■

7. Test Functions and Partitions of Unity

7.1. **Convolution and Young's Inequalities.** Letting δ_x denote the "delta–function" at x, we wish to define a product $(*)$ on functions on $\mathbb{R}^n$ such that $\delta_x * \delta_y = \delta_{x+y}$. Now formally any function f on $\mathbb{R}^n$ is of the form

$$f = \int_{\mathbb{R}^n} f(x)\delta_x dx$$

so we should have

$$\begin{aligned} f * g &= \int_{\mathbb{R}^n\times\mathbb{R}^n} f(x)g(y)\delta_x * \delta_y dxdy = \int_{\mathbb{R}^n\times\mathbb{R}^n} f(x)g(y)\delta_{x+y} dxdy \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} f(x-y)g(y)\delta_x dxdy \\ &= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} f(x-y)g(y)dy\right]\delta_x dx \end{aligned}$$

which suggests we make the following definition.

Definition 7.1. Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be measurable functions. We define

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

whenever the integral is defined, i.e. either $f(x-\cdot)g(\cdot) \in L^1(\mathbb{R}^n, m)$ or $f(x-\cdot)g(\cdot) \geq 0$. Notice that the condition that $f(x-\cdot)g(\cdot) \in L^1(\mathbb{R}^n, m)$ is equivalent to writing $|f| * |g|(x) < \infty$.

Notation 7.2. Given a multi-index $\alpha \in \mathbb{Z}_+^n$, let $|\alpha| = \alpha_1 + \cdots + \alpha_n$,

$$x^\alpha := \prod_{j=1}^n x_j^{\alpha_j}, \text{ and } \partial_x^\alpha = \left(\frac{\partial}{\partial x}\right)^\alpha := \prod_{j=1}^n \left(\frac{\partial}{\partial x_j}\right)^{\alpha_j}.$$

Remark 7.3 (The Significance of Convolution). Suppose that $L = \sum_{|\alpha|\leq k} a_\alpha \partial^\alpha$ is a constant coefficient differential operator and suppose that we can solve (uniquely) the equation $Lu = g$ in the form

$$u(x) = Kg(x) := \int_{\mathbb{R}^n} k(x,y)g(y)dy$$

where $k(x,y)$ is an "integral kernel." (This is a natural sort of assumption since, in view of the fundamental theorem of calculus, integration is the inverse operation to differentiation.) Since $\tau_z L = L\tau_z$ for all $z \in \mathbb{R}^n$, (this is another way to characterize constant coefficient differential operators) and $L^{-1} = K$ we should have $\tau_z K = K\tau_z$. Writing out this equation then says

$$\begin{aligned} \int_{\mathbb{R}^n} k(x-z,y)g(y)dy &= (Kg)(x-z) = \tau_z Kg(x) = (K\tau_z g)(x) \\ &= \int_{\mathbb{R}^n} k(x,y)g(y-z)dy = \int_{\mathbb{R}^n} k(x,y+z)g(y)dy. \end{aligned}$$

Since g is arbitrary we conclude that $k(x-z,y) = k(x,y+z)$. Taking $y = 0$ then gives

$$k(x,z) = k(x-z,0) =: \rho(x-z).$$

We thus find that $Kg = \rho * g$. Hence we expect the convolution operation to appear naturally when solving constant coefficient partial differential equations. More about this point later.

The following proposition is an easy consequence of Minkowski's inequality for integrals.

Proposition 7.4. *Suppose $q \in [1, \infty]$, $f \in L^1$ and $g \in L^q$, then $f * g(x)$ exists for almost every x, $f * g \in L^q$ and*

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p .$$

For $z \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{C}$, let $\tau_z f : \mathbb{R}^n \to \mathbb{C}$ be defined by $\tau_z f(x) = f(x - z)$.

Proposition 7.5. *Suppose that $p \in [1, \infty)$, then $\tau_z : L^p \to L^p$ is an isometric isomorphism and for $f \in L^p$, $z \in \mathbb{R}^n \to \tau_z f \in L^p$ is continuous.*

Proof. The assertion that $\tau_z : L^p \to L^p$ is an isometric isomorphism follows from translation invariance of Lebesgue measure and the fact that $\tau_{-z} \circ \tau_z = id$. For the continuity assertion, observe that

$$\|\tau_z f - \tau_y f\|_p = \|\tau_{-y} (\tau_z f - \tau_y f)\|_p = \|\tau_{z-y} f - f\|_p$$

from which it follows that it is enough to show $\tau_z f \to f$ in L^p as $z \to 0 \in \mathbb{R}^n$.

