential operator, $\tilde{A}(x) = \sum_{i=1}^n a_i(x)\partial_i$. With this notation we may write Eq. (2.1) as

$$\partial_t u = \tilde{A}u \text{ with } u(0,\cdot) = f. \tag{2.2}$$

The following lemma contains the key observation needed to solve Eq. (2.2).

Lemma 2.1. *Let A and $\tilde{A}$ be as above and $f \in C^1(\mathbb{R}^n, \mathbb{R})$, then*

$$\frac{d}{dt} f \circ e^{tA}(x) = \tilde{A}f \circ e^{tA}(x) = \tilde{A}\left(f \circ e^{tA}\right)(x). \tag{2.3}$$

Proof. By definition,

$$\frac{d}{dt} e^{tA}(x) = A(e^{tA}(x))$$

and so by the chain rule

$$\frac{d}{dt} f \circ e^{tA}(x) = \nabla f(e^{tA}(x)) \cdot A(e^{tA}(x)) = \tilde{A}f(e^{tA}(x))$$

which proves the first Equality in Eq. (2.3). For the second we will need to use the following two facts: 1) $e^{(t+s)A} = e^{tA} \circ e^{sZ}$ and 2) $e^{tA}(x)$ is smooth in x. Assuming this we find

$$\frac{d}{dt} f \circ e^{tA}(x) = \frac{d}{ds}|_0 f \circ e^{(t+s)A}(x) = \frac{d}{ds}|_0 \left[f \circ e^{tA} \circ e^{sZ}(x)\right] = \tilde{A}\left(f \circ e^{tA}\right)(x)$$

which is the second equality in Eq. (2.3). ■

Theorem 2.2. *The function $u \in C^1(\mathcal{D}(A), \mathbb{R})$ defined by*

$$u(t,x) := f(e^{tA}(x)) \tag{2.4}$$

solves Eq. (2.2). Moreover this is the unique function defined on $\mathcal{D}(A)$ which solves Eq. (2.3).

Proof. Suppose that $u \in C^1(\mathcal{D}(A), \mathbb{R})$ solves Eq. (2.2), then

$$\frac{d}{dt} u(t, e^{-tA}(x)) = u_t(t, e^{-tA}(x)) - \tilde{A}u(t, e^{-tA}(x)) = 0$$

and hence

$$u(t, e^{-tA}(x)) = u(0,x) = f(x).$$

Let $(t_0, x_0) \in \mathcal{D}(A)$ and apply the previous computations with $x = e^{tA}(x_0)$ to find $u(t_0, x) = f(e^{tA}(x_0))$. This proves the uniqueness assertion. The verification that u defined in Eq. (2.4) solves Eq. (2.2) is simply the second equality in Eq. (2.3). ■

Notation 2.3. Let $e^{t\tilde{A}}f(x) = u(t,x)$ where u solves Eq. (2.2), i.e.

$$e^{t\tilde{A}}f(x) = f(e^{tA}(x)).$$

The differential operator $\tilde{A} : C^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{C}(\mathbb{R}^n, \mathbb{R})$ is no longer bounded so it is not possible in general to conclude

$$e^{t\tilde{A}}f = \sum_{n=0}^{\infty} \frac{t^n}{n!}\tilde{A}^n f. \tag{2.5}$$

Indeed, to make sense out of the right side of Eq. (2.5) we must know f is infinitely differentiable and that the sum is convergent. This is typically not the case. because if f is only C^1. However there is still some truth to Eq. (2.5). For example if $f \in C^k(\mathbb{R}^n, \mathbb{R})$, then by Taylor's theorem with remainder,

$$e^{t\tilde{A}}f - \sum_{n=0}^{k} \frac{t^n}{n!}\tilde{A}^n f = o(t^k)$$

by which I mean, for any $x \in \mathbb{R}^n$,

$$t^{-k}\left[e^{t\tilde{A}}f(x) - \sum_{n=0}^{k} \frac{t^n}{n!}\tilde{A}^n f(x)\right] \to 0 \text{ as } t \to 0.$$

Example 2.4. Suppose $n = 1$ and $A(x) = 1$, $\tilde{A}(x) = \partial_x$ then $e^{tA}(x) = x + t$ and hence

$$e^{t\partial_x}f(x) = f(x+t).$$

It is interesting to notice that

$$e^{t\partial_x}f(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f^{(n)}(x)$$

is simply the Taylor series expansion of $f(x+t)$ centered at x. This series converges to the correct answer (i.e. $f(x+t)$) iff f is "real analytic." For more details see the Cauchy – Kovaleski Theorem in Section 4.

Example 2.5. Suppose $n = 1$ and $A(x) = x^2$, $\tilde{A}(x) = x^2\partial_x$ then $e^{tA}(x) = \frac{x}{1-tx}$ on $\mathcal{D}(A) = \{(t,x) : 1 - tx > 0\}$ and hence $e^{t\tilde{A}}f(x) = f(\frac{x}{1-tx}) = u(t,x)$ on $\mathcal{D}(A)$, where

$$u_t = x^2 u_x. \tag{2.6}$$

It may or may not be possible to extend this solution, $u(t,x)$, to a C^1 solution on all $\mathbb{R}^2$. For example if $\lim_{x\to\infty} f(x)$ does not exist, then $\lim_{t\uparrow x} u(t,x)$ does not exist for any $x > 0$ and so u can not be the restriction of C^1 – function on $\mathbb{R}^2$. On the other hand if there are constants $c_\pm$ and $M > 0$ such that $f(x) = c_+$ for $x > M$ and $f(x) = c_-$ for $x < -M$, then we may extend u to all $\mathbb{R}^2$ by defining

$$u(t,x) = \begin{cases} c_+ & \text{if} \quad x > 0 \text{ and } t > 1/x \\ c_- & \text{if} \quad x < 0 \text{ and } t < 1/x. \end{cases}$$

It is interesting to notice that $x(t) = 1/t$ solves $\dot{x}(t) = -x^2(t) = -A(x(t))$, so any solution $u \in C^1(\mathbb{R}^2, \mathbb{R})$ to Eq. (2.6) satisfies $\frac{d}{dt}u(t, 1/t) = 0$, i.e. u must be constant on the curves $x = 1/t$ for $t > 0$ and $x = 1/t$ for $t < 0$. See Example 2.13 below for a more detailed study of Eq. (2.6).

Example 2.6. Suppose $n = 2$.

(1) If $A(x,y) = (-y,x)$, i.e. $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ then
$$e^{tA}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
and hence
$$e^{t\tilde{A}}f(x,y) = f\left(x\cos t - y\sin t, y\cos t + x\sin t\right).$$
(2) If $A(x,y) = (x,y)$, i.e. $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ then
$$e^{tA}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
and hence
$$e^{t\tilde{A}}f(x,y) = f\left(xe^t, ye^t\right).$$

Theorem 2.7. *Given $A \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $h \in C^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$.*

(1) *(Duhamel' s Principle) The unique solution $u \in C^1(\mathcal{D}(A), \mathbb{R})$ to*
$$u_t = \tilde{A}u + h \text{ with } u(0,\cdot) = f \tag{2.7}$$
is given by
$$u(t,\cdot) = e^{t\tilde{A}}f + \int_0^t e^{(t-\tau)\tilde{A}}h(\tau,\cdot)d\tau$$
or more explicitly,
$$u(t,x) := f(e^{tA}(x)) + \int_0^t h(\tau, e^{(t-\tau)A}(x))d\tau. \tag{2.8}$$
(2) *The unique solution $u \in C^1(\mathcal{D}(A), \mathbb{R})$ to*
$$u_t = \tilde{A}u + hu \text{ with } u(0,\cdot) = f \tag{2.9}$$
is given by
$$u(t,\cdot) = e^{\int_0^t h(\tau, e^{(t-\tau)A}(x))d\tau}f(e^{tA}(x)) \tag{2.10}$$
which we abbreviate as
$$e^{t\left(\tilde{A}+M_h\right)}f(x) = e^{\int_0^t h(\tau, e^{(t-\tau)A}(x))d\tau}f(e^{tA}(x)). \tag{2.11}$$

Proof. We will verify the uniqueness assertions, leaving the routine check the Eqs. (2.8) and (2.9) solve the desired PDE's to the reader. Assuming u solves Eq. (2.7), we find
$$\frac{d}{dt}\left[e^{-t\tilde{A}}u(t,\cdot)\right](x) = \frac{d}{dt}u(t, e^{-tA}(x)) = \left(u_t - \tilde{A}u\right)(t, e^{-tA}(x)) = h(t, e^{-tA}(x))$$
and therefore
$$\left[e^{-t\tilde{A}}u(t,\cdot)\right](x) = u(t, e^{-tA}(x)) = f(x) + \int_0^t h(\tau, e^{-\tau A}(x))d\tau$$
and so replacing x by $e^{tA}(x)$ in this equation implies
$$u(t,x) = f(e^{tA}(x)) + \int_0^t h(\tau, e^{(t-\tau)A}(x))d\tau.$$

Similarly if u solves Eq. (2.9), we find with $z(t) := \left[e^{-t\tilde{A}}u(t,\cdot)\right](x) = u(t, e^{-tA}(x))$ that

$$\begin{aligned}\dot{z}(t) &= \frac{d}{dt}u(t, e^{-tA}(x)) = \left(u_t - \tilde{A}u\right)(t, e^{-tA}(x)) \\ &= h(t, e^{-tA}(x))u(t, e^{-tA}(x)) = h(t, e^{-tA}(x))z(t).\end{aligned}$$

Solving this equation for $z(t)$ then implies

$$u(t, e^{-tA}(x)) = z(t) = e^{\int_0^t h(\tau, e^{-\tau A}(x))d\tau} z(0) = e^{\int_0^t h(\tau, e^{-\tau A}(x))d\tau} f(x).$$

Replacing x by $e^{tA}(x)$ in this equation implies

$$u(t,x) = e^{\int_0^t h(\tau, e^{(t-\tau)A}(x))d\tau} f(e^{tA}(x)).$$

■

Remark 2.8. It is interesting to observe the key point to getting the simple expression in Eq. (2.11) is the fact that

$$e^{t\tilde{A}}(fg) = (fg) \circ e^{tA} = \left(f \circ e^{tA}\right) \cdot \left(g \circ e^{tA}\right) = e^{t\tilde{A}}f \cdot e^{t\tilde{A}}g.$$

That is to say $e^{t\tilde{A}}$ is an algebra homomorphism on functions. This property does not happen for any other type of differential operator. Indeed, if L is some operator on functions such that $e^{tL}(fg) = e^{tL}f \cdot e^{tL}g$, then differentiating at $t = 0$ implies

$$L(fg) = Lf \cdot g + f \cdot Lg,$$

i.e. L satisfies the product rule. One learns in differential geometry that this property implies L must be a vector field.

