A PRIMER ON RIEMANNIAN GEOMETRY AND STOCHASTIC ANALYSIS ON PATH SPACES

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Abstract. These notes represent an expanded version of the "mini course" that the author gave at the ETH (Zürich) and the University of Zürich in February of 1995. The purpose of these notes is to provide some basic background to Riemannian geometry, stochastic calculus on manifolds, and infinite dimensional analysis on path spaces. No differential geometry is assumed. However, it is assumed that the reader is comfortable with stochastic calculus and differential equations on Euclidean spaces.

Acknowledgement: It is pleasure to thank Professor A. Sznitman and the ETH for the opportunity to give these talks and for a very enjoyable stay in Zürich. I also would like to thank Professor E. Bolthausen for his hospitality and his role in arranging for the first lecture which was held at University of Zürich.

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Date: September 5, 1995. *File:ETHPRIME.tex* Last revised: January 29, 2003.
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1. Summary of ETH talk contents

In this section let me summarize the contents of the talks at the ETH and Zürich.

(1) The first talk was on an extension of the Cameron Martin quasi-invariance theorem to manifolds. This lecture is not contained in these notes. The interested reader may consult Driver [39, 40] for the original papers. For more expository papers on this topic see [41, 43]. (These papers are complimentary to these notes.) The reader should also consult Hsu [80], Norris [112], and Enchev and Stroock [56, 57] for the state of the art in this topic.
(2) The second lecture encompassed sections 1-2.3 of these notes. This is an introduction to embedded submanifolds and the Riemannian geometry on them which is induced from the ambient space.
(3) The third lecture covered sections 2.4-2.7. The topics were parallel translation, the development map, and the differential of the development map. This was all done for smooth paths.
(4) The fourth lecture covered parts of sections 3 and 4. Here we touched on stochastic development map and its differential. Integration by parts formula for the path space and some spectral properties of an "Ornstein-Uhlenbeck" like operator on the path space.

2. Manifold Primer

Conventions: Given two sets A and B, the notation $f : A \to B$ will mean that f is a function from a subset $\mathcal{D}(f) \subset A$ to B. (We will allow $\mathcal{D}(f)$ to be the empty set.) The set $\mathcal{D}(f) \subset A$ is called the domain of f and the subset $\mathcal{R}(f) \doteq f(\mathcal{D}(f)) \subset B$ is called the range of f. If f is injective we let $f^{-1} : B \to A$ denote the inverse function with domain $\mathcal{D}(f^{-1}) = \mathcal{R}(f)$ and range $\mathcal{R}(f^{-1}) = \mathcal{D}(f)$. If $f : A \to B$ and $g : B \to C$, the $g \circ f$ denotes the composite function from A to C with domain $\mathcal{D}(g \circ f) \doteq f^{-1}(\mathcal{D}(g))$ and range $\mathcal{R}(g \circ f) \doteq g \circ f(\mathcal{D}(g \circ f)) = g(\mathcal{R}(f) \cap \mathcal{D}(g))$.

Notation 2.1. Throughout these notes, let E and V denote finite dimensional vector spaces. A function $F : E \to V$ is said to be smooth if $\mathcal{D}(F)$ is open in E (empty set ok) and $F : \mathcal{D}(F) \to V$ is infinitely differentiable. Given a smooth function $F : E \to V$, let $F'(x)$ denote the differential of F at $x \in \mathcal{D}(F)$. Explicitly, $F'(x)$ denotes the linear map from E to V determined by

$$F'(x)a \doteq \frac{d}{dt}|_0 F(x + ta), \quad \forall a \in E. \tag{2.1}$$

2.1. **Embedded Submanifolds.** Rather than describe the most abstract setting for Riemannian geometry, for simplicity we choose to restrict our attention to embedded submanifolds of a Euclidean space E.[1] Let $N \doteq \dim(E)$.

Definition 2.2. A subset M of E (see Figure 1) is a d-dimensional embedded submanifold of E iff for all $m \in M$, there is a function $z : E \to \mathbb{R}^N$ such that:

(1) $\mathcal{D}(z)$ is an open neighborhood of E containing m,
(2) $\mathcal{R}(z)$ is an open subset of $\mathbb{R}^N$,
(3) $z : \mathcal{D}(z) \to \mathcal{R}(z)$ is a diffeomorphism (a smooth invertible map with smooth inverse), and
(4) $z(M \cap \mathcal{D}(z)) = \mathcal{R}(z) \cap (\mathbb{R}^d \times \{0\}) \subset \mathbb{R}^N$.

(We write M^d if we wish to emphasize that M is a d-dimensional manifold.)

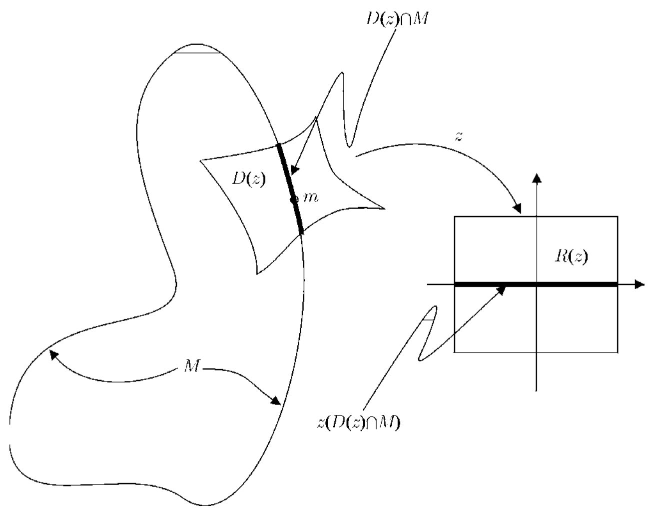

FIGURE 1. An embedded submanifold.

