9. Poisson and Laplace's Equation

For the majority of this section we will assume $\Omega \subset \mathbb{R}^n$ is a compact manifold with C^2 – boundary. Let us record a few consequences of the divergence theorem in Proposition 8.28 in this context. If $u, v \in C^2(\Omega^o) \cap C^1(\Omega)$ and $\int_\Omega |\triangle u| dx < \infty$ then

$$\int_\Omega \triangle u \cdot v dm = -\int_\Omega \nabla u \cdot \nabla v dm + \int_{\partial\Omega} v \frac{\partial u}{\partial n} d\sigma \tag{9.1}$$

and if further $\int_\Omega \{|\triangle u| + |\triangle v|\} dx < \infty$ then

$$\int_\Omega (\triangle uv - \triangle v\, u) dm = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} u \right) d\sigma. \tag{9.2}$$

Lemma 9.1. *Suppose $u \in C^2(\Omega^o) \cap C^1(\Omega)$, $\Delta u = 0$ on Ω^o and $u = 0$ on $\partial\Omega$. Then $u \equiv 0$. Similarly if $\Delta u = 0$ on Ω^o and $\partial_n u = 0$ on $\partial\Omega$, then u is constant on each connected component of Ω.*

Proof. Letting $v = u$ in Eq. (9.1) shows in either case that

$$0 = -\int_\Omega \nabla u \cdot \nabla u dm + \int_{\partial\Omega} u \frac{\partial u}{\partial n} d\sigma = -\int_\Omega |\nabla u|^2 \, dm.$$

This then implies $\nabla u = 0$ on Ω^o and hence u is constant on the connected component of Ω^o. If $u = 0$ on $\partial\Omega$, these constants must all be zero. ■

Proposition 9.2 (Laplacian on radial functions)**.** *Suppose $f(x) = F(|x|)$, then*

$$\triangle f(x) = \frac{1}{r^{n-1}} \frac{d}{dr}(r^{n-1} F'(r))\Big|_{r=|x|} = F''(|x|) + \frac{(n-1)}{|x|} F'(|x|). \tag{9.3}$$

In particular $\Delta F(|x|) = 0$ implies $\frac{d}{dr}(r^{n-1}F'(r)) = 0$ and hence $F'(r) = \tilde{A} r^{1-n}$. That is to say

$$F(r) = \begin{cases} Ar^{2-n} + B & \text{if} \quad n \neq 2 \\ A \ln r + B & \text{if} \quad n = 2. \end{cases}$$

Proof. Since $(\partial_v f)(x) = F'(|x|)\ \partial_v|x| = F'(|x|)\hat{x}\cdot v$ where $\hat{x} = \frac{x}{|x|}$, $\nabla f(x) = F'(|x|)\hat{x}$. Hence for $g \in C_c^1(\mathbb{R}^n)$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \triangle f(x) g(x) dx &= -\int_{\mathbb{R}^n} \nabla f(x)\cdot\nabla g(x)\ dx \\
&= -\int_{\mathbb{R}^n} F'(r)\hat{x}\cdot\nabla g(r\hat{x})\ dx \\
&= -\int_{S^{n-1}\times[0,\infty)} F'(r)\frac{d}{dr}g(r\omega)\ r^{n-1}dr\ d\sigma(\omega) \\
&= \int_{S^{n-1}\times[0,\infty)} \frac{d}{dr}(r^{n-1}F'(r))g(r\omega)dr\ d\sigma(\omega) \\
&= \int_{S^{n-1}\times[0,\infty)} \frac{1}{r^{n-1}}\ \frac{d}{dr}\ (r^{n-1}F'(r))\ g(r\omega)r^{n-1}\ dr\ d\sigma(\omega) \\
&= \int_{\mathbb{R}^n} \frac{1}{r^{n-1}}\ \frac{d}{dr}(r^{n-1}F'(r))\Big|_{r=|x|}\ g(x)\ dx.
\end{aligned}$$

Since this is valid for all $g \in C_c^1(\mathbb{R}^n)$, Eq. (9.3) is valid. Alternatively, we may simply compute directly as follows:

$$\begin{aligned}
\triangle f(x) &= \nabla\cdot[F'(|x|)\hat{x}] = \nabla F'(|x|)\cdot\hat{x} + F'(|x|)\nabla\cdot\hat{x} \\
&= F''(|x|)\hat{x}\cdot\hat{x} + F'(|x|)\nabla\cdot\frac{x}{|x|} = F''(|x|) + F'(|x|)\left\{\frac{n}{|x|} - \frac{x}{|x|^2}\cdot\hat{x}\right\} \\
&= F''(|x|) + \frac{(n-1)}{|x|}F'(|x|).
\end{aligned}$$

■

Notation 9.3. For $t > 0$, let

$$\alpha(t) := \alpha_n(t) := c_n \begin{cases} \frac{1}{t^{n-2}} & \text{if} \quad n \neq 2 \\ \ln t & \text{if} \quad n = 2, \end{cases} \tag{9.4}$$

where $c_n = \begin{cases} \frac{1}{(n-2)\sigma(S^{n-1})} & \text{if} \quad n \neq 2 \\ -\frac{1}{2\pi} & \text{if} \quad n = 2. \end{cases}$ Also let

$$\phi(y) = \phi_n(y) := \alpha(|y|) = c_n \begin{cases} \frac{1}{|y|^{n-2}} & \text{if} \quad n \neq 2 \\ \ln|y| & \text{if} \quad n = 2. \end{cases} \tag{9.5}$$

An important feature of α is that

$$\alpha'(t) = c_n \begin{cases} -(n-2)\frac{1}{t^{n-1}} & \text{if} \quad n \neq 2 \\ \frac{1}{t} & \text{if} \quad n = 2 \end{cases} = -\frac{1}{\sigma(S^{n-1})}\frac{1}{t^{n-1}} \tag{9.6}$$

for all n. This then implies, for all n, that

$$\nabla\phi(x) = \nabla[\alpha(|x|)] = \alpha'(|x|)\hat{x} = -\frac{1}{\sigma(S^{n-1})}\frac{1}{|x|^{n-1}}\hat{x} = -\frac{1}{\sigma(S^{n-1})}\frac{1}{|x|^n}x. \tag{9.7}$$

One more piece of notation will be useful in the sequel.

Notation 9.4 (Averaging operator)**.** Suppose μ is a finite measure on some space Ω, we will define

$$\fint_\Omega f d\mu := \frac{1}{\mu(\Omega)}\int_\Omega f d\mu.$$

For example if Ω is a compact manifold with C^2 – boundary in $\mathbb{R}^n$ then

$$\fint_\Omega f(x)dx = \frac{1}{m(\Omega)}\int_\Omega f(x)dx = \frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega f(x)dx$$

and

$$\fint_{\partial\Omega} f d\sigma = \frac{1}{\sigma(\partial\Omega)}\int_{\partial\Omega} f(x)dx = \frac{1}{\mathrm{Area}(\partial\Omega)}\int_{\partial\Omega} f(x)dx.$$

Theorem 9.5. *Let Ω be a compact manifold with C^2- boundary, $u \in C^2(\Omega^o) \cap C^1(\Omega)$ with $\int_\Omega |\Delta u(y)|\, dy < \infty$. Then for $x \in \Omega$*

$$u(x) = \int_{\partial\Omega}\left(\phi(x-y)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial\phi(x-y)}{\partial n_y}\right)d\sigma - \int_\Omega \phi(x-y)\triangle u(y)dy. \tag{9.8}$$

Proof. Let $\psi(y) := \phi(x-y)$ and $\epsilon > 0$ be small so that $B_x(\epsilon) \subset \Omega$ and let $\Omega_\epsilon := \Omega \setminus B_x(\epsilon)$, see Figure 27 below.

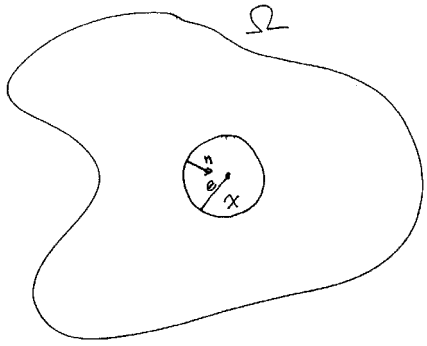

FIGURE 27. Removing the region where ψ is singular from Ω.

Let us begin by observing

$$\int_{|x-y|\le\epsilon} \psi(y)\, dy = \int_{|y|\le\epsilon} \frac{1}{|y|^{n-2}}\, dy = \sigma(S^{n-1})\int_0^\epsilon \frac{1}{r^{n-2}}\, r^{n-1}\, dr$$
$$= \sigma(S^{n-1})\int_0^\epsilon r\, dr = \sigma(S^{n-1})\frac{\epsilon^2}{2}$$

when $n \neq 2$ and for $n = 2$ that

$$\int_{|x-y|\le\epsilon} \psi(y)\, dy = \int_{|y|\le\epsilon} \ln|y|\ \, dy = \sigma(S^1)\int_0^\epsilon r\ln r\ dr$$
$$= 2\pi\left[\frac{1}{2}r^2\ln r - \frac{1}{4}r^2\right]_0^\epsilon = \pi\epsilon^2\left[\ln\epsilon - 1/2\right].$$

This shows $\psi \in L^1_{loc}(\Omega)$ and hence that $\psi\Delta u \in L^1(\Omega)$ and by dominated convergence theorem,

$$\int_\Omega \psi(y)\ \triangle u(y)\ dy = \lim_{\epsilon\downarrow 0}\int_{\Omega_\epsilon} \psi(y)\triangle u(y)\ dy.$$

Using Green's identity (Eq. (9.2) and Proposition 9.2) and $\Delta\psi = 0$ on Ω_ϵ, we find

$$\begin{aligned}\int_{\Omega_\epsilon} \triangle u(y)\psi(y)\, dy &= \int_{\Omega_\epsilon} \triangle\psi(y)u(y)\, dy + \int_{\partial\Omega_\epsilon} \left(\psi\frac{\partial u}{\partial n} - \frac{\partial\psi}{\partial n}u\right) d\sigma \\ &= \int_{\partial\Omega} \left(\psi\frac{\partial u}{\partial n} - \frac{\partial\psi}{\partial n}u\right) d\sigma + \int_{\partial\Omega_\epsilon\setminus\partial\Omega} \left(\psi\frac{\partial u}{\partial n} - \frac{\partial\psi}{\partial n}u\right) d\sigma. \end{aligned} \tag{9.9}$$

Working on the last term in Eq. (9.9) we have, for $n \neq 2$,

$$\begin{aligned}\int_{\partial B(x,\epsilon)} \psi(y)\, \frac{\partial u}{\partial \mathbf{n}}(y) &= \int_{|y|=\epsilon} \psi(x+\omega)\, \frac{\partial u}{\partial \mathbf{n}}(x+\omega) d\sigma(\omega) \\ &= \int_{|\omega|=1} \psi(x+\epsilon\omega)\, \frac{\partial u}{\partial \mathbf{n}}(x+\epsilon\omega)\epsilon^{n-1} d\sigma(\omega) \\ &= \int_{|\omega|=1} \frac{1}{\epsilon^{n-2}}\, \frac{\partial u}{\partial \mathbf{n}}(x+\epsilon\omega)\epsilon^{n-1} d\sigma(\omega) \\ &= \epsilon \int_{|\omega|} \frac{\partial u}{\partial \mathbf{n}}\, (x+\epsilon\omega) d\sigma(\omega) \to 0 \text{ as } \epsilon \downarrow 0. \end{aligned}$$

Similarly when $n = 2$,

$$\int_{\partial B(x,\epsilon)} \psi(y)\, \frac{\partial u}{\partial \mathbf{n}}(y) = \epsilon \ln \epsilon \int_{|\omega|=1} \frac{\partial u}{\partial \mathbf{n}}(x+\epsilon\omega) d\sigma(\omega) \to 0 \text{ as } \epsilon \downarrow 0.$$