Let us now use this result to find the solution to the wave equation

$$u_{tt} = u_{xx} \text{ with } u(0,\cdot) = f \text{ and } u_t(0,\cdot) = g. \tag{2.12}$$

To this end, let us notice the $u_{tt} = u_{xx}$ may be written as

$$\left(\partial_t - \partial_x\right)\left(\partial_t + \partial_x\right)u = 0$$

and therefore noting that

$$\left(\partial_t + \partial_x\right)u(t,x)|_{t=0} = g(x) + f'(x)$$

we have

$$\left(\partial_t + \partial_x\right)u(t,x) = e^{t\partial_x}\left(g + f'\right)(x) = \left(g + f'\right)(x+t).$$

The solution to this equation is then a consequence of Duhamel' s Principle which gives

$$\begin{aligned}u(t,x) &= e^{-t\partial_x}f(x) + \int_0^t e^{-(t-\tau)\partial_x}\left(g + f'\right)(x+\tau)d\tau \\ &= f(x-t) + \int_0^t \left(g + f'\right)(x + \tau - (t-\tau))d\tau \\ &= f(x-t) + \int_0^t \left(g + f'\right)(x + 2\tau - t)d\tau \\ &= f(x-t) + \int_0^t g(x + 2\tau - t)d\tau + \frac{1}{2}f(x + 2\tau - t)|_{\tau=0}^{\tau=t} \\ &= \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{-t}^t g(x+s)ds.\end{aligned}$$

The following theorem summarizes what we have proved.

Theorem 2.9. *If $f \in C^2(\mathbb{R},\mathbb{R})$ and $g \in C^1(\mathbb{R},\mathbb{R})$, then Eq. (2.12) has a unique solution given by*

$$u(t,x) = \frac{1}{2}\left[f(x+t)+f(x-t)\right] + \frac{1}{2}\int_{-t}^{t} g(x+s)ds. \tag{2.13}$$

Proof. We have already proved uniqueness above. The proof that u defined in Eq. (2.13) solves the wave equation is a routine computation. Perhaps the most instructive way to verify that u solves $u_{tt} = u_{xx}$ is to observe, letting $y = x+s$, that

$$\int_{-t}^{t} g(x+s)ds = \int_{x-t}^{x+t} g(y)dy = \int_{0}^{x+t} g(y)dy + \int_{x-t}^{0} g(y)dy = \int_{0}^{x+t} g(y)dy - \int_{0}^{x-t} g(y)dy.$$

From this observation it follows that

$$u(t,x) = F(x+t) + G(x-t)$$

where

$$F(x) = \frac{1}{2}\left(f(x) + \int_0^x g(y)dy\right) \text{ and } G(x) = \frac{1}{2}\left(f(x) - \int_0^x g(y)dy\right).$$

Now clearly F and G are C^2 – functions and

$$\left(\partial_t - \partial_x\right)F(x+t) = 0 \text{ and } \left(\partial_t + \partial_x\right)G(x-t) = 0$$

so that

$$\left(\partial_t^2 - \partial_x^2\right)u(t,x) = \left(\partial_t - \partial_x\right)\left(\partial_t + \partial_x\right)\left(F(x+t) + G(x-t)\right) = 0.$$

■

Now let us formally apply Exercise 2.16 to the wave equation $u_{tt} = u_{xx}$, in which case we should let $A^2 = -\partial_x^2$, and hence $A = \sqrt{-\partial_x^2}$. Evidently we should take

$$\cos\left(t\sqrt{-\partial_x^2}\right)f(x) = \frac{1}{2}\left[f(x+t)+f(x-t)\right] \text{ and}$$
$$\frac{\sin\left(t\sqrt{-\partial_x^2}\right)}{\sqrt{-\partial_x^2}}g(x) = \frac{1}{2}\int_{-t}^{t} g(x+s)ds = \frac{1}{2}\int_{x-t}^{x+t} g(y)dy$$

Thus with these definitions, we can try to solve the equation

$$u_{tt} = u_{xx} + h \text{ with } u(0,\cdot) = f \text{ and } u_t(0,\cdot) = g \tag{2.14}$$

by a formal application of Exercise 2.15. According to Eq. (2.72) we should have

$$u(t,\cdot) = \cos(tA)f + \frac{\sin(tA)}{A}g + \int_0^t \frac{\sin((t-\tau)A)}{A}h(\tau,\cdot)d\tau,$$

i.e.

$$u(t,x) = \frac{1}{2}\left[f(x+t)+f(x-t)\right] + \frac{1}{2}\int_{-t}^{t} g(x+s)ds + \frac{1}{2}\int_0^t d\tau \int_{x-t+\tau}^{x+t-\tau} dy\ h(\tau,y). \tag{2.15}$$

An alternative way to get to this equation is to rewrite Eq. (2.14) in first order (in time) form by introducing $v = u_t$ to find

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ h \end{pmatrix} \text{ with}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \text{ at } t = 0 \tag{2.16}$$

where

$$A := \begin{pmatrix} 0 & 1 \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}.$$

A restatement of Theorem 2.9 is simply

$$e^{tA}\begin{pmatrix} f \\ g \end{pmatrix}(x) = \begin{pmatrix} u(t,x) \\ u_t(t,x) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} f(x+t) + f(x-t) + \int_{-t}^{t} g(x+s)ds \\ f'(x+t) - f'(x-t) + g(x+t) + g(x-t) \end{pmatrix}.$$

According to Du hamel's principle the solution to Eq. (2.16) is given by

$$\begin{pmatrix} u(t,\cdot) \\ u_t(t,\cdot) \end{pmatrix} = e^{tA}\begin{pmatrix} f \\ g \end{pmatrix} + \int_0^t e^{(t-\tau)A}\begin{pmatrix} 0 \\ h(\tau,\cdot) \end{pmatrix} d\tau.$$

The first component of the last term is given by

$$\frac{1}{2}\int_0^t \left[\int_{\tau-t}^{t-\tau} h(\tau, x+s)ds\right] d\tau = \frac{1}{2}\int_0^t \left[\int_{x-t+\tau}^{x+t-\tau} h(\tau, y)dy\right] d\tau$$

which reproduces Eq. (2.15).

To check Eq. (2.15), it suffices to assume $f = g = 0$ so that

$$u(t,x) = \frac{1}{2}\int_0^t d\tau \int_{x-t+\tau}^{x+t-\tau} dy\ h(\tau, y).$$

Now

$$\begin{aligned}
u_t &= \frac{1}{2}\int_0^t \left[h(\tau, x+t-\tau) + h(\tau, x-t+\tau)\right] d\tau, \\
u_{tt} &= \frac{1}{2}\int_0^t \left[h_x(\tau, x+t-\tau) - h_x(\tau, x-t+\tau)\right] d\tau + h(t,x) \\
u_x(t,x) &= \frac{1}{2}\int_0^t d\tau \left[h(\tau, x+t-\tau) - h(\tau, x-t+\tau)\right] \text{ and} \\
u_{xx}(t,x) &= \frac{1}{2}\int_0^t d\tau \left[h_x(\tau, x+t-\tau) - h_x(\tau, x-t+\tau)\right]
\end{aligned}$$

so that $u_{tt} - u_{xx} = h$ and $u(0,x) = u_t(0,x) = 0$. We have proved the following theorem.

Theorem 2.10. *If $f \in C^2(\mathbb{R},\mathbb{R})$ and $g \in C^1(\mathbb{R},\mathbb{R})$, and $h \in C(\mathbb{R}^2,\mathbb{R})$ such that h_x exists and $h_x \in C(\mathbb{R}^2,\mathbb{R})$, then Eq. (2.14) has a unique solution $u(t,x)$ given by Eq. (2.14).*

Proof. The only thing left to prove is the uniqueness assertion. For this suppose that v is another solution, then $(u - v)$ solves the wave equation (2.12) with $f = g = 0$ and hence by the uniqueness assertion in Theorem 2.9, $u - v \equiv 0$. ■

2.1.1. *A 1-dimensional wave equation with non-constant coefficients.*

Theorem 2.11. *Let $c(x) > 0$ be a smooth function and $\tilde{C} = c(x)\partial_x$ and $f, g \in C^2(\mathbb{R})$. Then the unique solution to the wave equation*

$$u_{tt} = \tilde{C}^2 u = c u_{x\ x} + c' u_x \text{ with } u(0,\cdot) = f \text{ and } u_t(0,\cdot) = g \tag{2.17}$$

is

$$u(t,x) = \frac{1}{2}\left[f(e^{-tC}(x)) + f(e^{tC}(x))\right] + \frac{1}{2}\int_{-t}^{t} g(e^{sC}(x))ds. \tag{2.18}$$

defined for $(t,x) \in \mathcal{D}(C) \cap \mathcal{D}(-C)$.