When $f \in C_c(\mathbb{R}^n)$, $\tau_z f \to f$ uniformly and since the $K := \cup_{|z| \leq 1} \operatorname{supp}(\tau_z f)$ is compact, it follows by the dominated convergence theorem that $\tau_z f \to f$ in L^p as $z \to 0 \in \mathbb{R}^n$. For general $g \in L^p$ and $f \in C_c(\mathbb{R}^n)$,

$$\|\tau_z g - g\|_p \leq \|\tau_z g - \tau_z f\|_p + \|\tau_z f - f\|_p + \|f - g\|_p = \|\tau_z f - f\|_p + 2 \|f - g\|_p$$

and thus

$$\limsup_{z \to 0} \|\tau_z g - g\|_p \leq \limsup_{z \to 0} \|\tau_z f - f\|_p + 2 \|f - g\|_p = 2 \|f - g\|_p .$$

Because $C_c(\mathbb{R}^n)$ is dense in L^p, the term $\|f - g\|_p$ may be made as small as we please. ■

Definition 7.6. Suppose that (X, τ) is a topological space and μ is a measure on $\mathcal{B}_X = \sigma(\tau)$. For a measurable function $f : X \to \mathbb{C}$ we define the essential support of f by

(7.1)

$$\operatorname{supp}_\mu(f) = \{x \in U : \mu(\{y \in V : f(y) \neq 0\}) > 0 \text{ for all neighborhoods } V \text{ of } x\}.$$

Lemma 7.7. *Suppose (X, τ) is second countable and $f : X \to \mathbb{C}$ is a measurable function and μ is a measure on $\mathcal{B}_X$. Then $X := U \setminus \operatorname{supp}_\mu(f)$ may be described as the largest open set W such that $f 1_W(x) = 0$ for μ – a.e. x. Equivalently put, $C := \operatorname{supp}_\mu(f)$ is the smallest closed subset of X such that $f = f 1_C$ a.e.*

Proof. To verify that the two descriptions of $\operatorname{supp}_\mu(f)$ are equivalent, suppose $\operatorname{supp}_\mu(f)$ is defined as in Eq. (7.1) and $W := X \setminus \operatorname{supp}_\mu(f)$. Then

$$\begin{aligned} W &= \{x \in X : \mu(\{y \in V : f(y) \neq 0\}) = 0 \text{ for some neighborhood } V \text{ of } x\} \\ &= \cup \{V \subset_o X : \mu(f 1_V \neq 0) = 0\} \\ &= \cup \{V \subset_o X : f 1_V = 0 \text{ for } \mu - \text{a.e.}\} . \end{aligned}$$

So to finish the argument it suffices to show $\mu(f1_W \neq 0) = 0$. To to this let $\mathcal{U}$ be a countable base for τ and set

$$\mathcal{U}_f := \{V \in \mathcal{U} : f1_V = 0 \text{ a.e.}\}.$$

Then it is easily seen that $W = \cup \mathcal{U}_f$ and since $\mathcal{U}_f$ is countable $\mu(f1_W \neq 0) \leq \sum_{V \in \mathcal{U}_f} \mu(f1_V \neq 0) = 0$. ■

Lemma 7.8. *Suppose $f, g, h : \mathbb{R}^n \to \mathbb{C}$ are measurable functions and assume that x is a point in $\mathbb{R}^n$ such that $|f| * |g|(x) < \infty$ and $|f| * (|g| * |h|)(x) < \infty$, then*

1. $f * g(x) = g * f(x)$
2. $f * (g * h)(x) = (f * g) * h(x)$
3. *If $z \in \mathbb{R}^n$ and $\tau_z(|f| * |g|)(x) = |f| * |g|(x - z) < \infty$, then*
$$\tau_z(f * g)(x) = \tau_z f * g(x) = f * \tau_z g(x)$$
4. *If $x \notin \overline{\operatorname{supp}_m(f) + \operatorname{supp}_m(g)}$ then $f * g(x) = 0$ and in particular, $\operatorname{supp}_m(f * g) \subset \overline{\operatorname{supp}_m(f) + \operatorname{supp}_m(g)}$ where in defining $\operatorname{supp}_m(f * g)$ we will use the convention that "$f * g(x) \neq 0$" when $|f| * |g|(x) = \infty$.*

Proof. For item 1.,

$$|f| * |g|(x) = \int_{\mathbb{R}^n} |f|(x - y)|g|(y)dy = \int_{\mathbb{R}^n} |f|(y)|g|(y - x)dy = |g| * |f|(x)$$

where in the second equality we made use of the fact that Lebesgue measure invariant under the transformation $y \to x - y$. Similar computations prove all of the remaining assertions of the first three items of the lemma.