Using Eq. (9.7) and $n(y) = -\widehat{(y-x)}$ as in Figure 28 we find

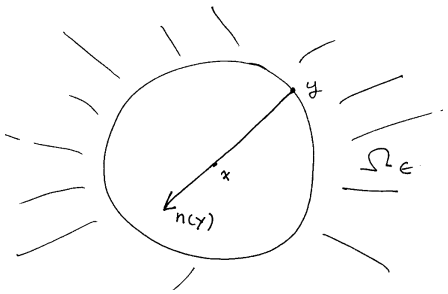

FIGURE 28. The outward normal to Ω_ϵ is the inward normal to $B(x,\epsilon)$.

$$\begin{aligned}\frac{\partial\psi}{\partial\mathbf{n}}(y) = \nabla_y\phi(y-x)\cdot n(y) &= -\frac{1}{\sigma(S^{n-1})}\frac{1}{|y-x|^n}\,(y-x)\cdot\left(-\widehat{(y-x)}\right) \\ &= \frac{1}{\sigma(S^{n-1})}\frac{1}{\epsilon^{n-1}} \end{aligned} \tag{9.10}$$

and therefore

$$-\int_{\partial\Omega_\epsilon\setminus\partial\Omega} u\,\frac{\partial\psi}{\partial\mathbf{n}}\,d\sigma(y) = -\frac{1}{\sigma(S^{n-1})}\frac{1}{\epsilon^{n-1}}\int_{\partial B(x,\epsilon)} u(y)d\sigma(y)$$
$$= -\fint_{|\omega|=1} u(x+\epsilon\omega)d\sigma(\omega) \to -u(x) \text{ as } \epsilon \downarrow 0$$

by the dominated convergence theorem. So we may pass to the limit in Eq. (9.9) to find

$$\int_\Omega \psi(y)\ \triangle u(y)dy = \int_{\partial\Omega}\left(\psi(y)\frac{\partial u}{\partial\mathbf{n}} - u\ \frac{\partial\psi}{\partial\mathbf{n}}\right)d\sigma(y) - u(x)$$

which is equivalent to Eq. (9.8). ■

The following Corollary gives an easy but useful extension of Theorem 9.5It will be us

Corollary 9.6. *Keeping the same notation as in Theorem 9.5. Further assume that $h \in C^2(\Omega^o)\cap C^1(\Omega)$ and $\Delta h = 0$ and set $G(y) := \phi(x-y)+h(y)$. Then we still have the representation formula*

$$u(x) = \int_{\partial\Omega}\left(G(y)\frac{\partial u}{\partial\mathbf{n}}(y) - u(y)\frac{\partial G(y)}{\partial\mathbf{n}}\right)d\sigma - \int_\Omega G(y)\triangle u(y)dy. \tag{9.11}$$

Proof. By Green's identity (Proposition 8.28) with $v = h$,

$$\int_\Omega \triangle u\ h dm = \int_\Omega (\triangle uh - \triangle h\ u)dm = \int_{\partial\Omega}\left(h\frac{\partial u}{\partial n} - \frac{\partial h}{\partial n}u\right)d\sigma,$$

i.e.

$$0 = -\int_\Omega \triangle u\ h\ dm + \int_{\partial\Omega}\left(h\frac{\partial u}{\partial n} - \frac{\partial h}{\partial n}u\right)d\sigma. \tag{9.12}$$

Eq. (9.11) now follows by adding Eqs. (9.8) and (9.12). ■

Corollary 9.7. *For all $u \in C_c^2(\mathbb{R}^n)$,*

$$-\int_{\mathbb{R}^n} \triangle u(y)\phi(y)dy = u(0). \tag{9.13}$$

Proof. Let $\Omega = B(0,R)$ where R is chosen so large that $\text{supp}(g)\subset\Omega$, then by Theorem 9.5,

$$u(0) = \int_{\partial\Omega}\left(\phi(y)\frac{\partial u}{\partial v}(y) - u(y)\frac{\partial\phi(y)}{\partial v_y}\right)d\sigma - \int_\Omega \phi(y)\triangle u(y)dy$$
$$= -\int_\Omega \phi(y)\triangle u(y)dy.$$

■

Remark 9.8. We summarize (9.13) by saying $-\triangle\phi = \delta$.

Formally we expect for reasonable functions ρ that

$$\Delta(\phi * \rho) = \Delta\phi * \rho = -\delta * \rho = -\rho.$$

Theorem 9.9. *Suppose* $\Omega \subset_o \mathbb{R}^n$, $\rho \in C^2(\Omega) \cap L^1(\Omega)$ *and*

$$u(x) := \int_\Omega \phi(x-y)\rho(y)dy = (\phi * 1_\Omega \rho)(x),$$

then

$$-\triangle u = \rho \text{ on } \Omega.$$

Proof. First assume that $\rho \in C_c^2(\Omega)$ in which case we may set $\rho := 1_\Omega \rho \in C_c^2(\mathbb{R}^n)$. Therefore

$$u(x) = \int_{\mathbb{R}^n} \rho(y) \frac{1}{|x-y|^{n-2}} dy = \int_{\mathbb{R}^n} \rho(x-y) \frac{1}{|y|^{n-2}} dy$$

and so we may differentiate under the integral to find

$$\triangle u(x) = \int_{\mathbb{R}^n} \triangle_x \rho(x-y) \ \frac{1}{|y|^{n-2}} \ dy = -\rho(x)$$

where the last equality follows from Corollary 9.7.

For $\rho \in C^2(\Omega) \cap L^1(\Omega)$ and $x_0 \in \Omega$, choose $\alpha \in C_c^\infty(\Omega, [0,1])$ such that $\alpha = 1$ in a neighborhood of x_0 and let $\beta := 1 - \alpha$. Then $u = (\phi * \alpha\rho) + (\phi * \beta 1_\Omega \rho)$ and so

$$\Delta u = \Delta(\phi * \alpha\rho) + \Delta(\phi * \beta 1_\Omega \rho). \tag{9.14}$$

By what we have just proved

$$\Delta(\phi * \alpha\rho)(x) = -(\alpha\rho)(x) = -\rho(x) \text{ for } x \text{ near } x_0. \tag{9.15}$$

Since $\beta = 0$ near x_0 and

$$(\phi * \beta 1_\Omega \rho)(x) = \int_\Omega \phi(x-y)\beta(y)\rho(y)dy,$$

we may differentiate past the integral to learn

$$\Delta(\phi * \beta 1_\Omega \rho)(x) = \int_\Omega \Delta_x \phi(x-y)\beta(y)\rho(y)dy = 0 \tag{9.16}$$

for x near x_0. and this completes the proof. The combination of Eqs. (9.14 – 9.16) completes the proof. ■

9.1. **Harmonic and Subharmonic Functions.**

Definition 9.10 (HarmonicFunctions)**.** Let $\Omega \subset_o \mathbb{R}^n$. A function $u \in C^2(\Omega)$ is said to be **harmonic (subharmonic)** on Ω if $\Delta u = 0$ ($\Delta u \geq 0$) on Ω.

Because of the Cauchy Riemann equations, the real and imaginary parts of holomorphic functions are harmonic. For example $z^2 = (x^2 - y^2) + 2ixy$ implies $(x^2 - y^2)$ and xy are harmonic functions on the plane. Similarly,

$$e^z = e^x \cos y + ie^x \sin y \text{ and}$$
$$\ln(z) = \ln r + i\theta$$

implies

$$e^x \cos y, \ e^x \sin y, \ \ \ln r, \text{ and } \theta(x,y)$$

are harmonic functions on their domains of definition.

Remark 9.11. If we can choose h in Corollary 9.6 so that $G = 0$ on $\partial\Omega$, then Eq. (9.11) gives

$$u(x) = -\int_\Omega G(y)\triangle u(y)dy - \int_{\partial\Omega} u\frac{\partial G(y)}{\partial v}d\sigma \tag{9.17}$$

which shows how to recover $u(x)$ from Δu on Ω and u on $\partial\Omega$. The next theorem is a consequence of this remark.

Theorem 9.12 (Mean Value Property)**.** *If $\triangle u = 0$ on Ω and $\overline{B(x,r)} \subset \Omega$ then*

$$u(x) = \frac{1}{\sigma(\partial B(x,r))}\int_{\partial B(x,r)} u(y)\,d\sigma(y) =: \fint_{\partial B(x,r)} u\ d\sigma \tag{9.18}$$

More generally if $\Delta u \geq 0$ on Ω, then

$$u(x) \leq \fint_{\partial B(x,r)} u\ d\sigma \tag{9.19}$$

Proof. For $y \in B(x,r)$,

$$G(y) = \phi(x-y) - \alpha(r) = \alpha\left(|x-y|\right) - \alpha(r)$$

where α is defined as in Eq. (9.4). Then $G(y) = 0$ for $y \in \partial B(x,r)$ and $G(y) > 0$ for all $y \in B(x,r)$ because α is decreasing as is seen from Eq. (9.6). From Eq. (9.10) (using now that n is the outward normal to $B(x,r)$),

$$\frac{\partial G}{\partial \mathbf{n}}(x+r\omega) = -\frac{1}{\sigma(S^{n-1})r^{n-1}} \text{ for } |\omega| = 1$$

and so according to Eq. (9.17) we have

$$\begin{aligned} u(x) &= \frac{1}{r^{n-1}\sigma(S^{n-1})}\int_{\partial B(x,r)} u d\sigma - \int_{B(x,r)} G(y)\triangle u(y)dy \\ &= \fint_{\partial B(x,r)} u\ d\sigma - \int_{B(x,r)} G(y)\triangle u(y)dy. \end{aligned} \tag{9.20}$$

This completes the proof since $G(y) > 0$ for all $y \in B(x,r)$. ■

Remark 9.13 (Mean value theorem). Assuming $\overline{B(x,R)} \subset \Omega$ and multiplying Eq. (9.18) (Eq. (9.19)) by

$$\sigma(\partial B(x,r)) = \sigma(S^{n-1})r^{n-1}$$

and then integrating on $0 \leq r \leq R$, implies

$$\begin{aligned} u(x)m(B(x,R)) = (\text{or } \leq) \int_0^R dr \int_{\partial B(x,r)} u(y)\,d\sigma(y) \\ = \int_0^R dr\ r^{n-1}\int_{S^{n-1}} u(x+r\omega)\,d\sigma(\omega) = \int_{B(x,R)} udm. \end{aligned}$$

Therefore if $\Delta u = 0$ or $\Delta u \geq 0$ then

$$u(x) = \fint_{B(x,R)} udm \text{ or } u(x) \leq \fint_{B(x,R)} udm \text{ respectively} \tag{9.21}$$

for all $\overline{B(x,R)} \subset \Omega$.

Proposition 9.14 (Converse of the mean value property)**.** *If $u \in C(\Omega)$ (or more generally measurable and locally bounded) and*

$$u(x) = \fint_{\partial B(x,r)} u(y)d\sigma(y) \tag{9.22}$$

for all $x \in \Omega$ and $r > 0$ such that $\overline{B(x,r)} \subset \Omega$, then $u \in C^\infty(\Omega)$ and $\triangle u = 0$. Similarly, if $u \in C^2(\Omega)$ and $x \in \Omega$ and

$$u(x) \leq \fint_{\partial B(x,r)} u(y)d\sigma(y) \tag{9.23}$$

for all r sufficiently small, then $\Delta u(x) \geq 0$.