Proof. (Uniqueness) If u is a C^2 – solution of Eq. (2.17), then

$$\left(\partial_t - \tilde{C}\right)\left(\partial_t + \tilde{C}\right) u = 0$$

and

$$\left(\partial_t + \tilde{C}\right) u(t,x)|_{t=0} = g(x) + \tilde{C} f(x).$$

Therefore

$$\left(\partial_t + \tilde{C}\right) u(t,x) = e^{t\tilde{C}}\left(g + f'\right)(x) = \left(g + f'\right)(e^{tC}(x))$$

which has solution given by Duhamel' s Principle as

$$\begin{aligned}
u(t,x) &= e^{-t\grave{A}} f(x) + \int_0^t e^{-(t-\tau)\tilde{C}}\left(g + \tilde{C}f\right)(e^{\tau C}(x))d\tau \\
&= f(e^{-tC}(x)) + \int_0^t \left(g + \tilde{C}f\right)(e^{(2\tau - t)C}(x))d\tau \\
&= f(e^{-tC}(x)) + \frac{1}{2}\int_{-t}^{t}\left(g + \tilde{C}f\right)(e^{sC}(x))ds \\
&= f(e^{-tC}(x)) + \frac{1}{2}\int_{-t}^{t} g(e^{sC}(x))ds + \frac{1}{2}\int_{-t}^{t}\frac{d}{ds}f(e^{sC}(x))ds \\
&= \frac{1}{2}\left[f(e^{-tC}(x)) + f(e^{tC}(x))\right] + \frac{1}{2}\int_{-t}^{t} g(e^{sC}(x))ds.
\end{aligned}$$

(Existence.) Let $y = e^{sC}(x)$ so $dy = c(e^{sC}(x))ds = c(y)ds$ in the integral in Eq. (2.18), then

$$\begin{aligned}
\int_{-t}^{t} g(e^{sC}(x))ds &= \int_{e^{-tC}(x)}^{e^{tC}(x)} g(y)\frac{dy}{c(y)} = \int_0^{e^{tC}(x)} g(y)\frac{dy}{c(y)} + \int_{e^{-tC}(x)}^{0} g(y)\frac{dy}{c(y)} \\
&= \int_0^{e^{tC}(x)} g(y)\frac{dy}{c(y)} - \int_0^{e^{-tC}(x)} g(y)\frac{dy}{c(y)}.
\end{aligned}$$

From this observation, it follows follows that

$$u(t,x) = F(e^{tC}(x)) + G(e^{-tC}(x))$$

where

$$F(x) = \frac{1}{2}\left(f(x) + \int_0^x g(y)\frac{dy}{c(y)}\right) \text{ and } G(x) = \frac{1}{2}\left(f(x) - \int_0^x g(y)\frac{dy}{c(y)}\right).$$

Now clearly F and G are C^2 – functions and

$$\left(\partial_t - \tilde{C}\right) F(e^{tC}(x)) = 0 \text{ and } \left(\partial_t + \tilde{C}\right) G(e^{-tC}(x)) = 0$$

so that

$$\left(\partial_t^2 - \tilde{C}^2\right) u(t,x) = \left(\partial_t - \tilde{C}\right)\left(\partial_t + \tilde{C}\right)\left[F(e^{tC}(x)) + G(e^{-tC}(x))\right] = 0.$$

■

By Du hamel's principle, we can similarly solve

$$u_{tt} = \tilde{C}^2 u + h \text{ with } u(0,\cdot) = 0 \text{ and } u_t(0,\cdot) = 0. \tag{2.19}$$

Corollary 2.12. *The solution to Eq. (2.19) is*

$$\begin{aligned} u(t,x) &= \frac{1}{2}\int_0^t \begin{pmatrix} \text{Solution to Eq. (2.17)} \\ \text{at time } t-\tau \\ \text{with } f=0 \text{ and } g = h(\tau,\cdot) \end{pmatrix} d\tau \\ &= \frac{1}{2}\int_0^t d\tau \int_{\tau-t}^{t-\tau} h(\tau, e^{sC}(x))ds. \end{aligned}$$

Proof. This is simply a matter of computing a number of derivatives:

$$\begin{aligned} u_t &= \frac{1}{2}\int_0^t d\tau \left[h(\tau, e^{(t-\tau)C}(x)) + h(\tau, e^{(\tau-t)C}(x))\right] \\ u_{tt} &= h(t,x) + \frac{1}{2}\int_0^t d\tau \left[\tilde{C}h(\tau, e^{(t-\tau)C}(x)) - \tilde{C}h(\tau, e^{(\tau-t)C}(x))\right] \\ \tilde{C}u &= \frac{1}{2}\int_0^t d\tau \int_{\tau-t}^{t-\tau} \tilde{C}h(\tau, e^{sC}(x))ds = \frac{1}{2}\int_0^t d\tau \int_{\tau-t}^{t-\tau} \frac{d}{ds}h(\tau, e^{sC}(x))ds \\ &= \frac{1}{2}\int_0^t d\tau \left[h(\tau, e^{(t-\tau)C}(x)) - h(\tau, e^{(\tau-t)C}(x))\right] \text{ and} \\ \tilde{C}^2 u &= \frac{1}{2}\int_0^t d\tau \left[\tilde{C}h(\tau, e^{(t-\tau)C}(x)) - \tilde{C}h(\tau, e^{(\tau-t)C}(x))\right]. \end{aligned}$$

Subtracting the second and last equation then shows $u_{tt} = \tilde{A}^2 u + h$ and it is clear that $u(0,\cdot) = 0$ and $u_t(0,\cdot) = 0$. ■

2.2. **General Linear First Order PDE.**

In this section we consider the following PDE,

$$\sum_{i=1}^n a_i(x)\partial_i u(x) = c(x)u(x) \tag{2.20}$$

where $a_i(x)$ and $c(x)$ are given functions. As above Eq. (2.20) may be written simply as

$$\tilde{A}u(x) = c(x)u(x). \tag{2.21}$$

The key observation to solving Eq. (2.21) is that the chain rule implies

$$\frac{d}{ds}u(e^{sA}(x)) = \tilde{A}u(e^{sA}(x)), \tag{2.22}$$

which we will write briefly as

$$\frac{d}{ds}u \circ e^{sA} = \tilde{A}u \circ e^{sA}.$$

Combining Eqs. (2.21) and (2.22) implies

$$\frac{d}{ds}u(e^{sA}(x)) = c(e^{sA}(x))u(e^{sA}(x))$$

which then gives

$$u(e^{sA}(x)) = e^{\int_0^s c(e^{\sigma A}(x))d\sigma} u(x). \tag{2.23}$$

Equation (2.22) shows that the values of u solving Eq. (2.21) along any flow line of A, are completely determined by the value of u at any point on this flow line. Hence we can expect to construct solutions to Eq. (2.21) by specifying u arbitrarily on any surface Σ which crosses the flow lines of A transversely, see Figure 3 below.

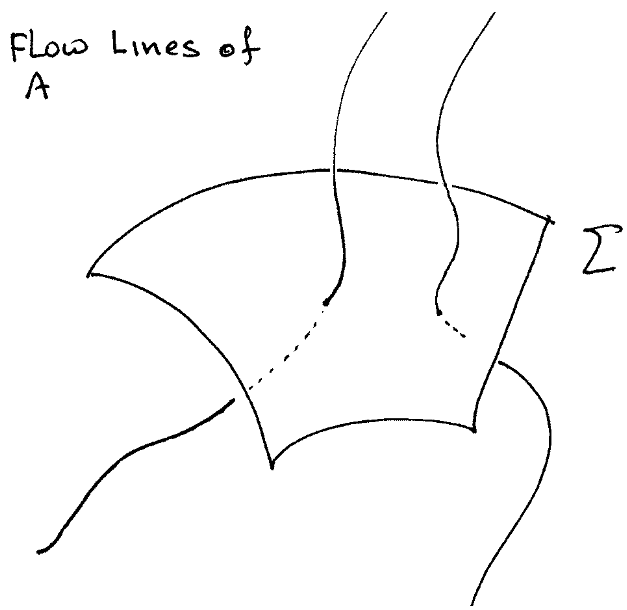


FIGURE 3. The flow lines of A through a non-characteristic surface Σ.

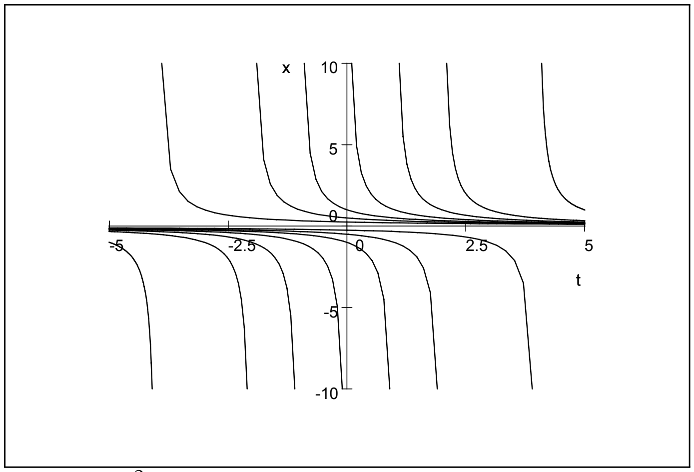
Example 2.13. Let us again consider the PDE in Eq. (2.6) above but now with initial data being given on the line $x = t$, i.e.

$$u_t = x^2 u_x \text{ with } u(\lambda, \lambda) = f(\lambda)$$

for some $f \in C^1(\mathbb{R}, \mathbb{R})$. The characteristic equations are given by

$$t'(s) = 1 \text{ and } x'(s) = -x^2(s) \tag{2.24}$$

and the flow lines of this equations must live on the solution curves to $\frac{dx}{dt} = -x^2$, i.e. on curves of the form $x(t) = \frac{1}{t-C}$ for $C \in \mathbb{R}$ and $x = 0$, see Figure 2.13.



Any solution to $u_t = x^2 u_x$ must be constant on these characteristic curves. Notice the line $x = t$ crosses each characteristic curve exactly once while the line $t = 0$ crosses some but not all of the characteristic curves.