Notation 2.3. Given an embedded submanifold and diffeomorphism z as in the above definition, we will write $z = (z_<, z_>)$ where $z_<$ is the first d components of z and $z_>$ consists of the last $N - d$ components of z. Also let $x : M \to \mathbb{R}^d$ denote the function defined by: $\mathcal{D}(x) \doteq M \cap \mathcal{D}(z)$, and $x \doteq z_<|_{D(x)}$. Notice that $\mathcal{R}(x) \doteq x(\mathcal{D}(x))$ is an open subset of $\mathbb{R}^d$ and that $x^{-1} : \mathcal{R}(x) \to \mathcal{D}(x)$, thought of as a function taking values in E, is smooth. The bijection $x : \mathcal{D}(x) \to \mathcal{R}(x)$ is called a chart on M. Let $\mathcal{A} = \mathcal{A}(M)$ denote the collection of charts on M. The collection of charts $\mathcal{A} = \mathcal{A}(M)$ is often referred to an **Atlas** for M.

Remark 2.4. The embedded submanifold M is made into a topological space using the induced topology from E. With this topology, each chart $x \in \mathcal{A}(M)$ is a homeomorphism from $\mathcal{D}(x) \subset_o M$ to $\mathcal{R}(x) \subset_o \mathbb{R}^d$.

Theorem 2.5 (A Basic Construction of Manifolds)**.** *Let $F : E \to \mathbb{R}^{N-d}$ be a smooth function and $M \doteq F^{-1}(\{0\}) \subset E$ which we assume to be non-empty. Suppose that*

[1]Because of the Whitney imbedding theorem (see for example Theorem 6-3 in Auslander and MacKenzie [17]), this is actually not a restriction.

$F'(m) : E \to \mathbb{R}^{N-d}$ is surjective for all $m \in M$, then M is a d – dimensional embedded submanifold of E.

Proof. We will begin by construction a smooth function $G : E \to \mathbb{R}^d$ such that $(G, F)'(m) : E \to \mathbb{R}^N = \mathbb{R}^d \times \mathbb{R}^{N-d}$ is invertible. To do this, let $X = \mathrm{Nul}(F'(m))$ and Y be a complementary subspace so that $E = X \oplus Y$ and let $P : E \to X$ be the associated projection map. Notice that $F'(m) : Y \to \mathbb{R}^{N-d}$ is a linear isomorphism of vector spaces and hence

$$\dim(X) = \dim(E) - \dim(Y) = N - (N - d) = d.$$

In particular, X and $\mathbb{R}^d$ are isomorphic as vector spaces. Set $G(m) = APm$ where $A : X \to \mathbb{R}^d$ is any linear isomorphism of vector spaces. Then for $x \in X$ and $y \in Y$,

$$\begin{aligned}(G, F)'(m)(x + y) &= (G'(m)(x + y), F'(m)(x + y)) \\ &= (AP(x + y), F'(m)y) = (Ax, F'(m)y) \in \mathbb{R}^d \times \mathbb{R}^{N-d}\end{aligned}$$

from which it follows that $(G, F)'(m)$ is an isomorphism.

By the implicit function theorem, there exists a neighborhood $U \subset_o E$ of m such that $V := (G, F)(U) \subset_o \mathbb{R}^N$ and $(G, F) : U \to V$ is a diffeomorphism. Let $z = (G, F)$ with $\mathcal{D}(z) = U$ and $\mathcal{R}(z) = V$ then z is a chart of E about m satisfying the conditions of Definition 2.2. Indeed, items 1) – 3) are clear by construction. If $p \in M \cap \mathcal{D}(z)$ then $z(p) = (G(p), F(p)) = (G(p), 0) \in \mathcal{R}(z) \cap (\mathbb{R}^d \times \{0\})$ and $p \in \mathcal{D}(z)$ is a point such that $z(p) = (G(p), F(p)) \in \mathcal{R}(z) \cap (\mathbb{R}^d \times \{0\})$, then $F(p) = 0$ and hence $p \in M \cap \mathcal{D}(z)$. ■

Example 2.6. Let $gl(n, \mathbb{R})$ denote the set of all $n \times n$ real matrices. The following are examples of embedded submanifolds.