Proof. First assume $u \in C(\Omega)$ and Eq. (9.22) hold which implies

$$u(x) = \fint_S u(x + r\omega)d\sigma(\omega) \tag{9.24}$$

for all $x \in \Omega$ and r sufficiently small, where $S = S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$. Let $\eta \in C_c^\infty(\mathbb{R}^n, [0,\infty))$ such that $\eta(0) > 0$ and

$$1 = \int_{\mathbb{R}^n} \eta(|x|^2)dx = \sigma(S)\int_0^\infty \eta(r^2)r^{n-1}dr$$

and for $\epsilon > 0$ let $\eta_\epsilon(x) = \epsilon^{-n}\eta\left(\frac{|x|^2}{\epsilon^2}\right) \in C_c^\infty(\mathbb{R}^n)$ and $u_\epsilon(x) = \eta_\epsilon * u(x)$. Then for any $x_0 \in \Omega$ and $\epsilon > 0$ sufficiently small, u_ϵ is a well defined smooth function near x_0. Moreover for x near x_0 we have

$$\begin{aligned} u_\epsilon(x) &= \int_{\mathbb{R}^n} \eta_\epsilon(x-y)u(y)dy = \int_0^\infty dr\ r^{n-1} \fint_{|\omega|=1} \eta_\epsilon(r\omega)u(x+r\omega)d\sigma(\omega) \\ &= \int_0^\infty dr r^{n-1} \fint_{|\omega|=1} \epsilon^{-n}\eta\left(\frac{r^2}{\epsilon^2}\right)u(x+r\omega)d\sigma(\omega) \\ &= u(x)\sigma(S)\int_0^\infty dr\ r^{n-1}\epsilon^{-n}\eta\left(\frac{r^2}{\epsilon^2}\right) = u(x) \end{aligned}$$

which shows u is smooth near x_0.

Now suppose that $u \in C^2$, and u satisfies Eq. (9.23), $x \in \Omega$ and $|r| < \epsilon$ with ϵ sufficiently small so that

$$f(r) := \fint_{\partial B(x,r)} u\ d\sigma = \fint_{S^{n-1}} u(x+r\omega)d\sigma(\omega)$$

is well defined. Clearly $f \in C^2(-\epsilon,\epsilon)$, f is an even function of r so $f'(0) = 0$, $f(0) = u(x)$ and $f(r) \geq f(0)$. From these conditions it follows that $f''(0) \geq 0$ for otherwise we would find from Taylor's theorem that $f(r) < f(0)$ for $0 < |r| < \epsilon$.

On the other hand

$$0 \leq f''(0) = \fint_{S^{n-1}} \left(\partial_\omega^2 u\right)(x) d\sigma(\omega) = \fint_{S^{n-1}} \left(\partial_i \partial_j u\right)(x) \omega_i \omega_j d\sigma(\omega)$$

$$= \left(\partial_i \partial_j u\right)(x) \delta_{ij} \fint_{S^{n-1}} \omega_i^2 d\sigma(\omega) = \frac{1}{n} \Delta u(x). \tag{9.25}$$

wherein we have used the symmetry of $d\sigma$ on S^{n-1} to conclude

$$\fint_{S^{n-1}} \omega_i \omega_j d\sigma(\omega) = 0 \text{ if } i \neq j$$

and

$$\fint_{S^{n-1}} \omega_i^2 d\sigma(\omega) = \frac{1}{n} \sum_{j=1}^n \fint_{S^{n-1}} \omega_j^2 d\sigma(\omega) = \frac{1}{n} \fint_{S^{n-1}} |\omega|^2 d\sigma(\omega) = \frac{1}{n} \ \forall \ i.$$

Alternatively, by the divergence theorem,

$$\fint_{S^{n-1}} \omega_i \omega_j d\sigma(\omega) = \fint_{S^{n-1}} \omega_i e_j \cdot n(\omega) d\sigma(\omega) = \frac{1}{\sigma(S^{n-1})} \int_{B(0,1)} \nabla \cdot (x_i e_j)\, dm$$

$$= \frac{1}{\sigma(S^{n-1})} m(B(0,1)) \delta_{ij} = \frac{1}{n} \delta_{ij}.$$

This completes the proof since if u satisfies (9.22) then f is constant and it follows from Eq. (9.25) that $\Delta u(x) = 0$.

Second proof of the last statement. Now that we know u is C^2 we have by Eq. (9.20) that

$$\fint_{B(x,r)} G(y) \triangle u(y) dy = \fint_{\partial B(x,r)} u \, d\sigma - u(x) \geq 0$$

and since with α as in Eq. (9.4),

$$\begin{aligned}
\fint_{B(x,r)} G(y) \triangle u(y) dy &= \fint_{B(0,r)} G(x+y) \triangle u(x+y) dy \\
&= \int_0^r \rho^{n-1} d\rho \int_{S^n} d\omega G(x+\rho\omega) \triangle u(x+\rho\omega) \\
&= \int_0^r \rho^{n-1} d\rho \left(\alpha(\rho) - \alpha(r)\right) \int_{S^n} d\omega \triangle u(x+\rho\omega) \\
&\cong \Delta u(x) \sigma(S^{n-1}) \int_0^r \rho^{n-1} d\rho \left(\alpha(\rho) - \alpha(r)\right) \\
&= \Delta u(x) \sigma(S^{n-1}) c_n \left\{ \frac{r^2}{2} - \frac{r^n}{n r^{n-2}} \right\} \\
&= b_n r^2 \Delta u(x)
\end{aligned}$$

where b_n is a positive constant. From this it follows that $\Delta u(x) \geq 0$.

Third proof of the last statement. If $u \in C^2(\Omega)$ satisfies expand $u(x + r\omega)$ in a Taylor series

$$u(x + r\omega) = u(x) + r \nabla u(x) \cdot \omega + \frac{r^2}{2} \partial_\omega^2 u(x) + o(r^3),$$

and integrate on ω to find

$$\begin{aligned}\fint_{\partial B(x,r)} u d\sigma &= \fint_{S^{n-1}} u(x+r\omega) d\sigma(\omega) \\ &= \fint_{S^{n-1}} \left[u(x) + r\nabla u(x)\cdot\omega + r^2 \frac{1}{2}\partial_\omega^2 u(x) + \dots \right] d\sigma(\omega) \\ &= u(x) + \frac{1}{2} r^2 \Delta u(x) + o(r^2).\end{aligned}$$

Thus if u satisfies Eq. (9.22) Eq. (9.23) we conclude

$$\begin{aligned}u(x) &= u(x) + \frac{1}{2} r^2 \Delta u(x) + o(r^2) \text{ or} \\ u(x) &\le u(x) + \frac{1}{2} r^2 \Delta u(x) + o(r^2)\end{aligned}$$

from which we conclude $\Delta u(x) = 0$ or $\Delta u(x) \ge 0$ respectively.

Fourth proof of the statement: If u satisfies Eq. (9.22) then $\Delta u = 0$. Since we already know u is smooth, it is permissible to differentiate Eq. (9.24) in r to learn,

$$\begin{aligned}0 &= \fint_{S^{n-1}} \nabla u(x+r\omega)\cdot\omega \, d\sigma(\omega) = \fint_{S^{n-1}} \frac{\partial u}{\partial \mathbf{n}}(x+r\omega) \, d\sigma(\omega) \\ &= \frac{1}{\sigma(S^{n-1}) r^{n-1}} \fint_{\partial B(x,r)} \nabla u \cdot \mathbf{n} \, d\sigma = \frac{1}{\sigma(S^{n-1}) r^{n-1}} \fint_{B(x,r)} \triangle u \, dm.\end{aligned}$$

Dividing this equation by r and letting $r \downarrow 0$ shows $\Delta u(x) = 0$. ■

Corollary 9.15 (Smoothness of Harmonic Functions)**.** *If $u \in C^2(\Omega)$ and $\triangle u = 0$ then $u \in C^\infty(\Omega)$. (Soon we will show u is real analytic, see Theorem 9.16 of Corollary 9.32 below.)*

Theorem 9.16 (Bounds on Harmonic functions)**.** *Suppose u is a Harmonic function on $\Omega \subset \mathbb{R}^n$, $x_0 \in \Omega$, α is a multi-index with $k := |\alpha|$ and $0 < r < \operatorname{dist}(x_o, \partial\Omega)$. Then*

$$|D^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))} \le \frac{C_k}{\operatorname{dist}(x_o,\partial\Omega)^{n+k}} \|u\|_{L^1(\Omega)} \tag{9.26}$$

where $C_k = \frac{(2^{n+1} n \ k)^k}{\alpha(n)}$. In particular one shows that u is real analytic in Ω.

Proof. Let $\eta_\epsilon(x)$ be constructed as in the proof of Proposition 9.14 so that $u(x) = u * \eta_\epsilon(x)$. Therefore, $D^\alpha u(x) = u_* D^\alpha \eta_\epsilon(x)$ and hence

$$|D^\alpha u(x_0)| \le \|u\|_{L^1(B(x_0,\epsilon))} \|D^\alpha \eta_\epsilon\|_{L^\infty}.$$

Now

$$D^\alpha \eta_\epsilon(x) = \epsilon^{-n} \frac{1}{\epsilon^{|\alpha|}} (D^\alpha \eta)(\frac{x}{\epsilon})$$

so that

$$|D^\alpha \eta_\epsilon(x)| = \epsilon^{-n} \frac{1}{\epsilon^{|\alpha|}} \left| (D^\alpha \eta)(\frac{x}{\epsilon}) \right| \le C_\alpha \frac{1}{\epsilon^{|\alpha|+n}} \cong C_\alpha \frac{1}{r^{|\alpha|+n}}$$

where the last identity is gotten by taking ϵ comparable to r. Putting this all together then implies that

$$|D^\alpha u(x_0)| \leq \frac{1}{r^{n+|\alpha|}} \|D^\alpha \eta\|_{L^\infty} \|u\|_{L^1(B(x_0,r))}$$

which is an inequality of the form in Eq. (9.26). To get the desired constant we will have to work harder. This is done in Theorem 7. on p. 29 of the book. The idea is to use $D^\alpha u$ is harmonic for all α and therefore,

$$\begin{aligned} D^\alpha u(x_0) &= \fint_{B(x_0,\rho)} D^\alpha u dm = \fint_{B(x_0,\rho)} \partial_i D^\beta u dm = \frac{n}{\sigma(S^{n-1})\rho^n} \int_{B(x_0,\rho)} \partial_i D^\beta u dm \\ &= \frac{n}{\sigma(S^{n-1})\rho^n} \int_{\partial B(x_0,\rho)} D^\beta u n_i d\sigma \end{aligned}$$

so that

$$|D^\alpha u(x_0)| \leq \frac{n}{\rho} \left\|D^\beta u\right\|_{L^\infty(B(x_0,\rho))}$$

and for $\alpha = 0$ and $x \in B(x_0, r/2)$ we have

$$|u(x)| \leq \fint_{B(x,r/2)} |u| \, dm \leq \frac{1}{|B(0,1)|} \left(\frac{2}{r}\right)^n \|u\|_{L^1(B(x_0,r)} .$$

Using this and similar inequalities along with a tricky induction argument one gets the desired constants. The details are in Theorem 7. p. 29 and Theorem 10 p.31 of the book. (See also Corollary 9.32 below for another proof of analyticity of u.) ■

Corollary 9.17 (Liouville's Theorem)**.** *Suppose $u \in C^2(\mathbb{R}^n)$, $\Delta u = 0$ on $\mathbb{R}^n$ and $|u(x)| \leq C(1 + |x|^N)$ for all $x \in \mathbb{R}^n$. Then u is a polynomial of degree at most N.*

Proof. We have seen there are constants $C_{|\alpha|} < \infty$ such that

$$\begin{aligned} |D^\alpha u(x_0)| &\leq C_{|\alpha|} \|u\|_{L^1(B(x_0,r))} \frac{1}{r^{n+|\alpha|}} \\ &\leq \widetilde{C}_{|\alpha|} r^n \|u\|_{L^\infty(B(x_0,r))} \cdot \frac{1}{r^{n+|\alpha|}} \\ &\cong C \frac{(1 + r^N)}{r^{|\alpha|}} \to 0 \text{ as } r \to \infty \end{aligned}$$

when if $|\alpha| > N$. Therefore $D^\alpha u := 0$ for all $|\alpha| > N$ and the the result follows by Taylor's Theorem with remainder,

$$u(x) = \sum_{|\alpha| \leq N} \frac{D^\alpha u(x_0)(x - x_0)^\alpha}{\alpha!}.$$

■

Corollary 9.18 (Compactness of Harmonic Functions)**.** *Suppose $\Omega \subset_o \mathbb{R}^n$ and $u_n \in C^2(\Omega)$ is a sequence of harmonic functions such that for each compact set $K \subset \Omega$,*

$$C_K := \sup \left\{ \int_K |u_n| \, dm : n \in \mathbb{N} \right\} < \infty.$$

Then there is a subsequence $\{v_n\} \subset \{u_n\}$ which converges, along with all of its derivatives, uniformly on compact subsets of Ω to a harmonic function u.