Item 4. Since $f * g(x) = \tilde{f} * \tilde{g}(x)$ if $f = \tilde{f}$ and $g = \tilde{g}$ a.e. we may, by replacing f by $f1_{\operatorname{supp}_m(f)}$ and g by $g1_{\operatorname{supp}_m(g)}$ if necessary, assume that $\{f \neq 0\} \subset \operatorname{supp}_m(f)$ and $\{g \neq 0\} \subset \operatorname{supp}_m(g)$. So if $x \notin (\operatorname{supp}_m(f) + \operatorname{supp}_m(g))$ then $x \notin (\{f \neq 0\} + \{g \neq 0\})$ and for all $y \in \mathbb{R}^n$, either $x - y \notin \{f \neq 0\}$ or $y \notin \{g \neq 0\}$. That is to say either $x - y \in \{f = 0\}$ or $y \in \{g = 0\}$ and hence $f(x - y)g(y) = 0$ for all y and therefore $f * g(x) = 0$. This shows that $f * g = 0$ on $\mathbb{R}^n \setminus \left(\overline{\operatorname{supp}_m(f) + \operatorname{supp}_m(g)}\right)$ and therefore

$$\mathbb{R}^n \setminus \left(\overline{\operatorname{supp}_m(f) + \operatorname{supp}_m(g)}\right) \subset \mathbb{R}^n \setminus \operatorname{supp}_m(f * g),$$

i.e. $\operatorname{supp}_m(f * g) \subset \overline{\operatorname{supp}_m(f) + \operatorname{supp}_m(g)}$. ■

Remark 7.9. Let A, B be closed sets of $\mathbb{R}^n$, it is not necessarily true that $A + B$ is still closed. For example, take

$$A = \{(x, y) : x > 0 \text{ and } y \geq 1/x\} \text{ and } B = \{(x, y) : x < 0 \text{ and } y \geq 1/|x|\},$$

then every point of $A + B$ has a positive y - component and hence is not zero. On the other hand, for $x > 0$ we have $(x, 1/x) + (-x, 1/x) = (0, 2/x) \in A + B$ for all x and hence $0 \in \overline{A + B}$ showing $A + B$ is not closed. Nevertheless if one of the sets A or B is compact, then $A + B$ is closed again. Indeed, if A is compact and $x_n = a_n + b_n \in A + B$ and $x_n \to x \in \mathbb{R}^n$, then by passing to a subsequence if necessary we may assume $\lim_{n\to\infty} a_n = a \in A$ exists. In this case

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} (x_n - a_n) = x - a \in B$$

exists as well, showing $x = a + b \in A + B$.

Proposition 7.10. *Suppose that $p, q \in [1, \infty]$ and p and q are conjugate exponents, $f \in L^p$ and $g \in L^q$, then $f * g \in BC(\mathbb{R}^n)$, $\|f * g\|_u \leq \|f\|_p \|g\|_q$ and if $p, q \in (1, \infty)$ then $f * g \in C_0(\mathbb{R}^n)$.*

Proof. The existence of $f * g(x)$ and the estimate $|f * g|(x) \leq \|f\|_p \|g\|_q$ for all $x \in \mathbb{R}^n$ is a simple consequence of Holders inequality and the translation invariance of Lebesgue measure. In particular this shows $\|f * g\|_u \leq \|f\|_p \|g\|_q$. By relabeling p and q if necessary we may assume that $p \in [1, \infty)$. Since

$$\|\tau_z (f * g) - f * g\|_u = \|\tau_z f * g - f * g\|_u \leq \|\tau_z f - f\|_p \|g\|_q \to 0 \text{ as } z \to 0$$

it follows that $f * g$ is uniformly continuous. Finally if $p, q \in (1, \infty)$, we learn from Lemma 7.8 and what we have just proved that $f_m * g_m \in C_c(\mathbb{R}^n)$ where $f_m = f 1_{|f| \leq m}$ and $g_m = g 1_{|g| \leq m}$. Moreover,

$$\begin{aligned} \|f * g - f_m * g_m\|_u &\leq \|f * g - f_m * g\|_u + \|f_m * g - f_m * g_m\|_u \\ &\leq \|f - f_m\|_p \|g\|_q + \|f_m\|_p \|g - g_m\|_q \\ &\leq \|f - f_m\|_p \|g\|_q + \|f\|_p \|g - g_m\|_q \to 0 \text{ as } m \to \infty \end{aligned}$$

showing $f * g \in C_0(\mathbb{R}^n)$. ■

Theorem 7.11 (Young's Inequality)**.** *Let $p, q, r \in [1, \infty]$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{7.2}$$

If $f \in L^p$ and $g \in L^q$ then $|f| * |g|(x) < \infty$ for m – a.e. x and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q. \tag{7.3}$$

In particular L^1 is closed under convolution. (The space $(L^1, *)$ is an example of a "Banach algebra" without unit.)