(1) Any open subset M of E.
(2) Graphs of smooth functions. (Why? You should produce a chart z.)
(3) $S^{N-1} \doteq \{x \in \mathbb{R}^{\mathbb{N}} | x \cdot x = 1\}$, take $E = \mathbb{R}^{\mathbb{N}}$ and $F(x) \doteq x \cdot x - 1$.
(4) $GL(n, \mathbb{R}) \doteq \{g \in gl(n, \mathbb{R}) | \det(g) \neq 0\}$, see item 1.
(5) $SL(n, \mathbb{R}) \doteq \{g \in gl(n, \mathbb{R}) | \det(g) = 1\}$, take $E = gl(n, \mathbb{R})$ and $F(g) \doteq \det(g)$. Recall that

$$\det{}'(g)A = \det(g)\mathrm{tr}(g^{-1}A) \tag{2.2}$$

for all $g \in GL(n, \mathbb{R})$. Let us recall the proof of Eq. (2.2). By definition we have

$$\det{}'(g)A = \frac{d}{dt}|_0 \det(g + tA) = \det(g)\frac{d}{dt}|_0 \det(I + tg^{-1}A).$$

So it suffices to prove $\frac{d}{dt}|_0 \det(I + tB) = \mathrm{tr}(B)$ for all matrices B. Now this is easily checked if B is upper triangular since then $\det(I + tB) = \prod_{i=1}^d (1 + tB_{ii})$ and hence by the product rule,

$$\frac{d}{dt}|_0 \det(I + tB) = \sum_{i=1}^d B_{ii} = \mathrm{tr}(B).$$

This completes the proof because: 1) every matrix can be put into upper triangular form by a similarity transformation and 2) det and tr are invariant under similarity transformations.

(6) $O(n) \doteq \{g \in gl(n,\mathbb{R}) | g^t g = I\}$, take $F(g) \doteq g^t g - I$ thought of as a function from $E = gl(n,\mathbb{R})$ to $\mathcal{S}(n)$, the symmetric matrices in $gl(n,\mathbb{R})$. To show $F'(g)$ is surjective, show

$$F'(g)(gB) = B + B^t \text{ for all } g \in O(n) \text{ and } B \in gl(n,\mathbb{R}).$$

(7) $SO(n) \doteq \{g \in O(n) | \det(g) = 1\}$, this is an open subset of $O(n)$.
(8) $M \times N$, where M and N are embedded submanifolds.
(9) $T^n \doteq \{z \in \mathbb{C}^n : |z^i| = 1 \text{ for } i = 1,2,\ldots,n\} = (S^1)^n$.

Definition 2.7. Let E and V be two finite dimensional vector spaces and $M^d \subset E$ and $N^k \subset V$ be two embedded submanifolds. A function $f : M \to N$ is said to be smooth if for all charts $x \in \mathcal{A}(M)$ and $y \in \mathcal{A}(N)$ the function $y \circ f \circ x^{-1} : \mathbb{R}^d \to \mathbb{R}^k$ is smooth.

Exercise 2.8. Let $M^d \subset E$ and $N^k \subset V$ be two embedded submanifolds as in Definition 2.7.

(1) Show that a function $f : \mathbb{R}^k \to M$ is smooth iff f is smooth when thought of as a function from $\mathbb{R}^k$ to E.
(2) If $F : E \to V$ is a smooth function such that $F(M \cap \mathcal{D}(F)) \subset N$, show that $f \doteq F|_M : M \to N$ is smooth.
(3) Show the composition of smooth maps between embedded submanifolds is smooth.

Suppose that $f : M \to N$ is smooth, $m \in M$ and $n = f(m)$. Since $M \subset E$ and $N \subset V$ are embeddded submanifolds, there are charts z and w on M and N respectively such that $m \in \mathcal{D}(z)$ and $n \in \mathcal{D}(w)$. By shrinking the domain of z if necessary, we may assmue that $\mathcal{R}(z) = U \times W$ where $U \subset_o \mathbb{R}^d$ and $W \subset_o \mathbb{R}^{N-d}$ in which case $z(M \cap \mathcal{D}(z)) = U \times \{0\}$. For $\xi \in \mathcal{D}(z)$, let $F(\xi) := f(z^{-1}(z_<(\xi), 0))$. Then $F : \mathcal{D}(z) \to N$ is a smooth function such that $F|_{M\cap\mathcal{D}(z)} = f|_{M\cap\mathcal{D}(z)}$. To see that F is smooth, we notice that

$$w_< \circ F = w_< \circ f(z^{-1}(z_<(\xi), 0)) = w_< \circ f \circ x^{-1} \circ (z_<(\cdot), 0)$$

where $x = z_<|_{\mathcal{D}(z)\cap M}$. By assumption $w_< \circ f \circ x^{-1}$ is smooth and since $\xi \to (z_<(\xi), 0)$, it follows $w_< \circ F$ is smooth showing F is smooth as claimed. Using a partition of unity argument (which we omit), one may use these ideas to prove the following fact.

Fact 2.9. Assuming the notation in Definition 2.7, a function $f : M \to N$ is smooth iff there is a smooth function $F : E \to V$ such that $f = F|_M$.

2.2. **Tangent Planes and Spaces.**

Definition 2.10. Given an embedded submanifold $M \subset E$ and $m \in M$, let $\tau_m M \subset E$ denote the collection of all vectors $v \in E$ such there exists a smooth curve $\sigma : (-\epsilon, \epsilon) \to M$ with $\sigma(0) = m$ and $v = \frac{d}{ds}|_0 \sigma(s)$. The subset $\tau_m M$ is called the tangent plane to M and m.