Proof. An application of Theorem 9.16 shows that for each compact set $K \subset \Omega$, $\sup_n |\nabla u_n|_{L^\infty(K)} < \infty$ and hence by the locally compact form of the Arzela-Ascolli theorem, there is a subsequence $\{v_n\} \subset \{u_n\}$ which converges uniformly on compact subsets of Ω to a continuous function $u \in C(\Omega)$. Passing to the limit in the mean value theorem for harmonic functions along with the converse to the mean value theorem, Proposition 9.14, shows u is harmonic on Ω. Since $v_m \to u$ uniformly on compacts it follows for any $K \sqsubset\sqsubset \Omega$ that $\int_K |u - v_n|\, dm \to 0$. Another application of Theorem 9.16 then shows $D^\alpha v_n \to D^\alpha u$ uniformly on compacts. ■

In light of Proposition 9.14, we will extend the notion of subharmonicity as follows.

Definition 9.19 (Subharmonic Functions)**.** A function $u \in C(\Omega)$ is said to be **subharmonic** if for all $x \in \Omega$ and all $r > 0$ sufficiently small,

$$u(x) \leq \fint_{\partial B(x,r)} u \, d\sigma.$$

The reason for the name subharmonic should become apparent from Corollary 9.25 below.

Remark 9.20. Suppose that $u, v \in C(\Omega)$ are subharmonic functions then so is $u+v$. Indeed,

$$u(x) + v(x) \leq \fint_{\partial B(x,r)} u \, d\sigma + \fint_{\partial B(x,r)} v \, d\sigma = \fint_{\partial B(x,r)} (u+v)\, d\sigma.$$

Theorem 9.21 (Harnack's Inequality)**.** *Let V be a precompact open and connected subset of Ω. Then there exists $C = C(V, \Omega)$ such that*

$$\sup_V u \leq C \inf_V u$$

for all non-negative sub-harmonic functions, u, on Ω.

Proof. Let $r = \frac{1}{4}\text{dist}(V, \Omega^c)$ and $x \in V$ (as in Figure 29) and $|y - x| \leq r$, then

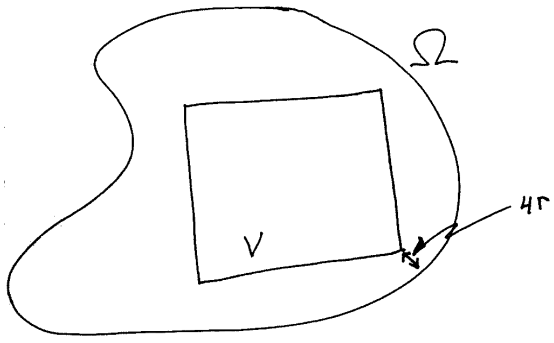

FIGURE 29. A pre-compact region $V \subset \Omega$.

by the mean value inequality in Eq. (9.21) of Remark 9.13,

$$u(x) = \fint_{B(x,2r)} u(z)dz = \frac{1}{m(B(0,1))(2r)^n} \int_{B(x,2r)} u(z)dz$$
$$\geq \frac{1}{m(B(0,1))(2r)^n} \int_{B(y,r)} u(z)dz = \frac{1}{2^n} \fint_{B(y,r)} u(z)dz = \frac{1}{2^n} u(y),$$

see Figure 30. Therefore $u(x) \geq \frac{1}{2^n} u(y)$ provided $x, y \in V$ with $|x - y| \leq r$. Since

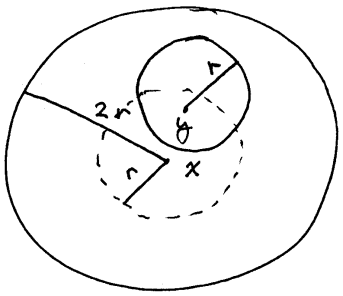

FIGURE 30. Nested balls.

$\bar{V}$ is compact there exists a finite cover $\mathcal{S} := \{W_i\}_{i=1}^M$ of $\bar{V}$ consisting of balls with of radius r with centers $x_i \in \bar{V}$. For all $x, y \in V$, there exists a path $\gamma : x \to y$ and hence a chain $B_i \in \mathcal{S}$ such that $x \in B_1$, $y \in B_k$ and $B_i \cap B_{i+1} \neq \phi$ for all $i = 1, \ldots, k-1$. It then follows from what we have just proved that

$$u(y) \leq (2^n)^k \; u(x) \leq 2^{Mn} \; u(x) =: Cu(x)$$

for all $x, y \in V$, i.e. $\sup_V u \leq C \inf_V u$ where $C := 2^{Mn}$. ■

Theorem 9.22 (Strong Maximum Principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be connected and open and $u \in C(\Omega)$ be a subharmonic function (see Definition 9.19). If $M = \sup_{x \in \Omega} u(x)$ is attained in Ω then $u := M$. (Notice that $u \in C^2(\Omega)$ and $\Delta u = 0$, then u is harmonic and hence in particular sub-harmonic.)*

Proof. Suppose there exists $x \in \Omega$ such that $M = u(x)$. If $\epsilon > 0$ is chosen so that $\overline{B(x,\epsilon)} \subset \Omega$ as in Figure 27 and $u(y) < M$ for some $y \in \partial B(x,\epsilon)$, then by the mean value inequality,

$$M = u(x) \leq \fint_{\partial B(x,\epsilon)} u(y)d\sigma(y) < M$$

which is nonsense. Therefore $u := M$ on $\partial B(x,\epsilon)$ and since $\epsilon \in (0, \operatorname{dist}(x, \partial\Omega))$ we concluded that $u := M$ on $B(x,R)$ provided $\overline{B(x,R)} \subset \Omega$. Therefore $\{x \in \Omega : u(x) = M\}$ is both open and relatively closed in Ω and hence $\{x \in \Omega : u(x) = M\} = \Omega$ because Ω is connected. ■

Corollary 9.23. *If Ω is bounded open set $u \in C(\overline{\Omega})$ is subharmonic, then*

$$M := \max_{x \in \overline{\Omega}} u(x) = \max_{x \in \operatorname{bd}(\Omega)} u(x).$$

Again this corollary applies to $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that $\Delta u = 0$.

Proof. By Theorem 9.22, if $x \in \Omega$ is an interior maximum of u, then $u = M$ on the connected component Ω_x of Ω which contains x. By continuity, u is constant on $\bar{\Omega}_x$ and in particular u takes on the value M on $\mathrm{bd}(\Omega)$. ■

Corollary 9.24. *Given $g \in C(\mathrm{bd}(\Omega))$, $f \in C(\Omega)$ there exists at most one function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\triangle u = f$ on Ω and $u = g$ on $\mathrm{bd}(\Omega)$.*

Proof. If $v \in C^2(\Omega) \cap C(\overline{\Omega})$ is another such function then $w := u - v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\Delta w = 0$ in Ω and $w = 0$ on $\mathrm{bd}(\Omega)$. Therefore applying Corollary 9.23 to w and $-w$ implies

$$\max_{x \in \overline{\Omega}} w(x) = \max_{x \in \mathrm{bd}(\Omega)} w(x) = 0 \text{ and } \min_{x \in \overline{\Omega}} w(x) = \min_{x \in \mathrm{bd}(\Omega)} w(x) = 0.$$

■

Corollary 9.25. *Suppose $g \in C(\mathrm{bd}(\Omega))$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\triangle u = 0$ on Ω. Then $w \leq u$ for any subharmonic function $w \in C(\bar{\Omega})$ such that $w \leq g$ on $\mathrm{bd}(\Omega)$.*

Proof. The function $-u$ is subharmonic and so is $v = w - u$ by Remark 9.20. Since $v = w - g \leq 0$ on $\mathrm{bd}(\Omega)$, it follows by Corollary 9.23 that $v \leq 0$ on Ω, i.e. $w \leq g$ on Ω. ■

9.2. **Green's Functions.**

Notation 9.26. Unless otherwise stated, for the rest of this section assume $\Omega \subset \mathbb{R}^n$ is a compact manifold with C^2 – boundary.

For $x \in \Omega$, suppose there exists $h \in C^2(\Omega^o) \cap C^1(\Omega)$ which solves

$$\triangle h_x = 0 \text{ on } \Omega \text{ with } h_x(y) = \phi(x - y) \text{ for } y \in \partial\Omega. \tag{9.27}$$

Hence if we define

$$G(x, y) = \phi_x(y) - h_x(y) \tag{9.28}$$

then by the representation formula (Eq. (9.11) also see Remark 9.11) implies

$$u(x) = -\int_\Omega G(x, y) \triangle u(y)\, dy - \int_{\partial\Omega} \frac{\partial G}{\partial \mathbf{n}_y}(x, y)\, u(y)\, d\sigma(y) \tag{9.29}$$

for all $u \in C^2(\Omega^o) \cap C^1(\Omega)$.

Throughout the rest of this subsection we will make the following assumption.

Assumption 2 (Solvability of Dirichlet Problem)**.** We assume that for each $g \in C(\partial\Omega)$ there exists $h = h_g \in C^2(\Omega^o) \cap C^1(\Omega)$ such that

$$\Delta h = 0 \text{ on } \Omega \text{ with } h = g \text{ on } \partial\Omega.$$

In this case we define $G(x, y)$ as in Eq. (9.28). We will (almost) verify that this assumption holds in Section 9.5 below. The full verification will come later when we study Hilbert space methods.

Theorem 9.27. *Let $G(x, y)$ be given as in Eq (9.28). Then*

1. *$G(x, y)$ is smooth on $(\Omega^o \times \Omega^o) \backslash \triangle$ where $\triangle = \{(x, x) : x \in \Omega^o\}$.*
2. *$G(x, y) = G(y, x)$ for all $x, y \in \Omega$. In particular the function $h(x, y) := h_x(y)$ is symmetric in x, y and $x \in \Omega^o \to h_x \in C(\Omega)$ is a smooth mapping.*

(3) *If Ω is connected, then $G(x,y) > 0$ for all $(x,y) \in (\Omega^o \times \Omega^o) \backslash \triangle$.*

Proof. Let $\epsilon > 0$ be small and $\Omega_\epsilon := \Omega \setminus (B(x,\epsilon) \cup B(z,\epsilon))$ as in Figure 31, then

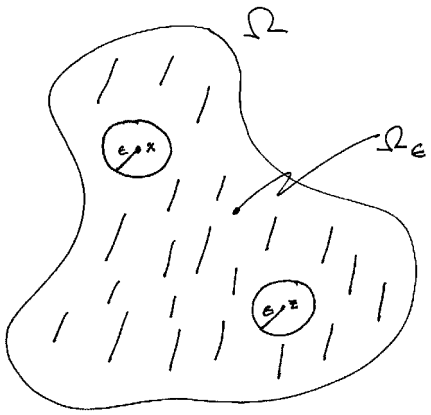

FIGURE 31. Excising the singular region from Ω.

by Green's theorem and the fact that $\Delta_y G(x,y) = 0$ if $y \neq x$,

$$\begin{aligned}
0 &= \int_{\Omega_\epsilon} \triangle_y G(x,y) G(z,y) dy \\
&= \int_{\partial\Omega_\epsilon} \left(\frac{\partial}{\partial \mathbf{n}} G(x,y) G(z,y) - G(x,y) \frac{\partial G}{\partial \mathbf{n}}(z,y) \right) d\sigma + \int_{\Omega_\epsilon} G(x,y) \triangle_y G(z,y) dy \\
&= \int_{\partial\Omega_\epsilon} \left(\frac{\partial}{\partial \mathbf{n}} G(x,y) G(z,y) - G(x,y) \frac{\partial G}{\partial \mathbf{n}}(z,y) \right) d\sigma
\end{aligned}$$