Remark 7.12. Before going to the formal proof, let us first understand Eq. (7.2) by the following scaling argument. For $\lambda > 0$, let $f_\lambda(x) := f(\lambda x)$, then after a few simple change of variables we find

$$\|f_\lambda\|_p = \lambda^{-1/p} \|f\| \text{ and } (f * g)_\lambda = \lambda f_\lambda * g_\lambda.$$

Therefore if Eq. (7.3) holds for some $p, q, r \in [1, \infty]$, we would also have

$$\|f * g\|_r = \lambda^{1/r} \|(f * g)_\lambda\|_r \leq \lambda^{1/r} \lambda \|f_\lambda\|_p \|g_\lambda\|_q = \lambda^{(1 + 1/r - 1/p - 1/q)} \|f\|_p \|g\|_q$$

for all $\lambda > 0$. This is only possible if Eq. (7.2) holds.

Proof. Let $\alpha, \beta \in [0, 1]$ and $p_1, p_2 \in [0, \infty]$ satisfy $p_1^{-1} + p_2^{-1} + r^{-1} = 1$. Then by Hölder's inequality,

$$\begin{aligned} |f * g(x)| &= \left| \int f(x - y) g(y) dy \right| \leq \int |f(x - y)|^{(1-\alpha)} |g(y)|^{(1-\beta)} |f(x - y)|^\alpha |g(y)|^\beta dy \\ &\leq \left(\int |f(x - y)|^{(1-\alpha)r} |g(y)|^{(1-\beta)r} dy \right)^{1/r} \left(\int |f(x - y)|^{\alpha p_1} dy \right)^{1/p_1} \left(\int |g(y)|^{\beta p_2} dy \right)^{1/p_2} \\ &= \left(\int |f(x - y)|^{(1-\alpha)r} |g(y)|^{(1-\beta)r} dy \right)^{1/r} \|f\|_{\alpha p_1}^\alpha \|g\|_{\beta p_2}^\beta. \end{aligned}$$

Taking the r^{th} power of this equation and integrating on x gives

$$\|f*g\|_r^r \le \int \left(\int |f(x-y)|^{(1-\alpha)r} |g(y)|^{(1-\beta)r} dy \right) dx \cdot \|f\|_{\alpha p_1}^{\alpha} \|g\|_{\beta p_2}^{\beta}$$

$$= \|f\|_{(1-\alpha)r}^{(1-\alpha)r} \|g\|_{(1-\beta)r}^{(1-\beta)r} \|f\|_{\alpha p_1}^{\alpha r} \|g\|_{\beta p_2}^{\beta r}. \tag{7.4}$$

Let us now suppose, $(1-\alpha)r = \alpha p_1$ and $(1-\beta)r = \beta p_2$, in which case Eq. (7.4) becomes,

$$\|f*g\|_r^r \le \|f\|_{\alpha p_1}^r \|g\|_{\beta p_2}^r$$

which is Eq. (7.3) with

$$p := (1-\alpha)r = \alpha p_1 \text{ and } q := (1-\beta)r = \beta p_2. \tag{7.5}$$

So to finish the proof, it suffices to show p and q are arbitrary indices in $[1,\infty]$ satisfying $p^{-1} + q^{-1} = 1 + r^{-1}$.

If α, β, p_1, p_2 satisfy the relations above, then

$$\alpha = \frac{r}{r+p_1} \text{ and } \beta = \frac{r}{r+p_2}$$

and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{p_1}\frac{r+p_1}{r} + \frac{1}{p_2}\frac{r+p_2}{r} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{2}{r} = 1 + \frac{1}{r}.$$

Conversely, if p, q, r satisfy Eq. (7.2), then let α and β satisfy $p = (1-\alpha)r$ and $q = (1-\beta)r$, i.e.

$$\alpha := \frac{r-p}{r} = 1 - \frac{p}{r} \le 1 \text{ and } \beta = \frac{r-q}{r} = 1 - \frac{q}{r} \le 1.$$