Theorem 2.11. *For each $m \in M$, $\tau_m M$ is a d-dimensional subspace of E. If $z : E \to \mathbb{R}^N$ is as in Definition 2.2, then $\tau_m M = nul(z'_>(m))$. If x is a chart on M such that $m \in \mathcal{D}(x)$, then*

$$\{\frac{d}{ds}|_0 x^{-1}(x(m) + se_i)\}_{i=1}^d$$

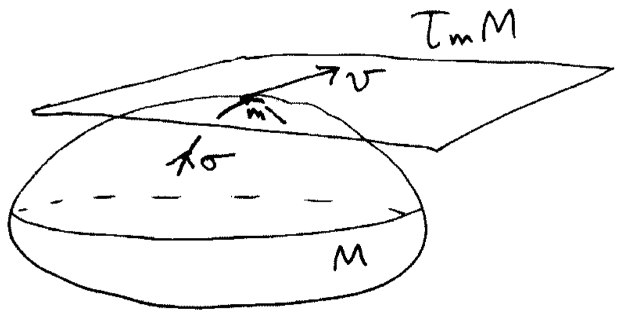

FIGURE 2. The tangent plane

is a basis for $\tau_m M$, *where* $\{e_i\}_{i=1}^d$ *is the standard basis for* $\mathbb{R}^d$.

Proof. Let $\sigma : (-\epsilon, \epsilon) \to M$ be a smooth curve with $\sigma(0) = m$ and $v = \frac{d}{ds}|_0 \sigma(s)$ and z be a chart around m as in Definition 2.2. Then $z_>(\sigma(s)) = 0$ for all s and therefore,

$$0 = \frac{d}{ds}|_0 z_>(\sigma(s)) = z'_>(m)v$$

which shows that $v \in \mathrm{nul}(z'_>(m))$, i.e. $\tau_m M \subset \mathrm{nul}(z'_>(m))$. Conversely, suppose that $v \in \mathrm{nul}(z'_>(m))$. Let $w = z'_<(m)v \in \mathbb{R}^d$ and $\sigma(s) := x^{-1}(z_<(m) + sw) \in M$ – defined for s near 0. Then by definition $\sigma'(0) \in \tau_m M$ which implies $\mathrm{nul}(z'_>(m)) \subset \tau_m M = \mathrm{nul}(z'_>(m))$ because $\sigma'(0) = v$. Indeed, differenitating the indentity $z^{-1} \circ z = id$ at m shows

$$\left(z^{-1}\right)'(z(m))z'(m) = I$$

and hence

$$\begin{aligned}\sigma'(0) &= \frac{d}{ds}|_0 x^{-1}(z_<(m) + sw) = \frac{d}{ds}|_0 z^{-1}(z_<(m) + sw, 0)\\ &= \left(z^{-1}\right)'((z_<(m), 0))(z'_<(m)v, 0) = \left(z^{-1}\right)'(z(m))z'(m)v\\ &= v.\end{aligned}$$

This completes the proof that $\tau_m M = \mathrm{nul}(z'_>(m))$.

Since $z'_<(m) : \tau_m M \to \mathbb{R}^d$ is a linear isomorphism, the above argument has also shown, for any $w \in \mathbb{R}^d$, that

$$\frac{d}{ds}|_0 x^{-1}(x(m) + sw) = \left(z'_<(m)|_{\tau_m M}\right)^{-1} w \in \tau_m M.$$

In particular it follows that

$$\{\frac{d}{ds}|_0 x^{-1}(x(m) + se_i)\}_{i=1}^d = \{\left(z'_<(m)|_{\tau_m M}\right)^{-1} e_i\}_{i=1}^d$$

is a is a basis for $\tau_m M$, ■

The following proposition is an easy consequence of Theorem 2.11 and the proof of Theorem 2.5.

Proposition 2.12. *Suppose that* M *is an embedded submanifold constructed as in Theorem 2.5, then*

$$\tau_m M = nul\{F'(m)\}.$$

Exercise 2.13. Show:

(1) $\tau_m M = E$, if M is an open subset of E.
(2) $\tau_g GL(n,\mathbb{R}) = gl(n,\mathbb{R})$, for all $g \in GL(n,\mathbb{R})$.
(3) $\tau_m S^{N-1} = \{m\}^\perp$ for all m in the $(N-1)$-dimensional sphere S^{N-1}.
(4) $\tau_g SL(n,\mathbb{R}) = \{A \in gl(n,\mathbb{R}) | \mathrm{tr}(g^{-1}A) = 0\}$.
(5) $\tau_g O(n) = \{A \in gl(n,\mathbb{R}) | g^{-1}A \text{ is skew symmetric}\}$. **Hint:** $g^{-1} = g^t$ for all $g \in O(n)$.
(6) if $M \subset E$ and $N \subset V$ are embedded submanifolds then

$$\tau_{(m,n)}(M \times N) = \tau_m M \times \tau_n N \subset E \times V.$$

Since it is quite possible that $\tau_m M = \tau_{m'} M$ for some $m \neq m'$, with m and m' in M (think of the sphere), it is helpful to label each of the tangent planes with their base point. For this reason we introduce the following definition.