Since $G(x,y)$ and $G(z,y) = 0$ for $y \in \partial\Omega$, the previous equation implies,

$$\int_{\partial(B(x,\epsilon)\cup B(z,\epsilon)).} \left\{ \frac{\partial}{\partial \mathbf{n}_y} G(x,y) G(z,y) - G(z,y) \frac{\partial}{\partial \mathbf{n}_y} G(z,y) \right\} d\sigma = 0.$$

We now let $\epsilon \downarrow 0$ in the above equations to find

$$\lim_{\epsilon \downarrow 0} \int_{\partial(B(x,\epsilon))} \frac{\partial \phi(x-y)}{\partial \mathbf{n}_y} G(z,y) d\sigma(y) = \lim_{\epsilon \downarrow 0} \int_{\partial B(z,\epsilon)} G(x,y) \frac{\partial}{\partial \mathbf{n}_y} \phi(z-y)\, d\sigma. \tag{9.30}$$

Moreover as we have seen above,

$$\lim_{\epsilon \downarrow 0} \int_{\partial B(z,\epsilon)} G(x,y) \frac{\partial}{\partial \mathbf{n}_y} \phi(z-y)\, d\sigma = G(x,z) \text{ and}$$

$$\lim_{\epsilon \downarrow 0} \int_{\partial(B(x,\epsilon))} \frac{\partial \phi(x-y)}{\partial \mathbf{n}_y} G(z,y) d\sigma(y) = G(z,x)$$

and hence $G(x,z) = G(z,x)$. Since $G(x,y) = \phi(x-y) - h_x(y)$ and $\phi(x-y) = \phi(y-x)$ it follows that $h_x(y) = h_y(x) =: h(x,y)$. Therefore $y \to h_x(y)$ and $x \to h_x(y)$ are

smooth functions. Now by the maximum principle:

$$|h_x(y)-h_z(y)| \le \max_{y\in\partial\Omega}|h_x(y)-h_z(y)| = \max_{y\in\partial\Omega}|\phi(x-y)-\phi(z-y)| \to 0 \text{ as } x\to z.$$

Therefore the map $x\in\Omega \to h_x \in C(\Omega)$ is continuous and in particular the map $(x,y)\to h(x,y)$ is jointly continuous. Finally letting η be as in the proof of Proposition 9.14, we find

$$\begin{aligned} h(x,y) &= \int_\Omega h(\widetilde{x},y)\eta(x-\widetilde{x})d\widetilde{x} \\ &= \int_{\Omega\times\Omega} h(\widetilde{x},\widetilde{y})\ \eta(y-\widetilde{y})\ \eta(x-\widetilde{x})\ d\widetilde{x}\ d\widetilde{y} \end{aligned}$$

from which it follows that in fact h is smooth on $\Omega\times\Omega$.

It only remains to show $x\to h_x\in C(\Omega)$ is smooth as well. Fix $x\in\Omega$ and for $v\in\mathbb{R}^n$, let $H_v\in C^2(\Omega^o)\cap C^1(\Omega)$ denote the solution to

$$\Delta H_v = 0 \text{ on } \Omega \text{ with } H_v(y) = v\cdot\nabla\phi(x-y) \text{ for } y\in\partial\Omega.$$

Notice that $v\to H_v$ is linear and by the maximum principle,

$$\begin{aligned} \|h_{x+v}-h_x-H_v\|_{L^\infty(\Omega)} &\le \|h_{x+v}-h_x-H_v\|_{L^\infty(\partial\Omega)} \\ &= \|\phi(x+v-\cdot)-\phi(x-\cdot)-v\cdot\nabla\phi(x-\cdot)\|_{L^\infty(\partial\Omega)}. \end{aligned}$$

Now,

$$\begin{aligned} &\phi(x+v-y)-\phi(x-y)-v\cdot\nabla\phi(x-y) \\ &\qquad = \int_0^1 [\nabla\phi(x+tv-y)-\nabla\phi(x-y)]\cdot v\,dt \end{aligned}$$

so that, by the dominated convergence theorem,

$$\begin{aligned} &||\phi(x+v-\cdot)-\phi(x-\cdot)-v\cdot\nabla\phi(x-\cdot)||_{L^\infty(\partial\Omega)} \\ &\qquad \le |v|\int_0^1 \|\nabla\phi(x+tv-\cdot)-\nabla\phi(x-\cdot)\|_{L^\infty(\partial\Omega)}\,dt = o(|v|). \end{aligned}$$

This proves $x\to h_x$ is differentiable and that $\partial_v h_x = H_v$. Similarly one shows that $x\to h_x$ has higher derivatives as well.

For the last item, let $x\in\Omega^o$ and choose $\epsilon>0$ sufficiently small so that $\overline{B(x,\epsilon)}\subset\Omega^o\setminus\{y\}$ and $G(x,z)>0$ for all $z\in\overline{B(x,\epsilon)}$. Then the function $u(y):=G(x,y)$ is Harmonic on $\Omega^0\setminus\overline{B(x,\epsilon)}$, $u\in C(\Omega\setminus B(x,\epsilon))$, $u=0$ on $\partial\Omega$ and $u>0$ on $\partial\overline{B(x,\epsilon)}$. Hence by the maximum principle, $0\le u$ on $\Omega\setminus B(x,\epsilon)$ and since u is not constant we must also have $u>0$ on $\Omega^0\setminus\overline{B(x,\epsilon)}$. Since $\epsilon>0$ was any sufficiently small number, it follows $G(x,y)>0$ for all $y\in\Omega^o\setminus\{x\}$. ■

Corollary 9.28. *Keeping the above hypothesis and assuming $\rho\in C^2(\Omega^o)\cap L^1(\Omega)$ and $g\in C(\partial\Omega)$, then there is (a necessarily unique) solution $u\in C^2(\Omega^o)\cap C(\Omega)$ to*

$$\Delta u = -\rho \text{ with } u = g \text{ on } \partial\Omega \tag{9.31}$$

which is given by Eq. (9.29).

Proof. According to the remarks just before Eq. (9.29), if a solution to Eq. (9.31) exists it must be given by

$$u(x) = \int_\Omega G(x,y)\rho(y)\ dy - \int_{\partial\Omega}\frac{\partial G}{\partial \mathbf{n}_y}(x,y)\ g(y)\ d\sigma(y). \tag{9.32}$$

From Assumption 2, there exists a solution $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that $\Delta v = 0$ and $v = g$ on $\partial\Omega$. So replacing u by $u - v$ if necessary, it suffices to prove there is a solution $u \in C^2(\Omega^o) \cap C(\Omega)$ such that Eq. (9.31) holds with $g \equiv 0$. To produce this solution, let

$$u(x) := \int_\Omega G(x,y)\rho(y)\, dy = \int_\Omega \phi(x-y)\rho(y)\, dy - H(x)$$

where

$$H(x) := \int_\Omega h(x,y)\rho(y)dy.$$

Using the result in Theorem 9.27, one easily shows $H \in C^\infty(\Omega^o) \cap C^1(\Omega)$and $\Delta H = 0$. By Theorem 9.9,

$$\Delta_x \int_\Omega \phi(x-y)\rho(y)\, dy = -\rho(x) \text{ for } x \in \Omega$$

and therefore $u \in C^2(\Omega)$ and $\Delta u = -\rho$. ■

Remark 9.29. Because of the maximum principle, for any $x \in \Omega$ the map $g \in C(\partial\Omega) \to h_g(x) \in C(\bar{\Omega})$ is a positive linear functional. So by the Riesz representation theorem, there exists a unique positive probability measure σ_x on $\partial\Omega$ such that

$$h_g(x) = \int_{\partial\Omega} g(y)d\sigma_x(y) \text{ for all } g \in C(\partial\Omega).$$

Evidently this measure is given by

$$d\sigma_x(y) = -\frac{\partial G}{\partial \mathbf{n}_y}(x,y)d\sigma(y)$$

and in particular $-\frac{\partial G}{\partial \mathbf{n}_y}(x,y) \geq 0$ for all $x \in \Omega$ and $y \in \partial\Omega$. It is in fact easy to see that $-\frac{\partial G}{\partial \mathbf{n}_y}(x,y) > 0$ for all $x \in \Omega$ and $y \in \partial\Omega$.

9.3. **Explicit Green's Functions and Poisson Kernels.** In this section we will use the *method of images* to construct explicit formula for the Green's functions and Poisson Kernels for the half plane[6], $\bar{\mathbb{H}}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}$and Balls $\overline{B(0,a)}$. For $x = (x', z) \in \mathbb{R}^{n-1} \times (0,\infty) = \mathbb{H}^n$ let $Rx := (x', -z)$. It is simple to verify $|x - y| = |Rx - y|$ for all $x \in \mathbb{H}^n$ and $y \in \partial\bar{\mathbb{H}}^n$. Form this and the properties of ϕ, one concluded, for $x \in \mathbb{H}^n$, that $h_x(y) := \phi(y - Rx)$ is Harmonic in $y \in \mathbb{H}^n$ and $h_x(y) = \phi(x-y)$ for all $y \in \partial\bar{\mathbb{H}}^n$. These remarks give rise to the following theorem.

Theorem 9.30. *For $x, y \in \bar{\mathbb{H}}^n$, let*

$$G(x,y) := \phi(y-x) - \phi(y-Rx) = \phi(y-x) - \phi(Ry-x).$$

Then G is the Greens function for Δ on $\mathbb{H}^n$ and

$$K(x,y) := -\frac{\partial G}{\partial \mathbf{n}}(x,y) = \frac{2x_n}{\sigma(S^{n-1})}\ \frac{1}{|x-y|^n} \text{ for } x \in \mathbb{H}^n \text{ and } y \in \partial\bar{\mathbb{H}}^n$$

[6]We will do this again later using the Fourier transform.

is the Poisson kernel for $\mathbb{H}^n$. *Furthermore if* $\rho \in C^2(\mathbb{H}^n) \cap L^1(\mathbb{H}^n)$ *and* $f \in BC\left(\partial\bar{\mathbb{H}}^n\right)$, *then*

$$u(x) = \int_{\mathbb{H}^n} G(x,y)\rho(y)dy + \int_{\partial\bar{\mathbb{H}}^n} K(x,y)f(y)d\sigma(y)$$

solves the equation

$$\Delta u = -\rho \text{ on } \mathbb{H}^n \text{ with } u = f \text{ on } \partial\bar{\mathbb{H}}^n.$$

Proof. First notice that

$$G(y,x) = \phi(x-y) - \phi(x-Ry) = \phi(x-y) - \phi(Rx - RRy) = G(x,y)$$

since ϕ is a function of $|\cdot|$. Therefore, if

$$u(x) = \int_{\mathbb{H}^n} G(x,y)\rho(y)dy = \int_{\mathbb{H}^n} \phi(x-y)\rho(y)dy - \int_{\mathbb{H}^n} \phi(x-Ry)\rho(y)dy,$$

we have from Theorem 9.9 that

$$\Delta u(x) = -\rho(x) - \int_{\mathbb{H}^n} \Delta_x \phi(x-Ry)\rho(y)dy = -\rho(x).$$

Since $G(x,y) = 0$ for $x \in \partial\bar{\mathbb{H}}^n$ and so $u(x) = 0$ for $x \in \partial\bar{\mathbb{H}}^n$. It is left to the reader to show u is continuous on $\overline{\mathbb{H}^n}$.