Solving Eqs. (2.24) with $t(0) = \lambda = x(0)$ gives

$$t(s) = s + \lambda \text{ and } x(s) = \frac{\lambda}{1 + s\lambda} \tag{2.25}$$

and hence

$$u(s + \lambda, \frac{\lambda}{1 + s\lambda}) = f(\lambda) \text{ for all } \lambda \text{ and } s > -1/\lambda.$$

(for a plot of some of the integral curves of Eq. (2.24).) Let

$$(t, x) = (s + \lambda, \frac{\lambda}{1 + s\lambda}) \tag{2.26}$$

and solve for $\lambda :$

$$x = \frac{\lambda}{1 + (t - \lambda)\lambda} \text{ or } x\lambda^2 - (xt - 1)\lambda - x = 0$$

which gives

$$\lambda = \frac{(xt - 1) \pm \sqrt{(xt - 1)^2 + 4x^2}}{2x}. \tag{2.27}$$

Now to determine the sign, notice that when $s = 0$ in Eq. (2.26) we have $t = \lambda = x$. So taking $t = x$ in the right side of Eq. (2.27) implies

$$\frac{\left(x^2 - 1\right) \pm \sqrt{\left(x^2 - 1\right)^2 + 4x^2}}{2x} = \frac{\left(x^2 - 1\right) \pm \left(x^2 + 1\right)}{2x} = \begin{cases} x & \text{with } + \\ -2/x & \text{with } - \end{cases}.$$

Therefore, we must take the plus sing in Eq. (2.27) to find

$$\lambda = \frac{(xt - 1) + \sqrt{(xt - 1)^2 + 4x^2}}{2x}$$

and hence

$$u(t, x) = f\left(\frac{(xt - 1) + \sqrt{(xt - 1)^2 + 4x^2}}{2x}\right). \tag{2.28}$$

When x is small,

$$\lambda = \frac{(xt-1) + (1-xt)\sqrt{1 + \frac{4x^2}{(xt-1)^2}}}{2x} \cong \frac{(1-xt)\frac{2x^2}{(xt-1)^2}}{2x} = \frac{x}{1-xt}$$

so that

$$u(t,x) \cong f\left(\frac{x}{1-xt}\right) \text{ when } x \text{ is small.}$$

Thus we see that $u(t,0) = f(0)$ and $u(t,x)$ is C^1 if f is C^1. Equation (2.28) sets up a one to one correspondence between solution u to $u_t = x^2 u_x$ and $f \in C^1(\mathbb{R},\mathbb{R})$.

Example 2.14. To solve

$$xu_x + yu_y = \lambda xyu \text{ with } u = f \text{ on } S^1, \tag{2.29}$$

let $A(x,y) = (x,y) = x\partial_x + y\partial_y$. The equations for $(x(s),y(s)) := e^{sA}(x,y)$ are

$$x'(s) = x(s) \text{ and } y'(s) = y(s)$$

from which we learn

$$e^{sA}(x,y) = e^s(x,y).$$

Then by Eq. (2.23),

$$u(e^s(x,y)) = e^{\lambda\int_0^s e^{2\sigma}xyd\sigma}u(x,y) = e^{\frac{\lambda}{2}\left(e^{2s}-1\right)xy}u(x,y).$$

Letting $(x,y) \to e^{-s}(x,y)$ in this equation gives

$$u(x,y) = e^{\frac{\lambda}{2}\left(1-e^{-2s}\right)xy}u(e^{-s}\,(x,y))$$

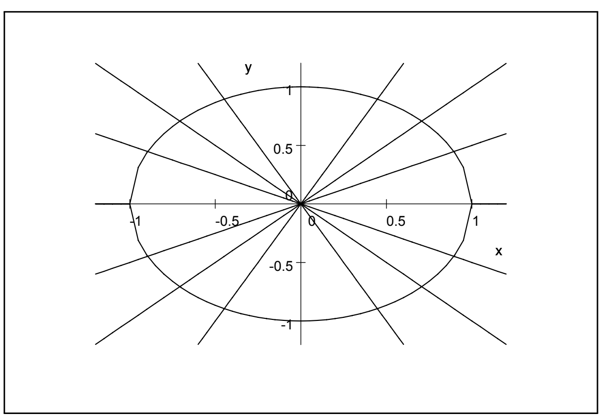
and then choosing s so that

$$1 = \left\|e^{-s}(x,y)\right\|^2 = e^{-2s}(x^2+y^2),$$

i.e. so that $s = \frac{1}{2}\ln\left(x^2+y^2\right)$. We then find

$$u(x,y) = \exp\left(\frac{\lambda}{2}\left(1 - \frac{1}{x^2+y^2}\right)xy\right) f\left(\frac{(x,y)}{\sqrt{x^2+y^2}}\right).$$

Notice that this solution always has a singularity at $(x,y) = (0,0)$ unless f is constant.



Characteristic curves for Eq. (2.29) along with the plot of S^1.

Example 2.15. The PDE,

$$e^x u_x + u_y = u \text{ with } u(x,0) = g(x), \tag{2.30}$$

has characteristic curves determined by $x' := e^x$ and $y' := 1$ and along these curves solutions u to Eq. (2.30) satisfy

$$\frac{d}{ds}u(x,y) = u(x,y). \tag{2.31}$$

Solving these "characteristic equations" gives

$$-e^{-x(s)} + e^{-x_0} = \int_0^s e^{-x}x'ds = \int_0^s 1ds = s \tag{2.32}$$

so that

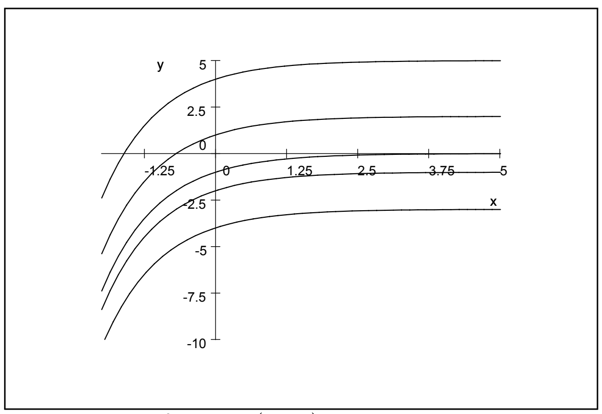
$$x(s) = -\ln(e^{-x_0} - s) \text{ and } y(s) = y_0 + s. \tag{2.33}$$

From Eqs. (2.32) and (2.33) one shows

$$y(s) = y_0 + e^{-x_0} - e^{-x(s)}$$

so the "characteristic curves" are contained in the graphs of the functions

$$y = C - e^{-x} \text{ for some constant } C.$$



Some characteristic curves for Eq. (2.30). Notice that the line $y = 0$ intersects some but not all of the characteristic curves. Therefore Eq. (2.30) does not uniquely determine a function u defined on all of $\mathbb{R}^2$. On the otherhand if the intial condition were $u(0,y) = g(y)$ the method would produce an everywhere defined solution.

Since the initial condition is at $y = 0$, set $y_0 = 0$ in Eq. (2.33) and notice from Eq.(2.31) that

$$u(-\ln(e^{-x_0} - s), s) = u(x(s), y(s)) = e^s u(x_0, 0) = e^s g(x_0). \tag{2.34}$$

Setting $(x,y) = (-\ln(e^{-x_0} - s), s)$ and solving for (x_0, s) implies

$$s = y \text{ and } x_0 = -\ln(e^{-x} + y)$$

and using this in Eq. (2.34) then implies

$$u(x,y) = e^y g\left(-\ln(y + e^{-x})\right).$$

This solution is only defined for $y > -e^{-x}$.

Example 2.16. In this example we will use the method of characteristics to solve the following non-linear equation,

$$x^2 u_x + y^2 \ u_y = u^2 \text{ with } u := 1 \text{ on } y = 2x. \tag{2.35}$$

As usual let (x, y) solve the characteristic equations, $x' = x^2$ and $y' = y^2$ so that

$$(x(s), y(s)) = \left(\frac{x_0}{1 - sx_0}, \frac{y_0}{1 - sy_0}\right).$$

Now let $(x_0, y_0) = (\lambda, 2\lambda)$ be a point on line $y = 2x$ and supposing u solves Eq. (2.35). Then $z(s) = u(x(s), y(s))$ solves

$$z' = \frac{d}{ds} u\,(x, y) = x^2 u_x + y^2 \ u_y = u^2(x, y) = z^2$$

with $z(0) = u(\lambda, 2\lambda) = 1$ and hence

$$u\left(\frac{\lambda}{1 - s\lambda}, \frac{2\lambda}{1 - 2s\lambda}\right) = u(x(s), y(s)) = z(s) = \frac{1}{1 - s}. \tag{2.36}$$

Let

$$(x, y) = \left(\frac{\lambda}{1 - s\lambda}, \frac{2\lambda}{1 - 2s\lambda}\right) = \left(\frac{1}{\lambda^{-1} - s}, \frac{1}{\lambda^{-1}/2 - s}\right) \tag{2.37}$$

and solve the resulting equations:

$$\lambda^{-1} - s = x^{-1} \text{ and } \lambda^{-1}/2 - s = y^{-1}$$

for s gives $s = x^{-1} - 2y^{-1}$ and hence

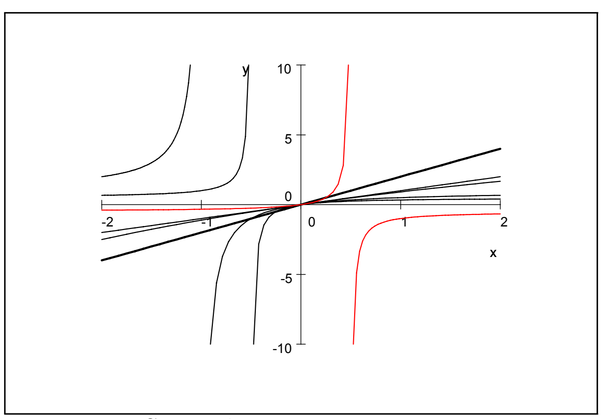
$$1 - s = 1 + 2y^{-1} - x^{-1} = x^{-1} y^{-1} \left(xy + 2x - y\right). \tag{2.38}$$