Definition 2.14. The **tangent space** $(T_m M)$ to M at m is given by

$$T_m M \doteq \{m\} \times \tau_m M \subset M \times E.$$

Let

$$TM \doteq \cup_{m \in M} T_m M,$$

and call TM the **tangent space (or tangent bundle)** of M. A **tangent vector** is a point $v_m \equiv (m, v) \in TM$. Each tangent space is made into a vector space using vector space operations: $c(v_m) \equiv (cv)_m$ and $v_m + w_m \doteq (v+w)_m$.

Exercise 2.15. Prove that TM is an embedded submanifold of $E \times E$. **Hint:** suppose that $z : E \to \mathbb{R}^N$ is a function as in the Definition 2.2. Define $\mathcal{D}(Z) \doteq \mathcal{D}(z) \times E$ and $Z : \mathcal{D}(Z) \to \mathbb{R}^N \times \mathbb{R}^N$ by $Z(x,a) \doteq (z(x), z'(x)a)$. Use Z's of this type to check TM satisfies Definition 2.2.

Given a smooth curve $\sigma : (-\epsilon, \epsilon) \to M$, let

$$\sigma'(0) \doteq (\sigma(0), \frac{d}{ds}|_0 \sigma(s)) \in T_{\sigma(0)} M.$$

By definition, we know that all tangent vectors are constructed this way. Given a chart $x = (x^1, x^2, \ldots, x^d)$ on M and $m \in \mathcal{D}(x)$, let $\partial/\partial x^i|_m$ denote the element $T_m M$ determined by $\partial/\partial x^i|_m = \sigma'(0)$, where $\sigma(s) \doteq x^{-1}(x(m) + se_i)$, i.e.

$$\partial/\partial x^i|_m = (m, \frac{d}{ds}|_0 x^{-1}(x(m) + se_i)), \tag{2.3}$$

see Figure 3. (The reason for this strange notation should become clear shortly.) Because of Theorem 2.11, $\{\partial/\partial x^i|_m\}_{i=1}^d$ is a basis for $T_m M$.

Definition 2.16. Suppose that $f : M \to V$ is a smooth function, $v_m \in T_m M$, and $m \in \mathcal{D}(f)$. Write

$$df\langle v_m \rangle = \frac{d}{ds}|_0 f(\sigma(s)),$$

where σ is any smooth curve in M such that $\sigma'(0) = v_m$. We also write $df\langle v_m \rangle$ as $v_m f$. The function $df : TM \to V$ will be called the differential of f.

To understand the notation in (2.3), suppose that $f = F \circ x = F(x^1, x^2, \ldots, x^d)$ where $F : \mathbb{R}^d \to \mathbb{R}$ is a smooth function and x is a chart on M. Then

$$\partial f(m)/\partial x^i = (D_i F)(x(m)),$$

where D_i denotes the ith partial derivative of F.

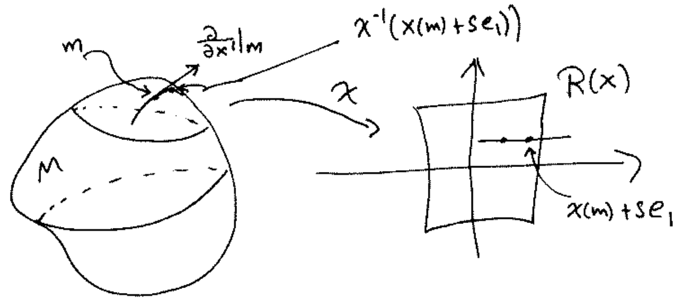

FIGURE 3. Forming a basis of tangent vectors.

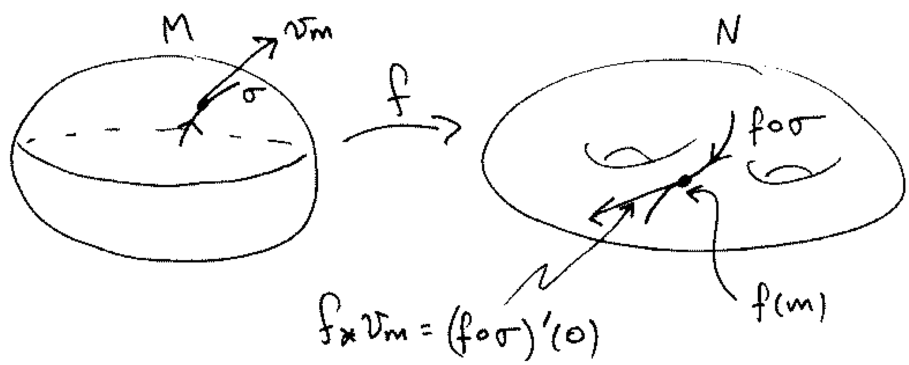

FIGURE 4. The differential of f.

Remark 2.17 (Product Rule). Suppose that $f : M \to V$ and $g : M \to \mathrm{End}(V)$ are smooth functions, then

$$v_m(gf) = \frac{d}{ds}|_0 g(\sigma(s))f(\sigma(s)) = v_m g \cdot f(m) + g(m)v_m f$$

or equivalently

$$d(gf)\langle v_m\rangle = dg\langle v_m\rangle f(m) + g(m)df\langle v_m\rangle.$$

This last equation will be abbreviated $d(gf) = dg \cdot f + gdf$.