For $x \in \mathbb{H}^n$ and $y \in \partial\bar{\mathbb{H}}^n$, we find form Eq. (9.7),

$$\begin{aligned} K(x,y) &:= -\frac{\partial G}{\partial \mathbf{n}_y}(x,y) = \frac{\partial}{\partial y_n} G(x,y) \\ &= \frac{\partial}{\partial y_n}\left[\phi(y-x) - \phi(y-Rx)\right] \\ &= -\frac{1}{\sigma(S^{n-1})}\frac{1}{|y-x|^n}(y-x)\cdot e_n + \frac{1}{\sigma(S^{n-1})}\frac{1}{|y-Rx|^n}(y-Rx)\cdot e_n \\ &= \frac{1}{\sigma(S^{n-1})}\frac{2x_n}{|y-x|^n}. \end{aligned}$$

Claim: For all $x \in \mathbb{H}^n$,

$$\int_{\partial\bar{\mathbb{H}}^n} K(x,y)dy = 1.$$

It is possible to prove this by direct computation, since (writing $x = (x', x_n)$ as above)

$$\begin{aligned} \int_{\partial\bar{\mathbb{H}}^n} K(x,y)dy &= \frac{2}{\sigma(S^{n-1})}\int_{\mathbb{R}^{n-1}} \frac{x_n}{\left(|x'-y|^2 + x_n^2\right)^{n/2}}dy \\ &= \frac{2}{\sigma(S^{n-1})}\int_{\mathbb{R}^{n-1}} \frac{1}{\left(|y|^2+1\right)^{n/2}}dy \\ &= \frac{2}{\sigma(S^{n-1})}\sigma(S^{n-2})\int_0^\infty r^{n-2}\frac{1}{\left(r^2+1\right)^{n/2}}dr \end{aligned}$$

where in the second equality we have made the change of variables $y \to x_n y$ and in the last we passed to polar coordinates. When $n = 2$ we find

$$\int_0^\infty r^{n-2}\frac{1}{\left(r^2+1\right)^{n/2}}dr = \int_0^\infty \frac{1}{r^2+1}dr = \pi/2$$

and for $n=3$ we may let $u=r^2$ to find

$$\int_0^\infty r^{n-2}\frac{1}{\left(r^2+1\right)^{n/2}}dr=\int_0^\infty r\frac{1}{\left(r^2+1\right)^{3/2}}dr=\frac{1}{2}\int_0^\infty\frac{1}{\left(u+1\right)^{3/2}}du=1.$$

These results along with

$$\begin{aligned}\int_0^\infty r^{n-2}\frac{1}{\left(r^2+1\right)^{n/2}}dr&=\int_0^\infty\left(r^2+1\right)^{-n/2}d\frac{r^{n-1}}{n-1}=\frac{n/2}{n-1}\int_0^\infty\left(r^2+1\right)^{-n/2-1}2r\ r^{n-1}dr\\&=\frac{n}{n-1}\int_0^\infty r^n\frac{1}{\left(r^2+1\right)^{\frac{n+2}{2}}}dr\end{aligned}$$

allows one to compute $\int_0^\infty r^{n-2}\frac{1}{(r^2+1)^{n/2}}dr$ inductively. I will not carry out the details of this method here. Rather, it is more instructive to use Corollary 9.6 to prove the claim. In order to do this let $u\in C_c^\infty(B(0,1),[0,1])$ such that $u(0)=1$, $u(x)=U(|x|)$ and $U(r)$ is decreasing as r decreases. Then by Corollary 9.6, with $u(x)=u_M(x):=u(x/M)$,

$$u_M(x)=\int\limits_{\partial\bar{\mathbb{H}}^n}K(x,y)u(y/M)d\sigma(y)-M^{-2}\int\limits_{\mathbb{H}^n}G(x,y)\left(\triangle u\right)(y/M)dy. \tag{9.33}$$

By the monotone convergence theorem,

$$\lim_{M\uparrow\infty}\int\limits_{\partial\bar{\mathbb{H}}^n}K(x,y)u(y/M)d\sigma(y)=\int\limits_{\partial\bar{\mathbb{H}}^n}K(x,y)d\sigma(y)$$

and therefore passing the limit in Eq. (9.33) gives

$$1=\int\limits_{\partial\bar{\mathbb{H}}^n}K(x,y)d\sigma(y)-\lim_{M\uparrow\infty}\left[M^{-2}\int\limits_{\mathbb{H}^n}G(x,y)\triangle u(y/M)dy\right].$$

This latter limit is zero, since

$$\begin{aligned}M^{-2}\int\limits_{\mathbb{H}^n}G(x,y)\triangle u(y/M)dy&=c_nM^{-2}\int\limits_{\mathbb{H}^n}\left[\frac{1}{|x-y|^{n-2}}-\frac{1}{|Rx-y|^{n-2}}\right]\left(\triangle u\right)(y/M)dy\\&=c_nM^{-2}M^n\int\limits_{\mathbb{H}^n}\left[\frac{1}{|x-My|^{n-2}}-\frac{1}{|Rx-My|^{n-2}}\right]\triangle u(y)dy\\&=c_n\int\limits_{\mathbb{H}^n}\left[\frac{1}{|x/M-y|^{n-2}}-\frac{1}{|Rx/M-y|^{n-2}}\right]\triangle u(y)dy.\end{aligned}$$

This latter expression tends to zero and $M \to \infty$ by the dominated convergence and this proves the claim. (Alternatively, for y large,

$$\begin{aligned}\frac{1}{|x-y|^{n-2}} - \frac{1}{|Rx-y|^{n-2}} &= \frac{1}{|y|^{n-2}}\left[\frac{1}{\left|\frac{x}{|y|}-\hat{y}\right|^{n-2}} - \frac{1}{\left|R\frac{x}{|y|}-\hat{y}\right|^{n-2}}\right] \\ &= \frac{1}{|y|^{n-2}}\left[\left(1+2\frac{x}{|y|}\cdot\hat{y}+\dots\right)-\left(1+2\frac{Rx}{|y|}\cdot\hat{y}+\dots\right)\right] \\ &= O(\frac{1}{|y|^{n-1}})\end{aligned}$$

and therefore

$$M^{-2}\int_{\mathbb{H}^n}\left[\frac{1}{|x-y|^{n-2}} - \frac{1}{|Rx-y|^{n-2}}\right](\triangle u)(y/M)dy = O\left(M^{-2}\frac{1}{M^{n-1}}M^n\right) = O(1/M) \to 0$$

as $M \to \infty$.

Since $G(x,y)$ is harmonic in x, it follows that $K(x,y) = -\frac{\partial}{\partial \mathbf{n}_y}K(x,y)$ is still Harmonic in x. and therefore

$$u(x) := \int_{\partial\bar{\mathbb{H}}^n} K(x,y)f(y)d\sigma(y) = \frac{2}{\sigma(S^{n-1})}\int_{\partial\bar{\mathbb{H}}^n}\frac{x_n}{|x-y|^n}f(y)d\sigma(y)$$

is harmonic as well. Since

$$\begin{aligned}u(x) &= \frac{2}{\sigma(S^{n-1})}\int_{\partial\bar{\mathbb{H}}^n}\frac{x_n}{\left(|x'-y|^2+x_n^2\right)^{n/2}}f(y)dy \\ &= \frac{2}{\sigma(S^{n-1})}\frac{1}{x_n^{n-1}}\int_{\partial\bar{\mathbb{H}}^n}\frac{1}{\left(|\frac{x'-y}{x_n}|^2+1\right)^{n/2}}f(y)dy\end{aligned}$$

it follows from Theorem 7.13 that $u((x',x_n)) \to f(x')$ as $x_n \downarrow 0$ uniformly for x' in compact subsets of $\partial\bar{\mathbb{H}}^n$. ■

9.4. **Green's function for Ball.** Let $r > 0$ be fixed, we will construct the Green's function for the ball $B(0,r)$. The idea for a given $x \in B(0,r)$, we should find a mirror location, say $\rho\hat{x}$ and a charge q so that

$$\phi(x-y) = q\phi(\rho\hat{x}-y) \text{ for all } |y| = r.$$

Assuming for the moment that $n \geq 3$ and writing $q = \beta^{(2-n)}$, this leads to the equations

$$|x-y|^2 = |\beta\rho\hat{x}-\beta y|^2 = \beta^2|\rho\hat{x}-y|^2$$

or equivalently squaring out both sides and using $|y| = r$,

$$|x|^2 - 2x\cdot y + r^2 = \beta^2\left(\rho^2 - 2\rho\hat{x}\cdot y + r^2\right).$$

Choosing $y \perp x$ and $y = r\hat{x}$ leads to the conditions

$$\begin{aligned}|x|^2 + r^2 &= \beta^2\left(\rho^2+r^2\right) \text{ and} \\ |x|^2 - 2r|x| + r^2 &= \beta^2\left(\rho^2 - 2\rho r + r^2\right).\end{aligned}$$

Subtracting these two equations implies $-2r|x| = -2\rho\beta^2 r$ or equivalently that $\rho = |x|/\beta^2$. Putting this into the first equation above then implies

$$|x|^2 + r^2 = \frac{|x|^2}{\beta^2} + \beta^2 r^2$$

or equivalently that

$$0 = r^2\beta^4 - \left(|x|^2 + r^2\right)\beta^2 + |x|^2 .$$

By the quadratic formula, this implies

$$\begin{aligned}\beta^2 &= \frac{\left(|x|^2 + r^2\right) \pm \sqrt{\left(|x|^2 + r^2\right)^2 - 4r^2|x|^2}}{2r^2} \\ &= \frac{\left(|x|^2 + r^2\right) \pm \sqrt{\left(|x|^2 - r^2\right)^2}}{2r^2} = \frac{\left(|x|^2 + r^2\right) \pm \left(r^2 - |x|^2\right)}{2r^2} \\ &= 1 \text{ or } \frac{|x|^2}{r^2}.\end{aligned}$$

Clearly the charge $\beta = 1$ will not work so we must take $\beta = |x|/r$ in which case, $\rho = r^2/|x|$ and hence

$$q\phi(\rho\hat{x} - y) = (|x|/r)^{(2-n)}\,\phi\left(r^2\frac{\hat{x}}{|x|} - y\right) = \phi\left(r\hat{x} - \frac{|x|}{r}y\right).$$

Let us now verify that our guess has worked. Let us begin by noting the following identities for $x, y \in \mathbb{R}^n$,

$$\left|r\hat{x} - r^{-1}|x|\,y\right|^2 = \left(r^2 - 2x\cdot y + r^{-2}|x|^2|y|^2\right) \tag{9.34}$$

and in particular when $|y| = r$ this implies

$$|\hat{x}r - |x|\hat{y}|^2 = \left(r^2 - 2x\cdot y + |x|^2\right) = |x-y|^2$$

so that

$$|x-y| = |\hat{x}r - |x|\hat{y}| = \left|\frac{x}{|x|}r - |x|\frac{y}{r}\right|. \tag{9.35}$$

Now the function

$$h_x(y) = \phi\left(r\hat{x} - \frac{|x|}{r}y\right) = \phi\left(\frac{|x|}{r}\left(y - r^2\frac{x}{|x|^2}\right)\right)$$

is harmonic in y and by Eq. (9.35),

$$h_x(y) = \phi\left(\hat{x}r - |x|\,\frac{y}{r}\right) = \phi\left(\hat{x}r - |x|\,\hat{y}\right) = \phi(x-y) \text{ when } |y| = r.$$

Hence we should define the Green's function for the ball to be given by

$$\begin{aligned}G(x,y) &= \phi(x-y) - h_x(y) = \phi(x-y) - \phi\left(\hat{x}r - |x|\frac{y}{r}\right) \\ &= \phi(x-y) - \phi\left(\hat{x}r - r^{-1}|x|\,|y|\,\hat{y}\right) \\ &= \phi(x-y) - \phi\left(\frac{|x|}{r}\left(y - r^2\frac{\hat{x}}{|x|}\right)\right).\end{aligned}$$

From Eq. (9.34), it follows that $h_x(y) = h_y(x)$ and therefore $G(x,y)$ is again symmetric under the interchange of x and y.