From Eq. (7.2), $\alpha = p(1-\frac{1}{q}) \ge 0$ and $\beta = q(1-\frac{1}{p}) \ge 0$, so that $\alpha, \beta \in [0,1]$. We then define $p_1 := p/\alpha$ and $p_2 := q/\beta$, then

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{r} = \beta\frac{1}{q} + \alpha\frac{1}{p} + \frac{1}{r} = \frac{1}{q} - \frac{1}{r} + \frac{1}{p} - \frac{1}{r} + \frac{1}{r} = 1$$

as desired. ■

Theorem 7.13 (Approximate δ – functions)**.** *Let $p \in [1,\infty]$, $\phi \in L^1(\mathbb{R}^n)$, $a := \int_{\mathbb{R}^n} f(x)dx$, and for $t > 0$ let $\phi_t(x) = t^{-n}\phi(x/t)$. Then*

1. *If $f \in L^p$ with $p < \infty$ then $\phi_t * f \to af$ in L^p as $t \downarrow 0$.*
2. *If $f \in BC(\mathbb{R}^n)$ and f is uniformly continuous then $\|\phi_t * f - f\|_\infty \to 0$ as $t \downarrow 0$.*
3. *If $f \in L^\infty$ and f is continuous on $U \subset_o \mathbb{R}^n$ then $\phi_t * f \to af$ uniformly on compact subsets of U as $t \downarrow 0$.*

See Theorem 8.15 if Folland for a statement about almost everywhere convergence.

Proof. Making the change of variables $y = tz$ implies

$$\phi_t * f(x) = \int_{\mathbb{R}^n} f(x-y)\phi_t(y)dy = \int_{\mathbb{R}^n} f(x-tz)\phi(z)dz$$

so that

$$\phi_t * f(x) - af(x) = \int_{\mathbb{R}^n} [f(x-tz) - f(x)]\,\phi(z)dz$$

$$= \int_{\mathbb{R}^n} [\tau_{tz}f(x) - f(x)]\,\phi(z)dz. \tag{7.6}$$

Hence by Minkowski's inequality for integrals, Proposition 7.5 and the dominated convergence theorem,

$$\|\phi_t * f - af\|_p \le \int_{\mathbb{R}^n} \|\tau_{tz} f - f\|_p \, |\phi(z)| \, dz \to 0 \text{ as } t \downarrow 0.$$

Item 2. is proved similarly. Indeed, form Eq. (7.6)

$$\|\phi_t * f - af\|_\infty \le \int_{\mathbb{R}^n} \|\tau_{tz} f - f\|_\infty \, |\phi(z)| \, dz$$

which again tends to zero by the dominated convergence theorem because $\lim_{t\downarrow 0} \|\tau_{tz} f - f\|_\infty = 0$ uniformly in z by the uniform continuity of f.

Item 3. Let $B_R = B(0, R)$ be a large ball in $\mathbb{R}^n$ and $K \sqsubset\sqsubset U$, then

$$\begin{aligned}
\sup_{x\in K} |\phi_t * f(x) - af(x)| &\le \left| \int_{B_R} [f(x - tz) - f(x)] \, \phi(z) dz \right| + \left| \int_{B_R^c} [f(x - tz) - f(x)] \, \phi(z) dz \right| \\
&\le \int_{B_R} |\phi(z)| \, dz \cdot \sup_{x\in K, z\in B_R} |f(x - tz) - f(x)| + 2 \|f\|_\infty \int_{B_R^c} |\phi(z)| \, dz \\
&\le \|\phi\|_1 \cdot \sup_{x\in K, z\in B_R} |f(x - tz) - f(x)| + 2 \|f\|_\infty \int_{|z|>R} |\phi(z)| \, dz
\end{aligned}$$

so that using the uniform continuity of f on compact subsets of U,

$$\limsup_{t\downarrow 0} \sup_{x\in K} |\phi_t * f(x) - af(x)| \le 2 \|f\|_\infty \int_{|z|>R} |\phi(z)| \, dz \to 0 \text{ as } R \to \infty.$$

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Remark 7.14 (Another Proof of part of Theorem 7.13). By definition of the convolution and Hölder's or Jensen's inequality we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |v * \phi_t(x)|^p dx &\le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (v(x - y)) |\phi_t(y)| dy \right)^p dx \\
&\le \int_{\mathbb{R}^n \times \mathbb{R}^n} |v(x - y)|^p \phi_t(y) dy \, dx = \|v\|_{L^p}^p.
\end{aligned}$$