Combining Eqs. (2.36) – (2.38) gives

$$u(x, y) = \frac{xy}{xy + 2x - y}.$$

Notice that the characteristic curves here lie on the trajectories determined by $\frac{dx}{x^2} = \frac{dy}{y^2}$, i.e. $y^{-1} = x^{-1} + C$ or equivalently

$$y = \frac{x}{1 + Cx}$$



Some characteristic curves

2.3. **Quasi-Linear Equations.** In this section we consider the following PDE,

$$A(x,z)\cdot\nabla_x u(t,x)=\sum_{i=1}^{n}a_i(x,u(x))\partial_i u(x)=c(x,u(x)) \tag{2.39}$$

where $a_i(x,z)$ and $c(x,z)$ are given functions on $(x,z)\in\mathbb{R}^n\times\mathbb{R}$ and $A(x,z):=(a_1(x,z),\dots,a_1(x,z))$. Assume u is a solution to Eq. (2.39) and suppose $x(s)$ solves $x'(s)=A(x(s),u(x(s)))$.Then from Eq. (2.39) we find

$$\frac{d}{ds}u(x(s))=\sum_{i=1}^{n}a_i(x(s),u(x(s)))\partial_i u(x(s))=c(x(s),u(x(s))),$$

see Figure 4 below. We have proved the following Lemma.

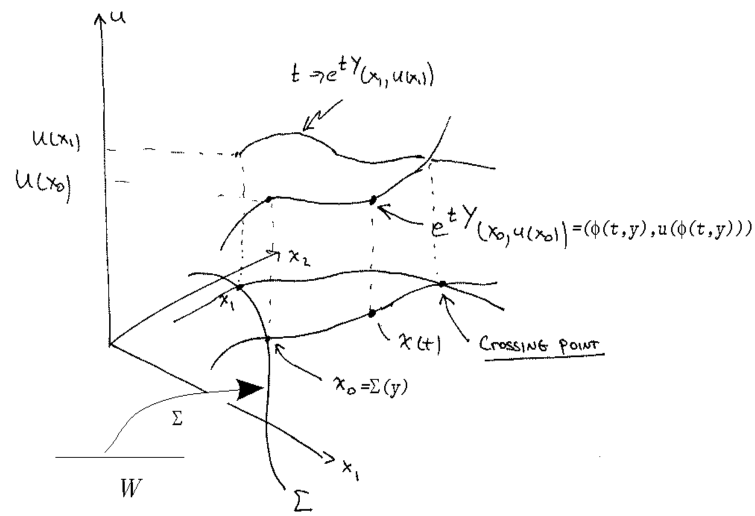


FIGURE 4. Determining the values of u by solving ODE's. Notice that potential problem though where the projection of characteristics cross in $x-$ space.

Lemma 2.17. *Let* $w=(x,z)$, $\pi_1(w)=x$, $\pi_2(w)=z$ *and* $Y(w)=(A(x,z),c(x,z))$. *If u is a solution to Eq. (2.39), then*

$$u(\pi_1\circ e^{sY}(x_0,u(x_0))=\pi_2\circ e^{sY}(x_0,u(x_0)).$$

Let Σ be a surface in $\mathbb{R}^n$ ($x-$ space), i.e. $\Sigma: U\subset_o\mathbb{R}^{n-1}\longrightarrow\mathbb{R}^n$ such that $\Sigma(0)=x_0$ and $D\Sigma(y)$ is injective for all $y\in U$. Now suppose $u_0:\Sigma\to\mathbb{R}$ is given we wish to solve for u such that (2.39) holds and $u=u_0$ on Σ. Let

$$\phi(s,y):=\pi_1\circ e^{sY}\left(\Sigma(y),u_0(\Sigma(y))\right) \tag{2.40}$$

then

$$\frac{\partial\phi}{\partial s}(0,0)=\pi_1\circ Y(x_0,u_0(x_0))=A(x_0,u_0(x_0))\text{ and}$$
$$D_y\phi(0,0)=D_y\Sigma(0).$$

Assume Σ is **non-characteristic at** x_0, that is $A(x_0, u_0(x_0)) \notin \operatorname{Ran} \Sigma'(0)$ where $\Sigma'(0) : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is defined by

$$\Sigma'(0)v = \partial_v \Sigma(0) = \frac{d}{ds}|_0 \Sigma(sv) \text{ for all } v \in \mathbb{R}^{n-1}.$$

Then $\left(\frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial y^1}, \dots, \frac{\partial \phi}{\partial y^{n-1}}\right)$ are all linearly independent vectors at $(0,0) \in \mathbb{R} \times \mathbb{R}^{n-1}$. So $\phi : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ has an invertible differential at $(0,0)$ and so the inverse function theorem gives the existence of open neighborhood $0 \in W \subset U$ and $0 \in J \subset \mathbb{R}$ such that $\phi\big|_{J \times W}$ is a homeomorphism onto an open set $V := \phi(J \times W) \subset \mathbb{R}^n$, see Figure 5. Because of Lemma 2.17, if we are going to have a C^1 – solution u to

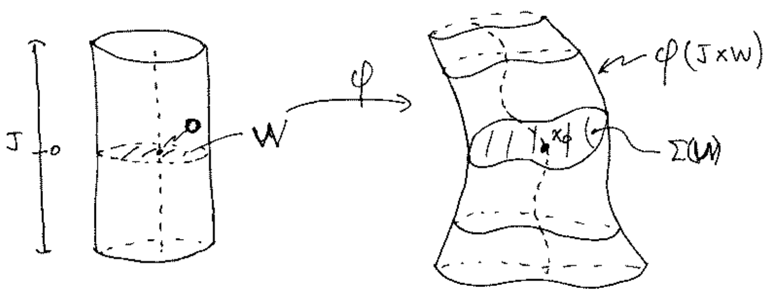


FIGURE 5. Constructing a neighborhood of the surface Σ near x_0 where we can solve the quasi-linear PDE.

Eq. (2.39) with $u = u_0$ on Σ it would have to satisfy

$$u(x) = \pi_2 \circ e^{sY}\left(\Sigma(y), u_0(\Sigma(y))\right) \text{ with } (s,y) := \phi^{-1}(x), \tag{2.41}$$

i.e. $x = \phi(s,y)$.

Proposition 2.18. *The function u in Eq. (2.41) solves Eq. (2.39) on V with $u = u_0$ on Σ.*

Proof. By definition of u in Eq. (2.41) and ϕ in Eq. (2.40),

$$\phi'(s,y) = \pi_1 Y \circ e^{sY}\left(\Sigma(y), u_0(\Sigma(y))\right) = A(\phi(s,y)), u(\phi(s,y))$$

and

$$\frac{d}{ds} u(\phi(s,y)) = \pi_2 Y\left(\phi(s,y), u(\phi(s,y))\right) = c(\phi(s,y), u(\phi(s,y))). \tag{2.42}$$

On the other hand by the chain rule,

$$\frac{d}{ds} u(\phi(s,y)) = \nabla u(\phi(s,u)) \cdot \phi'(s,y) = \nabla u(\phi(s,u)) \cdot A(\phi(s,y)), u(\phi(s,y)). \tag{2.43}$$

Comparing Eqs. (2.42) and (2.43) implies

$$\nabla u(\phi(s,y)) \cdot A(\phi(s,y), u(\phi(s,y))) = c(\phi(s,y), u(\phi(s,y))).$$

Since $\phi(J \times W) = V$, u solves Eq. (2.39) on V. Clearly $u(\phi(0,y)) = u_0(\Sigma(y))$ so $u = u_0$ on Σ. ■

Example 2.19 (Conservation Laws)**.** Let $F:\mathbb{R}\to\mathbb{R}$ be a smooth function, we wish to consider the PDE for $u=u(t,x)$,

$$0=u_t+\partial_x F(u)=u_t+F'(u)u_x \text{ with } u(0,x)=g(x). \tag{2.44}$$

The characteristic equations are given by

$$t'(s)=1,\ x'(s)=F'(z(s)) \text{ and } \frac{d}{ds}z(s)=0. \tag{2.45}$$

The solution to Eqs. (2.45) with $t(0)=0$, $x(0)=x$ and hence

$$z(0)=u(t(0),x(0))=u(0,x)=g(x),$$

are given by

$$t(s)=s,\ z(s)=g(x) \text{ and } x(s)=x+sF'(g(x)).$$

So we conclude that any solution to Eq. (2.44) must satisfy,

$$u(s,x+sF'(g(x))=g(x).$$

This implies, letting $\psi_s(x):=x+sF'(g(x))$, that

$$u(t,x)=g(\psi_t^{-1}(x)).$$

In order to find ψ_t^{-1} we need to know ψ_t is invertible, i.e. that ψ_t is monotonic in x. This becomes the condition

$$0<\psi_t'(x)=1+tF''(g(x))g'(x).$$

If this holds then we will have a solution.