Definition 2.18. Let $f : M \to N$ be a smooth map of embedded submanifolds. Define the differential (f_*) of f by

$$f_* v_m = (f \circ \sigma)'(0) \in T_{f(m)}N,$$

where $v_m = \sigma'(0) \in T_m M$, and $m \in \mathcal{D}(f)$.

Lemma 2.19. *The differentials defined in Definitions 2.16 and 2.18 are well defined linear maps on $T_m M$ for each $m \in \mathcal{D}(f)$.*

Proof. I will only prove that f_* is well defined, since the case of df is similar. By Fact 2.9, there is a smooth function $F : E \to V$, such that $f = F|_M$. Therefore by the chain rule

$$f_* v_m = (f \circ \sigma)'(0) \doteq (f(\sigma(0)), \frac{d}{ds}|_0 f(\sigma(s))) = (f(m), F'(m)v), \tag{2.4}$$

where σ is a smooth curve in M such that $\sigma'(0) = v_m$. It follows from (2.4) that $f_* v_m$ does not depend on the choice of the curve σ. It is also clear from (2.4), that f_* is linear on $T_m M$. ■

Remark 2.20. Suppose that $F : E \to V$ is a smooth function and that $f \doteq F|_M$. Then as in the proof of the above lemma,

$$df\langle v_m \rangle = F'(m)v \tag{2.5}$$

for all $v_m \in T_m M$, and $m \in \mathcal{D}(f)$. Incidentally, since the left hand sides of (2.4) and (2.5) are defined "intrinsically," the right members of (2.4) and (2.5) are independent of the choice of the functions F extending f.

Lemma 2.21 (Chain Rules)**.** *Suppose that M, N, and P are embedded submanifolds and V is a finite dimensional vector space. Let $f : M \to N$, $g : N \to P$, and $h : N \to V$ be smooth functions. Then:*

$$(g \circ f)_* v_m = g_*(f_* v_m), \quad \forall v_m \in TM \tag{2.6}$$

and

$$d(h \circ f)\langle v_m \rangle = dh\langle f_* v_m \rangle, \quad \forall v_m \in TM. \tag{2.7}$$

These equations may be written more concisely as $(g \circ f)_* = g_* f_*$ and $d(h \circ f) = dh f_*$ respectively.

Proof. Let σ be a smooth curve in M such that $v_m = \sigma'(0)$. Then, see Figure 5,

$$\begin{aligned}(g \circ f)_* v_m &\equiv (g \circ f \circ \sigma)'(0) = g_*(f \circ \sigma)'(0) \\ &= g_* f_* \sigma'(0) = g_* f_* v_m.\end{aligned}$$

Similarly,

$$\begin{aligned}d(h \circ f)\langle v_m \rangle &\equiv \frac{d}{ds}|_0 (h \circ f \circ \sigma)(s) = dh\langle (f \circ \sigma)'(0) \rangle \\ &= dh\langle f_* \sigma'(0) \rangle = dh\langle f_* v_m \rangle.\end{aligned}$$

■

If $f : M \to V$ is a smooth function, x is a chart on M, and $m \in \mathcal{D}(f) \cap \mathcal{D}(x)$, we will write $\partial f(m)/\partial x^i$ for $\langle df, \partial/\partial x^i|_m \rangle$. An easy computation using the definitions shows that $dx^i \langle \partial/\partial x^j|_m \rangle = \delta_{ij}$, from which it follows that $\{dx^i\}_{i=1}^d$ is the dual basis of $\{\partial/\partial x^i|_m\}_{i=1}^d$. Therefore

$$df\langle v_m \rangle = \sum_{i=1}^d \frac{\partial f(m)}{\partial x^i} dx^i \langle v_m \rangle,$$

which we will be abbreviated as

$$df = \sum_{i=1}^d \frac{\partial f}{\partial x^i} dx^i. \tag{2.8}$$

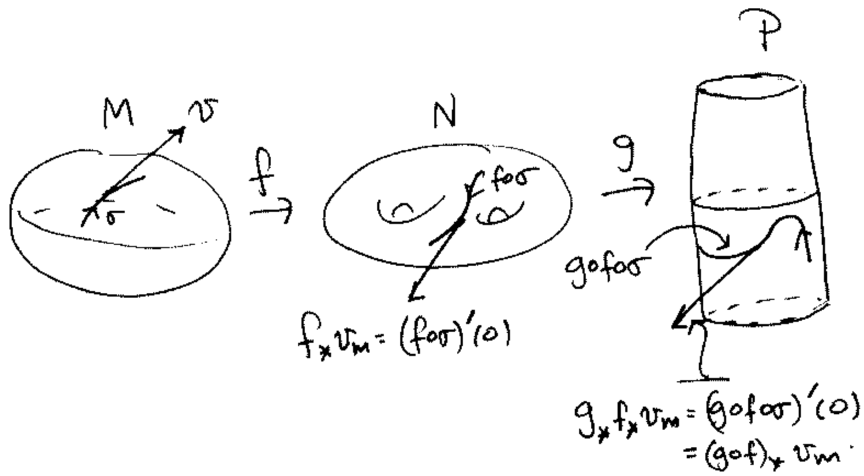

FIGURE 5. The chain rule.