Therefore $\|v * \phi_t\|_{L^p} \le \|v\|_{L^p}$ which implies $v * \phi_t \in L^p$. If $\phi_t \in C_c^\infty(\mathbb{R}^n)$, by differentiating under the integral (see Theorem 7.17 below) it is easily seen that $v * \phi_t \in C^\infty$. Finally for $u \in C_c(\mathbb{R}^n)$,

$$\begin{aligned}
\|v - v * \phi_t\|_{L^p} &\le \|v - u\|_{L^p} + \|u - u * \phi_t\|_{L^p} + \|u * \phi_t - v * \phi_t\|_{L^p} \\
&\le \|u - u * \phi_t\|_{L^p} + 2\|v - u\|_{L^p}
\end{aligned}$$

and hence

$$\limsup_{t\downarrow 0} \|v - v * \phi_t\|_{L^p} \le 2\|v - u\|_{L^p}$$

which may be made arbitrarily small since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, m)$.

Exercise 7.1. Let

$$f(t) = \begin{cases} e^{-1/t} & \text{if } \; t > 0 \\ 0 & \text{if } \; t \le 0. \end{cases}$$

Show $f \in C^\infty(\mathbb{R}, [0, 1])$.

Lemma 7.15. *There exists* $\phi \in C_c^\infty(\mathbb{R}^n, [0, \infty))$ *such that* $\phi(0) > 0$, $\operatorname{supp}(\phi) \subset \bar{B}(0, 1)$ *and* $\int_{\mathbb{R}^n} \phi(x) dx = 1$.

Proof. Define $h(t) = f(1-t)f(t+1)$ where f is as in Exercise 7.1. Then $h \in C_c^\infty(\mathbb{R},[0,1])$, $\operatorname{supp}(h) \subset [-1,1]$ and $h(0) = e^{-2} > 0$. Define $c = \int_{\mathbb{R}^n} h(|x|^2)dx$. Then $\phi(x) = c^{-1}h(|x|^2)$ is the desired function. ■

Definition 7.16. Let $X \subset \mathbb{R}^n$ be an open set. A **Radon** measure on $\mathcal{B}_X$ is a measure μ which is finite on compact subsets of X. For a Radon measure μ, we let $L^1_{loc}(\mu)$ consists of those measurable functions $f : X \to \mathbb{C}$ such that $\int_K |f|\, d\mu < \infty$ for all compact subsets $K \subset X$.

Theorem 7.17 (Differentiation under integral sign)**.** *Let $\Omega \subset \mathbb{R}^n$ and $f : \mathbb{R}^m \times \Omega \to \mathbb{R}$ be given.* ***Assume:***

(1) *$x \to f(x,y)$ is differentiable for all $y \in \Omega$.*
(2) *$\left|\frac{\partial f}{\partial x^i}\ (x,y)\right| \le g(y)$ for some g such that $\int\limits_\Omega |g(y)|dy < \infty$.*
(3) *$\int |f(x,y)|dy < \infty$.*

Then $\frac{\partial}{\partial x^i} \int\limits_\Omega f(x,y)dy = \int\limits_\Omega \frac{\partial f}{\partial x^i}\ (x,y)dy$ and moreover if $x \to \frac{\partial f}{\partial x^i}(x,y)$ is continuous then so is $x \to \int \frac{\partial f}{\partial x^i}\ (x,y)dy$.

The reader asked to use Theorem 7.17 to verify the following proposition.

Proposition 7.18. *Suppose that $f \in L^1_{loc}(\mathbb{R}^n, m)$ and $\phi \in C^1_c(\mathbb{R}^n)$, then $f * \phi \in C^1(\mathbb{R}^n)$ and $\partial_i(f*\phi) = f * \partial_i\phi$. Moreover if $\phi \in C_c^\infty(\mathbb{R}^n)$ then $f*\phi \in C^\infty(\mathbb{R}^n)$.*

Corollary 7.19 (C^∞ – Uryhson's Lemma)**.** *Given $K \sqsubset\sqsubset U \subset_o \mathbb{R}^n$, there exists $f \in C_c^\infty(\mathbb{R}^n,[0,1])$ such that* $\operatorname{supp}(f) \subset U$ *and $f = 1$ on K.*

Proof. Let ϕ be as in Lemma 7.15, $\phi_t(x) = t^{-n}\phi(x/t)$ be as in Theorem 7.13, d be the standard metric on $\mathbb{R}^n$ and $\epsilon = d(K,U^c)$. Since K is compact and U^c is closed, $\epsilon > 0$. Let $V_\delta = \{x \in \mathbb{R}^n : d(x,K) < \delta\}$ and $f = \phi_{\epsilon/3} * 1_{V_{\epsilon/3}}$, then

$$\operatorname{supp}(f) \subset \overline{\operatorname{supp}(\phi_{\epsilon/3}) + V_{\epsilon/3}} \subset \bar{V}_{2\epsilon/3} \subset U.$$