Example 2.20 (Conservation Laws in Higher Dimensions)**.** Let $F:\mathbb{R}\to\mathbb{R}^n$ be a smooth function, we wish to consider the PDE for $u=u(t,x)$,

$$0=u_t+\nabla\cdot F(u)=u_t+F'(u)\cdot\nabla u \text{ with } u(0,x)=g(x). \tag{2.46}$$

The characteristic equations are given by

$$t'(s)=1,\ x'(s)=F'(z(s)) \text{ and } \frac{d}{ds}z(s)=0. \tag{2.47}$$

The solution to Eqs. (2.47) with $t(0)=0$, $x(0)=x$ and hence

$$z(0)=u(t(0),x(0))=u(0,x)=g(x),$$

are given by

$$t(s)=s,\ z(s)=g(x) \text{ and } x(s)=x+sF'(g(x)).$$

So we conclude that any solution to Eq. (2.46) must satisfy,

$$u(s,x+sF'(g(x)))=g(x). \tag{2.48}$$

This implies, letting $\psi_s(x):=x+sF'(g(x))$, that

$$u(t,x)=g(\psi_t^{-1}(x)).$$

In order to find ψ_t^{-1} we need to know ψ_t is invertible. Locally by the implicit function theorem it suffices to know,

$$\psi_t'(x)v=v+tF''(g(x))\partial_v g(x)=\left[I+tF''(g(x))\nabla g(x)\cdot\right]v$$

is invertible. Alternatively, let $y=x+sF'(g(x))$, (so $x=y-sF'(g(x))$) in Eq. (2.48) to learn, using Eq. (2.48) which asserts $g(x)=u(s,x+sF'(g(x)))=u(s,y)$,

$$u(s,y)=g\left(y-sF'(g(x))\right)=g\left(y-sF'(u(s,y))\right).$$

This equation describes the solution u implicitly.

Example 2.21 (Burger's Equation)**.** Recall Burger's equation is the PDE,

$$u_t + uu_x = 0 \text{ with } u(0,x) = g(x) \tag{2.49}$$

where g is a given function. Also recall that if we view $u(t,x)$ as a time dependent vector field on $\mathbb{R}$ and let $x(t)$ solve

$$\dot{x}(t) = u(t, x(t)),$$

then

$$\ddot{x}(t) = u_t + u_x\dot{x} = u_t + u_x u = 0.$$

Therefore x has constant acceleration and

$$x(t) = x(0) + \dot{x}(0)t = x(0) + g(x(0))t.$$

This equation contains the same information as the characteristic equations. Indeed, the characteristic equations are

$$t'(s) = 1, \quad x'(s) = z(s) \text{ and } z'(s) = 0. \tag{2.50}$$

Taking initial condition $t(0) = 0$, $x(0) = x_0$ and $z(0) = u(0, x_0) = g(x_0)$ we find

$$t(s) = s, \quad z(s) = g(x_0) \text{ and } x(s) = x_0 + sg(x_0).$$

According to Proposition 2.18, we must have

$$u((s, x_0 + sg(x_0)) = u(s, x(s)) = u(0, x(0)) = g\left(x_0\right). \tag{2.51}$$

Letting $\psi_t(x_0) := x_0 + tg(x_0)$, "the" solution to $(t,x) = (s, x_0 + sg(x_0))$ is given by $s = t$ and $x_0 = \psi_t^{-1}(x)$. Therefore, we find from Eq. (2.51) that

$$u(t,x) = g\left(\psi_t^{-1}(x)\right). \tag{2.52}$$

This gives the desired solution provided ψ_t^{-1} is well defined.

Example 2.22 (Burger's Equation Continued)**.** Continuing Example 2.21. Suppose that $g \geq 0$ is an increasing function (i.e. the faster cars start to the right), then ψ_t is strictly increasing and for any $t \geq 0$ and therefore Eq. (2.52) gives a solution for all $t \geq 0$. For a specific example take $g(x) = \max(x, 0)$, then

$$\psi_t(x) = \begin{cases} (1+t)x & \text{if} \quad x \geq 0 \\ x & \text{if} \quad x \leq 0 \end{cases}$$

and therefore,

$$\psi_t^{-1}(x) = \begin{cases} (1+t)^{-1}x & \text{if} \quad x \geq 0 \\ x & \text{if} \quad x \leq 0 \end{cases}$$

$$u(t,x) = g\left(\psi_t^{-1}(x)\right) = \begin{cases} (1+t)^{-1}x & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x \leq 0. \end{cases}$$

Notice that $u(t,x) \to 0$ as $t \to \infty$ since all the fast cars move off to the right leaving only slower and slower cars passing $x \in \mathbb{R}$.

Example 2.23. Now suppose $g \geq 0$ and that $g'(x_0) < 0$ at some point $x_0 \in \mathbb{R}$, i.e. there are faster cars to the left of x_0 then there are to the right of x_0, see Figure 6. Without loss of generality we may assume that $x_0 = 0$. The projection of a number of characteristics to the (t,x) plane for this velocity profile are given in Figure 7 below. Since any C^2 – solution to Eq.(2.49) must be constant on these lines with the value given by the slope, it is impossible to get a C^2 – solution on all of $\mathbb{R}^2$ with this initial condition. Physically, there are collisions taking place which causes the formation of a shock wave in the system.

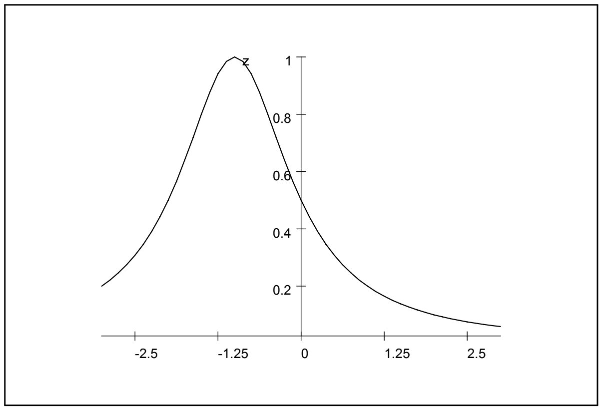


FIGURE 6. An intial velocity profile where collisions are going to occur. This is the graph of $g(x) = 1/\left(1 + (x+1)^2\right)$.

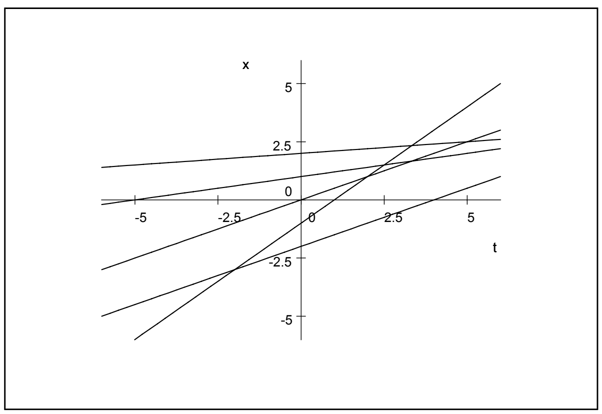


FIGURE 7. Crossing of projected characteristics for Burger's equation.

2.4. **Distribution Solutions for Conservation Laws.** Let us again consider the conservation law in Example 2.19 above. We will now restrict our attention to non-negative times. Suppose that u is a C^1 − solution to

$$u_t + (F(u))_x = 0 \text{ with } u(0,x) = g(x) \tag{2.53}$$

and $\phi \in C_c^2([0,\infty) \times \mathbb{R})$. Then by integration by parts,

$$\begin{aligned}
0 &= -\int_{\mathbb{R}} dx \int_{t \geq 0} dt (u_t + F(u)_x)\varphi \\
&= -\int_{\mathbb{R}} [u\varphi]\Big|_{t=0}^{t=\infty} dx + \int_{\mathbb{R}} dx \int_{t\geq 0} dt(u\varphi_t + F(u)\phi_x) \\
&= \int_{\mathbb{R}} g(x)\varphi(0,x)dx + \int_{\mathbb{R}} dx \int_{t \geq 0} dt(u(t,x)\phi_t(t,x) + F(u(t,x))\phi_x(t,x)).
\end{aligned}$$

Definition 2.24. A bounded measurable function $u(t,x)$ is a **distributional solution** to Eq. (2.53) iff

$$0=\int_{\mathbb{R}} g(x)\varphi(0,x)dx+\int_{\mathbb{R}} dx\int_{t\geq 0} dt(u(t,x)\phi_t(t,x)+F(u(t,x))\phi_x(t,x))$$

for all test functions $\phi\in C_c^2(D)$ where $D=[0,\infty)\times\mathbb{R}$.

Proposition 2.25. *If u is a distributional solution of Eq. (2.53) and u is C^1 then u is a solution in the classical sense. More generally if $u\in C^1(R)$ where R is an open region contained in $D^0:=(0,\infty)\times\mathbb{R}$ and*

$$\int_{\mathbb{R}} dx\int_{t\geq 0} dt(u(t,x)\phi_t(t,x)+F(u(t,x))\phi_x(t,x))=0 \tag{2.54}$$

for all $\phi\in C_c^2(R)$ then $u_t+(F(u))_x:=0$ on R.

Proof. Undo the integration by parts argument to show Eq. (2.54) implies

$$\int_R (u_t+(F(u))_x)\varphi dxdt=0$$

for all $\phi\in C_c^1(R)$. This then implies $u_t+(F(u))_x=0$ on R. ■

Theorem 2.26 (Rankine-Hugoniot Condition)**.** *Let R be a region in D^0 and $x=c(t)$ for $t\in[a,b]$ be a C^1 curve in R as pictured below in Figure 8.*

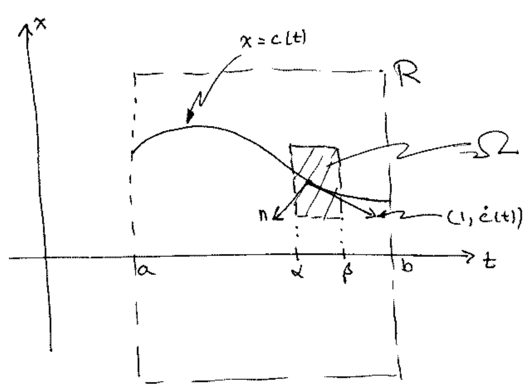


FIGURE 8. A curve of discontinuities of u.