For $y \in \partial\overline{B(0,r)}$, using Eq. (9.7) we find

$$\begin{aligned}
-K(x,y) &= \frac{\partial G}{\partial \mathbf{n}_y}(x,y) = \partial_{\hat{y}} G(x,y) = \nabla_y G(x,y)\cdot\hat{y} \\
&= \nabla_y \left[\phi(x-y) - \phi\left(\frac{|x|}{r}\left(y - r^2\frac{\hat{x}}{|x|}\right)\right)\right]\cdot\hat{y} \\
&= -\frac{1}{\sigma(S^{n-1})}\left[\frac{1}{|y-x|^n}(y-x) - \frac{|x|}{r}\frac{1}{\left(\frac{|x|}{r}\right)^n\left|y - r^2\frac{\hat{x}}{|x|}\right|^n}\frac{|x|}{r}\left(y - r^2\frac{\hat{x}}{|x|}\right)\right]\cdot\hat{y} \\
&= -\frac{1}{\sigma(S^{n-1})}\left[\frac{1}{|y-x|^n}(y-x) - \frac{|x|}{r}\frac{1}{\left|\frac{|x|}{r}y - r\hat{x}\right|^n}\left(\frac{|x|}{r}y - r\hat{x}\right)\right]\cdot\hat{y} \\
&= -\frac{1}{\sigma(S^{n-1})}\left[\frac{1}{|y-x|^n}(y-x) - \frac{|x|}{r}\frac{1}{|x-y|^n}(|x|\hat{y} - r\hat{x})\right]\cdot\hat{y} \\
&= -\frac{1}{\sigma(S^{n-1})|y-x|^n}\left[(y-x) - \left(\frac{|x|^2}{r}\hat{y} - x\right)\right]\cdot\hat{y} \\
&= -\frac{1}{\sigma(S^{n-1})|y-x|^n}\left[y - \frac{|x|^2}{r}\hat{y}\right]\cdot\hat{y} \\
&= -\frac{1}{\sigma(S^{n-1})r|y-x|^n}\left[r^2 - |x|^2\right].
\end{aligned}$$

These computations lead to the following theorem.

Theorem 9.31. *For $x, y \in B(0,r)$, let*

$$G(x,y) := \phi(x-y) - \phi\left(\hat{x}r - |x|\frac{y}{r}\right)$$

and if $y \in \partial\overline{B(0,r)}$, let

$$K(x,y) := -\frac{\partial G}{\partial \mathbf{n}}(x,y) = \frac{r^2 - |x|^2}{\sigma(S^{n-1})r}|x-y|^{-n}.$$

Then $\rho \in C^2(B(0,r)) \cap L^1(B(0,r))$ and $f \in C\left(\partial\overline{B(0,r)}\right)$, then

$$u(x) = \int_{B(0,r)} G(x,y)\rho(y)dy + \int_{\partial\overline{B(0,r)}} K(x,y)f(y)d\sigma(y) \tag{9.36}$$

solves the equation

$$\Delta u = -\rho \text{ on } B(0,r) \text{ with } u = f \text{ on } \partial\overline{B(0,r)}.$$

Proof. The proof is essentially the same as Theorem 9.30 but a bit easier. From Theorem 9.5 with $u = 1$ it follows again that

$$\int_{\partial\overline{B(0,r)}} K(x,y)d\sigma(y) = 1.$$

As $x \to x_0 \in \partial\overline{B(0,r)}$, the function $K(x,y)$ becomes peaked for y near x_0 and goes to zero away from x_0, it follows by the standard approximate δ – function arguments that

$$\int_{\partial\overline{B(0,r)}} K(x,y)f(y)d\sigma(y) \to f(x_0) \text{ as } x \to x_0.$$

The rest of the argument is the same as before. ■

Corollary 9.32. *Suppose that u is a harmonic function on Ω, then u is real analytic on Ω.*

Proof. The condition of being real analytic is local and invariant under translations as is the notion of being harmonic. Hence we may assume $0 \in \overline{B(0,r)} \subset \Omega$ for some $r > 0$, in which case we have, for $|x| < r$ and $f = u|_{\partial\overline{B(0,r)}}$, that

$$u(x) = \int_{\partial\overline{B(0,r)}} K(x,y) f(y) d\sigma(y) = \frac{r^2 - |x|^2}{\sigma(S^{n-1})r} \int_{\partial\overline{B(0,r)}} |x-y|^{-n} f(y) d\sigma(y)$$

$$= \frac{r^2 - |x|^2}{\sigma(S^{n-1})r} \int_{\partial\overline{B(0,r)}} |x - \hat{y}r|^{-n} f(y) d\sigma(y). \tag{9.37}$$

Now

$$\begin{aligned} |x-y|^{-n} = |x-\hat{y}r|^{-n} &= r^{-n} \left|\hat{y} - r^{-1}x\right|^{-n} \\ &= r^{-n}\left(1 - 2r^{-1}\hat{y}\cdot x + \frac{|x|^2}{r^2}\right)^{-n/2} \\ &=: r^{-n}\left(1 - \alpha(x,y)\right)^{-n/2} \end{aligned}$$

where

$$\alpha(x,y) := 2r^{-1}\hat{y}\cdot x - \frac{|x|^2}{r^2}.$$

Since

$$|\alpha(x,y)| \le 2r^{-1}|x| + \frac{|x|^2}{r^2} \le 2\alpha_0 + \alpha_0^2 < 1$$

if $|x| \le \alpha_0 r$ and $\alpha_0 < \sqrt{2} - 1$, we find that $|x-y|^{-n}$ has a convergent power series expansion,

$$|x-y|^{-n} = r^{-n} \sum_{m=0}^{\infty} a_m \alpha(x,y)^m \text{ for } |x| \le \alpha_0 r.$$

Plugging this into Eq. (9.37) shows $u(x)$ has a convergent power series expansion in x for $|x| \le \left(\sqrt{2}-1\right) r$. ■

9.5. **Perron's Method for solving the Dirichlet Problem.** For this section let $\Omega \subset_o \mathbb{R}^n$ be a bounded open set and $f \in C\left(\mathrm{bd}(\Omega), \mathbb{R}\right)$ be a given function. We are going to investigate the solvability of the Dirichlet problem:

$$\Delta u = 0 \text{ on } \Omega \text{ with } u = f \text{ on } \mathrm{bd}(\Omega). \tag{9.38}$$

Let $\mathcal{S}(\Omega)$ denote those $w \in C\left(\bar{\Omega}\right)$ such that w is subharmonic on Ω and let $\mathcal{S}_f(\Omega)$ denote those $w \in \mathcal{S}(\Omega)$ such that $w \le f$ on $\mathrm{bd}(\Omega)$. As we have seen in Corollary 9.25, if there is a solution to $u \in C^2(\Omega) \cap C\left(\bar{\Omega}\right)$, then $w \le u$ for all $w \in \mathcal{S}_f(\Omega)$. This suggests we try to define

$$u(x) := u_f(x) := \sup\left\{w(x) : w \in \mathcal{S}_f(\Omega)\right\} \text{ for all } x \in \bar{\Omega}. \tag{9.39}$$

Notation 9.33. Given $w \in \mathcal{S}(\Omega)$, $\xi \in \Omega$ and $r > 0$ such that $\overline{B(\xi,r)} \subset \Omega$, let (see Figure 32)

$$w_{\xi,r}(y) = \begin{cases} w(y) & \text{for} \quad y \in \Omega \setminus B(\xi,r) \\ h(y) & \text{for} \quad y \in \overline{B(\xi,r)} \end{cases}$$

where $h \in C\left(\overline{B(\xi,r)}\right)$ is the unique solution to

$$\Delta h = 0 \text{ on } B(\xi,r) \text{ with } h = w \text{ on } \partial B(\xi,r).$$

The existence of h is guaranteed by Theorem 9.31.

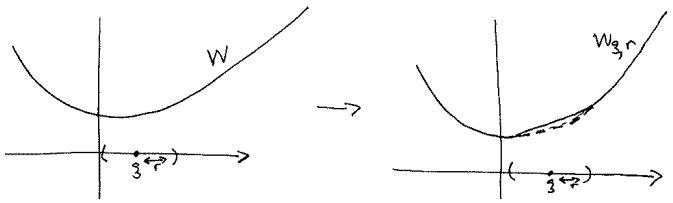

FIGURE 32. The construction of $w_{\xi,r}$ in the one-dimensional case.

Proposition 9.34. *Let $w \in \mathcal{S}(\Omega)$ and $w_{\xi,r}$ be as above. Then*

1. $w \le w_{\xi,r}$.
2. $w_{\xi,r} \in \mathcal{S}(\Omega)$, *i.e.* $w_{\xi,r}$ *is subharmonic.*
3. *We have*

$$w(\xi) \le \fint_{\partial B(\xi,r)} w d\sigma.$$

Proof. 1. Since $w = w_{\xi,r}$ on $\Omega \setminus B(\xi,r)$, it suffices to show $w \le h$ on $B(\xi,r)$. But this follows from Corollary 9.25.

2. Since $w_{\xi,r}$ is harmonic on $B(\xi,r)$ and subharmonic on $\Omega \setminus \overline{B(\xi,r)}$, we need only show

$$w_{\xi,r}(y) \le \fint_{\partial B(y,\rho)} w_{\xi,r} d\sigma$$

for all $y \in \partial B(\xi,r)$ and ρ sufficiently small. This is easily checked, since w is subharmonic,

$$w_{\xi,r}(y) = w(y) \le \fint_{\partial B(y,\rho)} w d\sigma \le \fint_{\partial B(y,\rho)} w_{\xi,r} d\sigma$$

wherein the last equality we made use of Item 1.

3. By item 1. and the mean value property for the harmonic function, $w_{\xi,r}$, we have

$$w(\xi) \le w_{\xi,r}(\xi) = \fint_{\partial B(\xi,r)} w_{\xi,r} d\sigma = \fint_{\partial B(\xi,r)} w d\sigma.$$

■

Theorem 9.35. *The function $u = u_f$ defined in Eq. (9.39) is harmonic on Ω and $u \le g$ on* $\mathrm{bd}(\Omega)$.

Proof. Let us begin with a couple of observations. In what follows

$$m := \min\{f(x) : x \in \mathrm{bd}(\Omega)\} \text{ and } M := \min\{f(x) : x \in \mathrm{bd}(\Omega)\}.$$

1. The function $u = u_f \ge m$ on Ω since $m \in \mathcal{S}_f(\Omega)$.

(2) By the maximum principle $w \leq M$ on Ω for all $w \in \mathcal{S}_f(\Omega)$ and therefore $u_f \leq M$ on Ω.
(3) If $w_1, \dots, w_m \in \mathcal{S}_f(\Omega)$, then $w = \max\{w_1, \dots, w_m\} \in \mathcal{S}_f(\Omega)$. Indeed for $\xi \in \Omega$ and r small,
$$\fint_{\partial B(\xi,r)} w d\sigma \geq \fint_{\partial B(\xi,r)} w_i d\sigma \geq w_i(\xi)$$
for all i.
(4) Now suppose $\xi \in \Omega$ and $R > 0$ be chosen so that $\overline{B(\xi,R)} \subset \Omega$ and $D \subset B(\xi,R)$ is a countable set. Then there is a harmonic function w_D on $B(\xi,R)$ such that $w_D = u_f$ on D.

To prove this last item let $D := \{y_k\}_{k=1}^{\infty}$ and choose $\{w_k^m\} \subset \mathcal{S}_f(\Omega)$ such that $w_k^m(y_k) \to u(y_k)$ as $m \to \infty$ for each k. By replacing w_k^m by $\max\{w_k^1, \dots, w_k^m\}$ if necessary we may assume for each k that w_k^m is increasing in m for each k. Letting $W_m := \max\{w_1^m, \dots, w_k^m\}$ we find an increasing sequence $\{W_m\} \subset \mathcal{S}_f(\Omega)$ such that $W_m(y) \uparrow u_f(y)$ for all $y \in D$. Finally define a sequence $\{w_m\} \subset \mathcal{S}_f(\Omega)$ by $w_m := (W_m)_{\xi,2R}$. By the maximum principle, w_m is still increasing and since $W_m \leq w_m$ and we still have $w_m(y) \uparrow u_f(y)$ for all $y \in D$. We now define $w_D := \lim_{m\to\infty} w_m|_{B(\xi,R)}$ which exists because w_m is increasing and w. We have $w_D = u_f$ on D and because $\{w_m\}$ is a bounded and convergent sequence of harmonic functions on $B(\xi,R)$, it follows from Corollary 9.18 that w is harmonic on $B(\xi,R)$. This completes the proof of item 4.