Since $\bar{V}_{2\epsilon/3}$ is closed and bounded, $f \in C_c^\infty(U)$ and for $x \in K$,

$$f(x) = \int_{\mathbb{R}^n} 1_{d(y,K)<\epsilon/3} \cdot \phi_{\epsilon/3}(x-y)dy = \int_{\mathbb{R}^n} \phi_{\epsilon/3}(x-y)dy = 1.$$

The proof will be finished after the reader (easily) verifies $0 \le f \le 1$. ■

Here is an application of this corollary whose proof is left to the reader.

Lemma 7.20 (Integration by Parts)**.** *Suppose f and g are measurable functions on $\mathbb{R}^n$ such that $t \to f(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)$ and $t \to g(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)$ are continuously differentiable functions on $\mathbb{R}$ for each fixed $x = (x_1,\dots,x_n) \in \mathbb{R}^n$. Moreover assume $f \cdot g$, $\frac{\partial f}{\partial x_i} \cdot g$ and $f \cdot \frac{\partial g}{\partial x_i}$ are in $L^1(\mathbb{R}^n, m)$. Then*

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i} \cdot g dm = -\int_{\mathbb{R}^n} f \cdot \frac{\partial g}{\partial x_i} dm.$$

With this result we may give another proof of the Riemann Lebesgue Lemma.

Lemma 7.21. *For $f \in L^1(\mathbb{R}^n, m)$ let*

$$\hat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-i\xi\cdot x}dm(x)$$

be the Fourier transform of f. Then $\hat{f} \in C_0(\mathbb{R}^n)$ and $\left\|\hat{f}\right\|_u \le (2\pi)^{-n/2}\|f\|_1$. (The choice of the normalization factor, $(2\pi)^{-n/2}$, in $\hat{f}$ is for later convenience.)

Proof. The fact that $\hat{f}$ is continuous is a simple application of the dominated convergence theorem. Moreover,

$$\left|\hat{f}(\xi)\right| \leq \int |f(x)|\, dm(x) \leq (2\pi)^{-n/2} \|f\|_1$$

so it only remains to see that $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

First suppose that $f \in C_c^\infty(\mathbb{R}^n)$ and let $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ be the Laplacian on $\mathbb{R}^n$. Notice that $\frac{\partial}{\partial x_j} e^{-i\xi \cdot x} = -i\xi_j e^{-i\xi\cdot x}$ and $\Delta e^{-i\xi\cdot x} = -|\xi|^2 e^{-i\xi\cdot x}$. Using Lemma 7.20 repeatedly,

$$\begin{aligned}\int \Delta^k f(x) e^{-i\xi\cdot x} dm(x) &= \int f(x) \Delta_x^k e^{-i\xi\cdot x} dm(x) = -|\xi|^{2k} \int f(x) e^{-i\xi\cdot x} dm(x) \\ &= -(2\pi)^{n/2} |\xi|^{2k} \hat{f}(\xi)\end{aligned}$$

for any $k \in \mathbb{N}$. Hence $(2\pi)^{n/2}\left|\hat{f}(\xi)\right| \leq |\xi|^{-2k} \left\|\Delta^k f\right\|_1 \to 0$ as $|\xi| \to \infty$ and $\hat{f} \in C_0(\mathbb{R}^n)$. Suppose that $f \in L^1(m)$ and $f_k \in C_c^\infty(\mathbb{R}^n)$ is a sequence such that $\lim_{k\to\infty} \|f - f_k\|_1 = 0$, then $\lim_{k\to\infty} \left\|\hat{f} - \hat{f}_k\right\|_u = 0$ and hence Hence $\hat{f} \in C_0(\mathbb{R}^n)$ because $C_0(\mathbb{R}^n)$ is complete. ■

Corollary 7.22. *Let $X \subset \mathbb{R}^n$ be an open set and μ be a Radon measure on $\mathcal{B}_X$.*

(1) *Then $C_c^\infty(X)$ is dense in $L^p(\mu)$ for all $1 \leq p < \infty$.*
(2) *If $h \in L^1_{loc}(\mu)$ satisfies*

$$\int_X fh d\mu = 0 \text{ for all } f \in C_c^\infty(X) \tag{7.7}$$

then $h(x) = 0$ for μ – a.e. x.