Suppose that $f : M^d \to N^k$ is a smooth map of embedded submanifolds, $m \in M$, x is a chart on M such that $m \in \mathcal{D}(x)$, and y is a chart on N such that $f(m) \in \mathcal{D}(y)$. Then the matrix of

$$f_{*m} \equiv f_*|_{T_m M} : T_m M \to T_{f(m)} N$$

relative to the basis $\{\partial/\partial x^i|_m\}_{i=1}^d$ of $T_m M$ and $\{\partial/\partial y^j|_{f(m)}\}_{j=1}^k$ of $T_{f(m)}N$ is $(\partial(y^j \circ f)(m)/\partial x^i)$. Indeed, if $v_m = \sum v^i \partial/\partial x^i|_m$, then

$$\begin{aligned} f_* v_m &= \sum_{j=1}^k dy^j \langle f_* v_m \rangle \partial/\partial y^j|_{f(m)} \\ &= \sum_{j=1}^k d(y^j \circ f)\langle v_m \rangle \partial/\partial y^j|_{f(m)} \qquad \text{(by (2.7))} \\ &= \sum_{j=1}^k \sum_{i=1}^d \partial(y^j \circ f)(m)/\partial x^i \cdot dx^i \langle v_m \rangle \partial/\partial y^j|_{f(m)} \qquad \text{(by (2.8))} \\ &= \sum_{j=1}^k \sum_{i=1}^d [\partial(y^j \circ f)(m)/\partial x^i] v^i \partial/\partial y^j|_{f(m)}. \end{aligned}$$

Example 2.22. Let $M = O(n)$, $k \in O(n)$, and $f : O(n) \to O(n)$ be defined by $f(g) \equiv kg$. Then f is a smooth function on $O(n)$ because it is the restriction of a smooth function on $gl(n, \mathbb{R})$. Given $A_g \in T_g O(n)$, by Eq. (2.4),

$$f_* A_g = (kg, kA) = (kA)_{kg}$$

(In the future we denote f by L_k, L_k is left translation by $k \in O(n)$.)

Exercise 2.23 (Continuation of Exercise 2.15)**.** Show for each chart x on M that the function

$$\phi(v_m) \doteq (x(m), dx\langle v_m \rangle) = x_* v_m$$

is a chart on TM. Note that $\mathcal{D}(\phi) \doteq \cup_{m \in \mathcal{D}(x)} T_m M$.

The following lemma gives an important example of a smooth function on M which will be needed when we consider the Riemannian geometry of M.

Lemma 2.24. *Suppose that $(E,(\cdot,\cdot))$ is an inner product space and the $M \subset E$ is an embedded submanifold. For each $m \in M$, let $P(m)$ denote the orthogonal projection of E onto $\tau_m M$ (the tangent plane to M and m) and $Q(m) \equiv Id - P(m)$ denote the orthogonal projection onto $\tau_m M^{\perp}$. Then P and Q are smooth functions from M to $gl(E)$, where $gl(E)$ denotes the vector space of linear maps from E to E.*

Proof. Let $z : E \to \mathbb{R}^N$ be as in Definition 2.2. To simplify notation, let $F(p) \equiv z_>(p)$ for all $p \in \mathcal{D}(z)$, so that $\tau_m M = \mathrm{nul} F'(m)$ for all $m \in \mathcal{D}(x) = \mathcal{D}(z) \cap M$. It is easy to check that $F'(m) : E \to \mathbb{R}^{N-d}$ is surjective for all $m \in \mathcal{D}(x)$. It is now an exercise in linear algebra to show that

$$(F'(m)F'(m)^*) : \mathbb{R}^{N-d} \to \mathbb{R}^{N-d}$$

is invertible for all $m \in \mathcal{D}(x)$ and that

$$Q(m) = F'(m)^*(F'(m)F'(m)^*)^{-1}F'(m). \tag{2.9}$$

Since being invertible is an open condition, $(F'(\cdot)F'(\cdot)^*)$ is invertible in an open neighborhood $\mathcal{N} \subset E$ of $\mathcal{D}(x)$. Hence Q has a smooth extension $\tilde{Q}$ to $\mathcal{N}$ given by

$$\tilde{Q}(x) \equiv F'(x)^*(F'(x)F'(x)^*)^{-1}F'(x).$$

Since $Q|_{\mathcal{D}(x)} = \tilde{Q}|_{\mathcal{D}(x)}$ and $\tilde{Q}$ is smooth on $\mathcal{N}$, $Q|_{\mathcal{D}(x)}$ is also smooth. Since z as in Definition 2.2 was arbitrary, it follows that Q is smooth on M. Clearly, $P \equiv id - Q$ is also a smooth function on M. ■

Definition 2.25. A local vector field Y on M is a smooth function $Y : M \to TM$ such that $Y(m) \in T_m M$ for all $m \in \mathcal{D}(Y)$, where $\mathcal{D}(Y)$ is assumed to be an open subset of M. Let $\Gamma(TM)$ denote the collection of globally defined (i.e. $\mathcal{D}(Y) = M$) smooth vector-fields Y on M.

Note that $\partial/\partial x^i$ are local vector-fields on M for each chart $x \in \mathcal{A}(M)$ and $i = 1, 2, \ldots, d$. The next exercise asserts that these vector fields are smooth.