We now use item 4. to prove u_f is continuous at $\xi \in \Omega$. To do this let $\{y_k\}_{k=1}^{\infty} \subset B(\xi,R)$ be any sequence such that $y_k \to \xi$ as $k \to \infty$ and let $D = \{\xi\} \cup \{y_k\}_{k=1}^{\infty} \subset B(\xi,R)$. Since w_D is harmonic and hence continuous,
$$\lim_{k\to\infty} u_f(y_k) = \lim_{k\to\infty} w_D(y_k) = w_D(\xi) = u_f(\xi)$$
showing u_f is continuous.

To show u_f is harmonic on $B(\xi,R)$, let D be a countable dense subset of $B(\xi,R)$. Then the continuity of u_f and the fact that $u_f = w_D$ on D, it follows that $u_f = w_D$ on $B(\xi,R)$. In particular u_f is harmonic on $B(\xi,R)$. Since ξ is arbitrary, we have shown u_f is harmonic. ■

To complete our program, we want to show that u_f extends to a function in $C(\bar{\Omega})$ and that $u_f = f$ on $\mathrm{bd}(\Omega)$. For this we will need some assumption on $\mathrm{bd}(\Omega)$.

Definition 9.36. A function $Q \in C(\bar{\Omega})$ is a **barrier function** for $\eta \in \mathrm{bd}(\Omega)$ if Q is subharmonic on Ω, $Q(\eta) = 0$ and $Q(x) < 0$ for all $x \in \mathrm{bd}(\Omega) \setminus \{\eta\}$.

Example 9.37. Suppose that $\eta \in \mathrm{bd}(\Omega)$ and there exists $\xi \in \mathbb{R}^n$ such that $(x-\eta)\cdot\xi < 0$ for all $x \in \mathrm{bd}(\Omega) \setminus \{\eta\}$ (see Figure 33 below), then the function $Q(x) := (x-\eta)\cdot\xi$ is a barrier function of η.

Example 9.38. Suppose that $\eta \in \mathrm{bd}(\Omega)$ and there exists a ball $\overline{B(\xi,r)} \cap \bar{\Omega} = \{\eta\}$ (see Figure 34), then $Q(x) := \alpha(r) - \alpha(|x-\xi|)$ is a barrier function for η, where α is defined in Eq. 9.4.

Theorem 9.39. *Suppose $f \in C(\mathrm{bd}(\Omega))$ and $u = u_f$ is the harmonic function defined by Eq. (9.39) and there exists a barrier function Q for $\eta \in \mathrm{bd}(\Omega)$. Then*

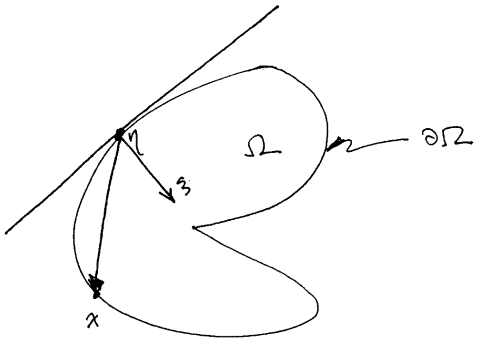

FIGURE 33. Constructing a barrier function at point where η where $\partial\Omega$ lies in a half plane.

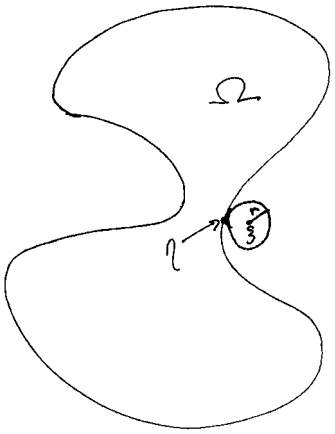

FIGURE 34. Another η for which there exists a barrier function.

$\lim_{x\to\eta} u_f(x) = f(\eta)$. *In particular if every point* $\eta \in \mathrm{bd}(\Omega)$ *admits a barrier function, then there is a unique solution* $u \in C\left(\bar{\Omega}\right) \cap C^2(\Omega)$ *to* $\Delta u = 0$ *with* $u = f$ *on* $\mathrm{bd}(\Omega)$.

Proof. Given $\epsilon > 0$ and $K > 0$, let $w(x) := f(\eta) - \epsilon - KQ(x)$ for all $x \in \bar{\Omega}$. For any $\epsilon > 0$ we may choose (using continuity of f and compactness of $\mathrm{bd}(\Omega)$) K sufficiently large so that $w \le f$ on $\mathrm{bd}(\Omega)$, i.e. $w \in \mathcal{S}_f(\Omega)$. Therefore $w \le u_f$ and hence

$$f(\eta) - \epsilon = w(\eta) = \lim_{x\to\eta} w(x) \le \liminf_{x\to\eta} u_f(x).$$

Since $\epsilon > 0$ is arbitrary, this shows

$$\liminf_{x\to\eta} u_f(x) \ge f(\eta). \tag{9.40}$$

We now consider the function

$$\begin{aligned} -u_{-f}(x) &= -\sup\{w(x) : w \in \mathcal{S}_{-f}(\Omega)\} = \inf\{-w(x) : w \in \mathcal{S}_{-f}(\Omega)\} \\ &= \inf\{W(x) : -W \in \mathcal{S}_{-f}(\Omega)\}. \end{aligned} \tag{9.41}$$

If $w \in \mathcal{S}_f(\Omega)$ and $-W \in \mathcal{S}_{-f}(\Omega)$, then $w-W$ is sub harmonic and $w-W \leq f-f = 0$ on $\mathrm{bd}(\Omega)$, therefore by the maximum principle it follows that $w \leq W$ on $\bar{\Omega}$. Using this in Eq. (9.41) shows

$$-u_{-f}(x) \geq \inf\{w(x) : w \in \mathcal{S}_f(\Omega)\} = u_f(x).$$

Therefore,

(9.42)

$$\limsup_{x\to\eta} u_f(x) \leq \limsup_{x\to\eta} (-u_{-f}(x)) = -\liminf_{x\to\eta} (u_{-f}(x)) = -(-f(\eta)) = f(\eta)$$

which combined with Eq. (9.40) shows

$$\lim_{x\to\eta} u_f(x) = f(\eta).$$

■

Exercise 9.1. Suppose that R is an $n \times n$ orthogonal matrix ($R^{\mathrm{tr}}R = I = RR^{\mathrm{tr}}$) viewed as a linear transformation on $\mathbb{R}^n$. Show for $f \in C^2(\mathbb{R}^n)$ that $\Delta(f \circ R) = \Delta f \circ R$, i.e. Δ is invariant under rotations.

Exercise 9.2. Show that every point $\eta \in \mathrm{bd}(\Omega)$ has a barrier function when $\mathrm{bd}(\Omega)$ is C^2. **Hint:** By making a change of coordinated involving rotations and translations change of coordinates, it suffices to assume $\eta = 0 \in \mathrm{bd}(\Omega)$ and that $B(0,r) \cap \mathrm{bd}(\Omega)$ is the graph of a C^2 – function $g : B(0,r) \cap \mathbb{R}^{n-1} \to \mathbb{R}^n$ such that $g(0) = 0$ and $\nabla g(0) = 0$. Show for $\delta > 0$ sufficiently small that

$$d_\delta(x) := |\delta e_n - x|^2 \text{ for } x \in \mathrm{bd}(\Omega)$$

has a unique global minimum at $x = 0$. Use this fact and Example 9.38 to complete the proof.

9.6. **Solving the Dirichlet Problem by Integral Equations.** Another method for solving the Dirichlet problem to reduce it to a question of solvability of a certain integral equation in $\mathrm{bd}(\Omega)$. For a nice sketch of how this goes the reader is referred to Reed and Simon [2], included below. For a more detailed account the reader may consult Sobolev [3] or Guenther and Lee [1].

The following text is taken from Reed and Simon Volume 1.

uses Results for the Banach space C

Example (Dirichlet problem) The main impetus for the study of compact operators arose from the use of integral equations in attempting to solve the classical boundary value problems of mathematical physics. We briefly describe this method. Let D be an open bounded region in $\mathbb{R}^3$ with a smooth boundary surface ∂D. The Dirichlet problem for Laplace's equation is: given a continuous function f on ∂D, find a function u, twice differentiable in D and continuous on $\bar{D}$, which satisfies

$$\begin{aligned} \Delta u(x) &= 0 \qquad x \in D \\ u(x) &= f(x) \qquad x \in \partial D \end{aligned}$$

Let $K(x, y) = (x - y, n_y)/2\pi|x - y|^3$ where n_y is the outer normal to ∂D at the point $y \in \partial D$. Then, as a function of x, $K(x, y)$ satisfies $\Delta_x K(x, y) = 0$ in the interior which suggests that we try to write u as a superposition

$$u(x) = \int_{\partial D} K(x, y)\varphi(y)\, dS(y) \qquad \text{(VI.6a)}$$

where $\varphi(y)$ is some continuous function on ∂D and dS is the usual surface measure. Indeed, for $x \in D$, the integral makes perfectly good sense and

$\Delta u(x) = 0$ in D. Furthermore, if x_0 is any point in ∂D and $x \to x_0$ from *inside* D, it can be proven that

$$u(x) \to -\varphi(x_0) + \int_{\partial D} K(x_0, y)\varphi(y)\, dS(y) \qquad \text{(VI.6b)}$$

If $x \to x_0$ from outside D, the minus is replaced by a plus. Also,

$$\int_{\partial D} K(x_0, y)\varphi(y)\, dS(y)$$

ists and is a continuous function on ∂D if φ is a continuous function on ∂D. The proof depends on the fact that the boundary of D is smooth which implies that for $x, y \in \partial D$, $(x - y, n_y) \approx c|x - y|^2$ as $x \to y$.

Since we wish $u(x) = f(x)$ on ∂D, the whole question reduces to whether we can find φ so that

$$f(x) = -\varphi(x) + \int_{\partial D} K(x, y)\varphi(y)\, dS(y), \qquad x \in \partial D$$

Let T: $C(\partial D) \to C(\partial D)$ be defined by

$$T\varphi = \int_{\partial D} K(x, y)\varphi(y)\, dS(y)$$

Not only is T bounded but (as we will shortly see) T is also compact. Thus, by the Fredholm alternative, either $\lambda = 1$ is in the point spectrum of T in which case there is a $\psi \in C(\partial D)$ such that $(I - T)\psi = 0$, or $-f = (I - T)\varphi$ has a unique solution for each $f \in C(\partial D)$. If u is defined by (VI.6a) with ψ replacing φ, then $u \equiv 0$ in D by the maximum principle. Further, $\partial u/\partial n$ is continuous across ∂D and therefore equals zero on ∂D. By an integration by parts this implies that $u \equiv 0$ outside ∂D. Therefore, by (VI.6b), $2\psi(x) \equiv 0$ on ∂D, so the first alternative does not hold.

The idea of the compactness proof is the following. Let

$$K_\delta(x, z) = \frac{(x - z, n_z)}{|x - z|^3 + \delta}$$

If $\delta > 0$, the kernel K_δ is continuous, so, by the discussion at the beginning of this section, the corresponding integral operators T_δ, are compact. To prove that T is compact, we need only show that $\|T - T_\delta\| \to 0$ as $\delta \to 0$. By the estimate

$$|(T_\delta f)(x) - (Tf)(x)| \le \|f\|_\infty \int_{\partial D} |K(x, z) - K_\delta(x, z)|\, dS(z)$$

we must only show that the integral converges to zero uniformly in x as $\delta \to 0$. To prove this, divide the integration region into the set where $|x - z| \ge \varepsilon$ and its complement. For fixed ε, the kernels converge uniformly on the first region. By using the fact that K is integrable, the contribution from the second region can be made arbitrarily small for ε sufficiently small.