Proof. Let $f \in C_c(X)$, ϕ be as in Lemma 7.15, ϕ_t be as in Theorem 7.13 and set $\psi_t := \phi_t * (f 1_X)$. Then by Proposition 7.18 $\psi_t \in C^\infty(X)$ and by Lemma 7.8 there exists a compact set $K \subset X$ such that $\text{supp}(\psi_t) \subset K$ for all t sufficiently small. By Theorem 7.13, $\psi_t \to f$ uniformly on X as $t \downarrow 0$

(1) The dominated convergence theorem (with dominating function being $\|f\|_\infty 1_K$), shows $\psi_t \to f$ in $L^p(\mu)$ as $t \downarrow 0$. This proves Item 1. because of the measure theoretic fact that $C_c(X)$ is dense in $L^p(\mu)$.
(2) Keeping the same notation as above, the dominated convergence theorem (with dominating function being $\|f\|_\infty |h| 1_K$) implies

$$0 = \lim_{t\downarrow 0} \int_X \psi_t h d\mu = \int_X \lim_{t\downarrow 0} \psi_t h d\mu = \int_X fh d\mu.$$

Since this is true for all $f \in C_c(X)$, it follows by measure theoretic arguments that $h = 0$ a.e.

■

7.2. **Smooth Partitions of Unity.**

Theorem 7.23. *Let $V_1, \dots, V_k \subset_0 \mathbb{R}^n$ and $\phi \in C_c^\infty\left(\cup_{i=1}^k V_i\right)$. Then there exists $\phi_j \in C_c^\infty(V_j)$ such that $\phi = \sum_i^k \phi_j$. If $\phi \geq 0$ one can choose $\phi_j \geq 0$.*

Proof. The proof will be by a number of steps.

(1) There exists $K_j \sqsubset\sqsubset V_j$ such that $\operatorname{supp}\phi \subset \cup K_j$. Indeed, for all $x \in \operatorname{supp}\phi$ there exists an open neighborhood N_x of x such that $\overline{N}_x \subset V_j$ for some j and $\overline{N}_x$ is compact. Now $\{N_x\}_{x\in\operatorname{supp}\phi}$ covers $K := \operatorname{supp}\phi$ and hence there exists a finite set $\Lambda \subset\subset K$ such that $K \subset \cup_{x\in\Lambda} N_x$.. Let $K_j := \cup\left\{\overline{N}_x : x \in \Lambda \text{ and } \overline{N}_x \subset V_j\right\}$. Then each K_j is compact, $K_j \subset V_j$ and $\operatorname{supp}\phi = K \subset \bigcup_{j=1}^{k} K_j$.

(2) By Corollary 7.19 there exists $\psi_j \in C_c^\infty(V_j, [0,1])$ such that $\psi_j := 1$ in the neighborhood of K_j. Now define

$$
\begin{aligned}
\phi_1 &= \phi\psi_1\\
\phi_2 &= (\phi - \phi_1)\psi_2 = \phi(1-\psi_1)\psi_2\\
\phi_3 &= (\phi - \phi_1 - \phi_2)\psi_3 = \phi\{(1-\psi_1) - (1-\psi_1)\psi_2\}\psi_3\\
&= \phi(1-\psi_1)(1-\psi_2)\psi_3\\
&\vdots\\
\phi_k &= (\phi - \phi_1 - \phi_2 - \cdots - \phi_{k-1})\psi_k = \phi(1-\psi_1)(1-\psi_2)\ldots(1-\psi_{k-1})\psi_k
\end{aligned}
$$

By the above computations one finds that (a) $\phi_i \geq 0$ if $\phi \geq 0$ and (b)

$$\phi - \phi_1 - \phi_2 - \cdots - \phi_k = \phi(1-\psi_1)(1-\psi_2)\ldots(1-\psi_k) = 0.$$

since either $\phi(x) = 0$ or $x \notin \operatorname{supp}\phi = K$ and $1 - \psi_i(x) = 0$ for some i.

■

Corollary 7.24. *Let $V_1, \ldots, V_k \subset_0 \mathbb{R}^n$ and K be a compact subset of $\cup_{i=1}^k V_i$. Then there exists $\phi_i \in C_c^\infty(V_i, [0,1])$ such $\sum_{i=1}^k \phi_i \leq 1$ with $\sum_{i=1}^k \phi_i = 1$ on a neighborhood of K.*

Proof. By Corollary 7.19 there exists $\phi \in C_c^\infty(\cup_{i=1}^k V_i, [0,1])$ such that $\phi = 1$ on a neighborhood of K. Now let $\{\phi_i\}_{i=1}^k$ be the functions constructed in Theorem 7.23. ■