Suppose $u\in C^1(R\setminus c([a,b]))$ and u is bounded and has limiting values u^+ and u^- on $x=c(t)$ when approaching from above and below respectively. Then u is a distributional solution of $u_t+(F(u))_x=0$ in R if and only if

$$u_t+\frac{\partial}{\partial x}F(u):=0 \text{ on } R\setminus c([a,b]) \tag{2.55}$$

and

$$\dot{c}(t)\left[u^+(t,c(t))-u^-(t,c(t))\right]=F(u^+(t,c(t)))-F(u^-(t,c(t)) \text{ for all } t\in[a,b]. \tag{2.56}$$

Proof. The fact that Equation 2.55 holds has already been proved in the previous proposition. For (2.56) let Ω be a region as pictured in Figure 8 above and $\phi \in C_c^1(\Omega)$. Then

(2.57)
$$0 = \int_{\Omega} (u\,\phi_t + F(u)\phi_x)dt\ dx = \int_{\Omega_+} (u\phi_t + F(u)\phi_x)dt\ dx + \int_{\Omega_-} (u\phi_t + F(u)\phi_x)dt\ dx$$

where
$$\Omega_{\pm} = \left\{(t,x) \in \Omega : \begin{array}{c} x > c(t) \\ x < c(t) \end{array}\right\}.$$

Now the outward normal to $\Omega_{\pm}$ along c is
$$n(t) = \pm\frac{(\dot{c}(t), -1)}{\sqrt{1+\dot{c}(t)^2}}$$

and the "surface measure" along c is given by $d\sigma(t) = \sqrt{1+\dot{c}(t)^2}dt$. Therefore
$$n(t)\ d\sigma(t) = \pm(\dot{c}(t), -1)dt$$

where the sign is chosen according to the sign in $\Omega_{\pm}$. Hence by the divergence theorem,
$$\begin{aligned}\int_{\Omega_{\pm}} (u\,\phi_t + F(u)\phi_x)dt\ dx &= \int_{\Omega_{\pm}} (u, F(u)) \cdot (\,\phi_t, \phi_x)dt\ dx = \int_{\partial\Omega_{\pm}} \phi(u, F(u)) \cdot n(t)\ d\sigma(t) \\ &= \pm\int_{\alpha}^{\beta} \phi(t, c(t))(u_t^{\pm}(t, c(t))\dot{c}(t) - F(u_t^{\pm}(t, c(t))))dt.\end{aligned}$$

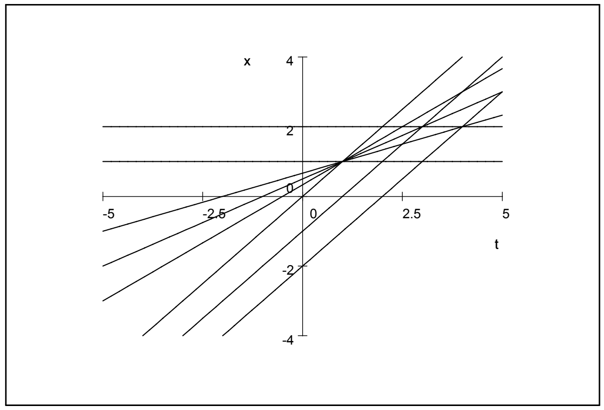
Putting these results into Eq. (2.57) gives
$$0 = \int_{\alpha}^{\beta} \{\dot{c}(t)\left[u^+(t,c(t)) - u^-(t,c(t))\right] - (F(u^+(t,c(t))) - F(u^-(t,c(t)))\}\phi(t,c(t))dt$$

for all ϕ which implies
$$\dot{c}(t)\left[u^+(t,c(t)) - u^-(t,c(t))\right] = F(u^+(t,c(t))) - F(u^-(t,c(t)).$$

■

Example 2.27. In this example we will find an integral solution to Burger's Equation, $u_t + uu_x = 0$ with initial condition
$$u(0,x) = \begin{cases} 0 & x \geq 1 \\ 1-x & 0 \leq x \leq 1 \\ 1 & x \leq 0. \end{cases}$$

The characteristics are given from above by
$$\begin{aligned}&x(t) = (1-x_0)t + x_0 \quad x_0 \in (0,1) \\ &x(t) = x_0 + t \text{ if } x_0 \leq 0 \text{ and} \\ &x(t) = x_0 \text{ if } x_0 \geq 1.\end{aligned}$$



Projected characteristics

For the region bounded determined by $t \le x \le 1$ and $t \le 1$ we have $u(t,x)$ is equal to the slope of the line through (t,x) and $(1,1)$, i.e.

$$u(t,x) = \frac{x-1}{t-1}.$$

Notice that the solution is not well define in the region where characteristics cross, i.e. in the shock zone,

$$R_2 := \{(t,x) : t \ge 1,\ x \ge 1 \text{ and } x \le t\},$$

see Figure 9. Let us now look for a distributional solution of the equation valid for

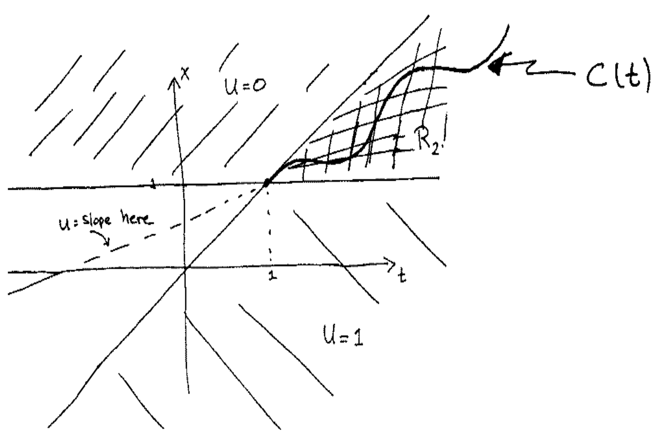


FIGURE 9. The schock zone and the values of u away from the shock zone.

all (x,t) by looking for a curve $c(t)$ in R_2 such that above $c(t)$, $u = 0$ while below $c(t)$, $u = 1$.

To this end we will employ the Rankine-Hugoniot Condition of Theorem 2.26. To do this observe that Burger's Equation may be written as $u_t + (F(u))_x = 0$ where $F(u) = \frac{u^2}{2}$. So the Jump condition is

$$\dot{c}(u_+ - u_-) = (F(u_+) - F(u_-))$$

that is

$$(0-1)\dot{c} = \left(\frac{0^2}{2} - \frac{1^2}{2}\right) = -\frac{1}{2}.$$

Hence $\dot{c}(t) = \frac{1}{2}$ and therefore $c(t) = \frac{1}{2}t + 1$ for $t \geq 0$. So we find a distributional solution given by the values in shown in Figure 10.

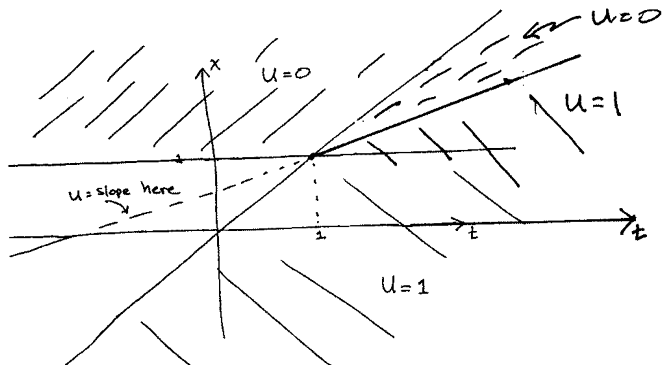


FIGURE 10. A distributional solution to Burger's equation.

2.5. **Exercises.**

Exercise 2.1. For $A \in L(X)$, let

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n. \tag{2.58}$$

Show directly that:

(1) e^{tA} is convergent in $L(X)$ when equipped with the operator norm.
(2) e^{tA} is differentiable in t and that $\frac{d}{dt} e^{tA} = Ae^{tA}$.

Exercise 2.2. Suppose that $A \in L(X)$ and $v \in X$ is an eigenvector of A with eigenvalue λ, i.e. that $Av = \lambda v$. Show $e^{tA}v = e^{t\lambda}v$. Also show that $X = \mathbb{R}^n$ and A is a diagonalizable $n \times n$ matrix with

$$A = SDS^{-1} \text{ with } D = diag(\lambda_1, \ldots, \lambda_n)$$

then $e^{tA} = Se^{tD}S^{-1}$ where $e^{tD} = diag(e^{t\lambda_1}, \ldots, e^{t\lambda_n})$.

Exercise 2.3. Suppose that $A, B \in L(X)$ and $[A, B] \equiv AB - BA = 0$. Show that $e^{(A+B)} = e^A e^B$.

Exercise 2.4. Suppose $A \in C(\mathbb{R}, L(X))$ satisfies $[A(t), A(s)] = 0$ for all $s, t \in \mathbb{R}$. Show

$$y(t) := e^{\left(\int_0^t A(\tau)d\tau\right)} x$$

is the unique solution to $\dot{y}(t) = A(t)y(t)$ with $y(0) = x$.

Exercise 2.5. Compute e^{tA} when

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and use the result to prove the formula

$$\cos(s+t) = \cos s \cos t - \sin s \sin t.$$

Hint: Sum the series and use $e^{tA}e^{sA} = e^{(t+s)A}$.

Exercise 2.6. Compute e^{tA} when

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

with $a, b, c \in \mathbb{R}$. Use your result to compute $e^{t(\lambda I + A)}$ where $\lambda \in \mathbb{R}$ and I is the 3×3 identity matrix. **Hint:** Sum the series.

Theorem 2.28. *Suppose that $T_t \in L(X)$ for $t \geq 0$ satisfies*

(1) *(Semi-group property.) $T_0 = Id_X$ and $T_t T_s = T_{t+s}$ for all $s, t \geq 0$.*
(2) *(Norm Continuity) $t \to T_t$ is continuous at 0, i.e. $\|T_t - I\|_{L(X)} \to 0$ as $t \downarrow 0$.*

Then there exists $A \in L(X)$ such that $T_t = e^{tA}$ where e^{tA} is defined in Eq. (2.58).

Exercise 2.7. Prove Theorem 2.28 using the following outline.

(1) First show $t \in [0, \infty) \to T_t \in L(X)$ is continuous.
(2) For $\epsilon > 0$, let $S_\epsilon := \frac{1}{\epsilon}\int_0^\epsilon T_\tau d\tau \in L(X)$. Show $S_\epsilon \to I$ as $\epsilon \downarrow 0$ and conclude from this that S_ϵ is invertible when $\epsilon > 0$ is sufficiently small. For the remainder of the proof fix such a small $\epsilon > 0$.
(3) Show
$$T_t S_\epsilon = \frac{1}{\epsilon}\int_t^{t+\epsilon} T_\tau d\tau$$
and conclude from this that
$$\lim_{t\downarrow 0} t^{-1}(T_t - I) S_\epsilon = \frac{1}{\epsilon}(T_\epsilon - Id_X).$$
(4) Using the fact that S_ϵ is invertible, conclude $A = \lim_{t\downarrow 0} t^{-1}(T_t - I)$ exists in $L(X)$ and that
$$A = \frac{1}{\epsilon}(T_\epsilon - I) S_\epsilon^{-1}.$$
(5) Now show using the semigroup property and step 4. that $\frac{d}{dt}T_t = AT_t$ for all $t > 0$.
(6) Using step 5, show $\frac{d}{dt}e^{-tA}T_t = 0$ for all $t > 0$ and therefore $e^{-tA}T_t = e^{-0A}T_0 = I$.

Exercise 2.8 (Higher Order ODE)**.** Let X be a Banach space, , $\mathcal{U} \subset_o X^n$ and $f \in C(J \times \mathcal{U}, X)$ be a Locally Lipschitz function in $\mathbf{x} = (x_1, \ldots, x_n)$. Show the n^{th} ordinary differential equation,

(2.59)
$$y^{(n)}(t) = f(t, y(t), \dot{y}(t), \ldots y^{(n-1)}(t)) \text{ with } y^{(k)}(0) = y_0^k \text{ for } k = 0, 1, 2 \ldots, n-1$$

where $(y_0^0, \ldots, y_0^{n-1})$ is given in $\mathcal{U}$, has a unique solution for small $t \in J$. **Hint**: let $\mathbf{y}(t) = \left(y(t), \dot{y}(t), \ldots y^{(n-1)}(t)\right)$ and rewrite Eq. (2.59) as a first order ODE of the form

$$\dot{\mathbf{y}}(t) = Z(t, \mathbf{y}(t)) \text{ with } \mathbf{y}(0) = (y_0^0, \ldots, y_0^{n-1}).$$

Exercise 2.9. Use the results of Exercises 2.6 and 2.8 to solve

$$\ddot{y}(t) - 2\dot{y}(t) + y(t) = 0 \text{ with } y(0) = a \text{ and } \dot{y}(0) = b.$$

Hint: The 2×2 matrix associated to this system, A, has only one eigenvalue 1 and may be written as $A = I + B$ where $B^2 = 0$.

Exercise 2.10. Suppose that $A : \mathbb{R} \to L(X)$ is a continuous function and $U, V : \mathbb{R} \to L(X)$ are the unique solution to the linear differential equations

$$\dot{V}(t) = A(t)V(t) \text{ with } V(0) = I$$

and

$$\dot{U}(t) = -U(t)A(t) \text{ with } U(0) = I. \tag{2.60}$$

Prove that $V(t)$ is invertible and that $V^{-1}(t) = U(t)$. **Hint:** 1) show $\frac{d}{dt}\left[U(t)V(t)\right] = 0$ (which is sufficient if $\dim(X) < \infty$) and 2) show compute $y(t) := V(t)U(t)$ solves a linear differential ordinary differential equation that has $y \equiv 0$ as an obvious solution. Then use the uniqueness of solutions to ODEs. (The fact that $U(t)$ must be defined as in Eq. (2.60) is the content of Exercise **??** below.)

Exercise 2.11 (Duhamel' s Principle I)**.** Suppose that $A : \mathbb{R} \to L(X)$ is a continuous function and $V : \mathbb{R} \to L(X)$ is the unique solution to the linear differential equation in Eq. (**??**). Let $x \in X$ and $h \in C(\mathbb{R}, X)$ be given. Show that the unique solution to the differential equation:

$$\dot{y}(t) = A(t)y(t) + h(t) \text{ with } y(0) = x \tag{2.61}$$

is given by

$$y(t) = V(t)x + V(t)\int_0^t V(\tau)^{-1}h(\tau)\,d\tau. \tag{2.62}$$

Hint: compute $\frac{d}{dt}[V^{-1}(t)y(t)]$ when y solves Eq. (2.61).

Exercise 2.12 (Duhamel' s Principle II)**.** Suppose that $A : \mathbb{R} \to L(X)$ is a continuous function and $V : \mathbb{R} \to L(X)$ is the unique solution to the linear differential equation in Eq. (**??**). Let $W_0 \in L(X)$ and $H \in C(\mathbb{R}, L(X))$ be given. Show that the unique solution to the differential equation:

$$\dot{W}(t) = A(t)W(t) + H(t) \text{ with } W(0) = W_0 \tag{2.63}$$

is given by

$$W(t) = V(t)W_0 + V(t)\int_0^t V(\tau)^{-1}H(\tau)\,d\tau. \tag{2.64}$$

Exercise 2.13 (Non-Homogeneous ODE)**.** Suppose that $U \subset_o X$ is open and $Z : \mathbb{R} \times U \to X$ is a continuous function. Let $J = (a, b)$ be an interval and $t_0 \in J$. Suppose that $y \in C^1(J, U)$ is a solution to the "non-homogeneous" differential equation:

$$\dot{y}(t) = Z(t, y(t)) \text{ with } y(t_o) = x \in U. \tag{2.65}$$

Define $Y \in C^1(J - t_0, \mathbb{R} \times U)$ by $Y(t) \equiv (t + t_0, y(t + t_0))$. Show that Y solves the "homogeneous" differential equation

$$\dot{Y}(t) = \tilde{A}(Y(t)) \text{ with } Y(0) = (t_0, y_0), \tag{2.66}$$

where $\tilde{A}(t,x) \equiv (1, Z(x))$. Conversely, suppose that $Y \in C^1(J - t_0, \mathbb{R} \times U)$ is a solution to Eq. (2.66). Show that $Y(t) = (t + t_0, y(t + t_0))$ for some $y \in C^1(J, U)$ satisfying Eq. (2.65). (In this way the theory of non-homogeneous ode's may be reduced to the theory of homogeneous ode's.)

Exercise 2.14 (Differential Equations with Parameters)**.** Let W be another Banach space, $U \times V \subset_o X \times W$ and $Z \in C(U \times V, X)$ be a locally Lipschitz function on $U \times V$. For each $(x, w) \in U \times V$, let $t \in J_{x,w} \to \phi(t, x, w)$ denote the maximal solution to the ODE

$$\dot{y}(t) = Z(y(t), w) \text{ with } y(0) = x. \tag{2.67}$$

Prove

$$\mathcal{D} := \{(t, x, w) \in \mathbb{R} \times U \times V : t \in J_{x,w}\} \tag{2.68}$$

is open in $\mathbb{R} \times U \times V$ and ϕ and $\dot{\phi}$ are continuous functions on $\mathcal{D}$.

Hint: If $y(t)$ solves the differential equation in (2.67), then $v(t) \equiv (y(t), w)$ solves the differential equation,

$$\dot{v}(t) = \tilde{A}(v(t)) \text{ with } v(0) = (x, w), \tag{2.69}$$

where $\tilde{A}(x, w) \equiv (Z(x, w), 0) \in X \times W$ and let $\psi(t, (x, w)) := v(t)$. Now apply the Theorem **??** to the differential equation (2.69).

Exercise 2.15 (Abstract Wave Equation)**.** For $A \in L(X)$ and $t \in \mathbb{R}$, let

$$\cos(tA) := \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{(2n)!} t^{2n} A^{2n} \text{ and}$$

$$\frac{\sin(tA)}{A} := \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)!} t^{2n+1} A^{2n}.$$

Show that the unique solution $y \in C^2(\mathbb{R}, X)$ to

$$\ddot{y}(t) + A^2 y(t) = 0 \text{ with } y(0) = y_0 \text{ and } \dot{y}(0) = \dot{y}_0 \in X \tag{2.70}$$

is given by

$$y(t) = \cos(tA) y_0 + \frac{\sin(tA)}{A} \dot{y}_0.$$

Remark 2.29*.* Exercise 2.15 can be done by direct verification. Alternatively and more instructively, rewrite Eq. (2.70) as a first order ODE using Exercise 2.8. In doing so you will be lead to compute e^{tB} where $B \in L(X \times X)$ is given by

$$B = \begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix},$$

where we are writing elements of $X \times X$ as column vectors, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. You should then show

$$e^{tB} = \begin{pmatrix} \cos(tA) & \frac{\sin(tA)}{A} \\ -A\sin(tA) & \cos(tA) \end{pmatrix}$$

where

$$A\sin(tA) := \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)!} t^{2n+1} A^{2(n+1)}.$$

Exercise 2.16 (Duhamel's Principle for the Abstract Wave Equation)**.** Continue the notation in Exercise 2.15, but now consider the ODE,

$$\ddot{y}(t) + A^2 y(t) = f(t) \text{ with } y(0) = y_0 \text{ and } \dot{y}(0) = \dot{y}_0 \in X \tag{2.71}$$

where $f \in C(\mathbb{R}, X)$. Show the unique solution to Eq. (2.71) is given by

$$y(t) = \cos(tA) y_0 + \frac{\sin(tA)}{A} \dot{y}_0 + \int_0^t \frac{\sin((t-\tau)A)}{A} f(\tau) d\tau \tag{2.72}$$

Hint: Again this could be proved by direct calculation. However it is more instructive to deduce Eq. (2.72) from Exercise 2.11 and the comments in Remark 2.29.

Exercise 2.17. Number 3 on p. 163 of Evans.