Exercise 2.26. Let Y be a vector field on M and $x \in \mathcal{A}(M)$ be a chart on M. Then

$$Y(m) \equiv \sum dx^i \langle Y(m) \rangle \partial/\partial x^i|_m,$$

which we abbreviate as $Y = \sum Y^i \partial/\partial x^i$. Show that the condition that Y is smooth translates into the statement that the functions $Y^i \equiv dx^i\langle Y \rangle$ are smooth on M.

Exercise 2.27. Let $Y : M \to TM$, be a vector field. Then $Y(m) = (m, y(m)) = y(m)_m$ for some function $y : M \to E$ such that $y(m) \in \tau_m M$ for all $m \in \mathcal{D}(Y) = \mathcal{D}(y)$. Show that Y is smooth iff $y : M \to E$ is smooth.

Example 2.28. Let $M = SL(n, \mathbb{R})$, and $A \in gl(n, \mathbb{R})$ such that $\mathrm{tr} A = 0$. Then $\tilde{A}(g) \equiv (g, gA)$ for $g \in M$ is a smooth vector field on M.

Example 2.29. Keep the notation of Lemma 2.24. Let $y : M \to E$ be any smooth function. Then $Y(m) \equiv (m, P(m)y(m))$ for all $m \in M$ is a smooth vector-field on M.

Definition 2.30. Given $Y \in \Gamma(TM)$ and $f \in C^\infty(M)$, let $Yf \in C^\infty(M)$ be defined by $(Yf)(m) \equiv df\langle Y(m)\rangle$, for all $m \in \mathcal{D}(f) \cap \mathcal{D}(Y)$. In this way the vector-field Y may be viewed as a first order differential operator on $C^\infty(M)$.

Exercise 2.31. Let Y and W be two smooth vector-fields on M. Let $[Y,W]$ denote the linear operator on $C^\infty(M)$ determined by

$$[Y,W]f \equiv Y(Wf) - W(Yf), \quad \forall f \in C^\infty(M). \tag{2.10}$$

Show that $[Y,W]$ is again a first order differential operator on $C^\infty(M)$ coming from a vector-field. In particular, suppose that x is a chart on M and $Y = \sum Y^i \partial/\partial x^i$ and $W = \sum W^i \partial/\partial x^i$, then

$$[Y,W] = \sum (YW^i - WY^i)\partial/\partial x^i \quad \text{on } \mathcal{D}(x). \tag{2.11}$$

Also prove

$$[Y,W](m) = (m, (Yw - Wy)(m)) = (m, dw\langle Y(m)\rangle - dy\langle W(m)\rangle), \tag{2.12}$$

where $Y(m) = (m, y(m))$, $W(m) = (m, w(m))$ and $y, w : M \to E$ are smooth functions such that $y(m), w(m) \in \tau_m M$.

Hint: To prove (2.12): recall that f, y, and w have extensions to smooth functions on E. To see that $(Yw - Wy)(m) \in \tau_m M$ for all $m \in M$, let $z = (z_<, z_>)$ be as in Definition 2.2. Then using $0 = (YW - WY)z_>$ and the fact that mixed partial derivatives commute, one learns that $z'_>(m)\{Y(m)w - W(m)y\} = z'_>(m)\{dw\langle Y(m)\rangle - dy\langle W(m)\rangle\} = 0$.

3. Riemannian Geometry Primer

In this section, we consider the following objects: 1) Riemannian metrics, 2) Riemannian volume forms, 3) gradients, 4) divergences, 5) Laplacians, 6) covariant derivatives, 7) parallel translations, and 8) curvatures.

3.1. **Riemannian Metrics.**

Definition 3.1. A Riemannian metric, $\langle\cdot,\cdot\rangle$, on M is a smoothly varying choice of inner product, $\langle\cdot,\cdot\rangle_m$, on each of the tangent spaces T_mM, $m \in M$. Where $\langle\cdot,\cdot\rangle$ is said to be smooth provided that the function $(m \to \langle X(m), Y(m)\rangle_m) : M \to \mathbb{R}$ is smooth for all smooth vector fields X and Y on M.

It is customary to write ds^2 for the function on TM defined by

$$ds^2\langle v_m\rangle \doteq \langle v_m, v_m\rangle_m.$$

Clearly, the Riemannian metric $\langle\cdot,\cdot\rangle$ is uniquely determined by the function ds^2. Given a chart x on M and

$$v_m = \sum dx^i\langle v_m\rangle \partial/\partial x^i|_m \in T_mM,$$

then

$$ds^2\langle v_m\rangle = \sum_{i,j} \langle \partial/\partial x^i|_m, \partial/\partial x^j|_m\rangle_m dx^i\langle v_m\rangle dx^j\langle v_m\rangle. \tag{3.1}$$

We will abbreviate this equation in the future by writing

$$ds^2 = \sum g^x_{ij} dx^i dx^j \tag{3.2}$$

where $g^x_{i,j}(m) \doteq \langle \partial/\partial x^i|_m, \partial/\partial x^j|_m\rangle_m$. Typically $g^x_{i,j}$ will be abbreviated by g_{ij} if no confusion is likely to arise.

Example 3.2. Let $M = \mathbb{R}^N$ and let $x = (x^1, x^2, \dots, x^N)$ denote the standard chart on M, i.e. $x(m) = m$ for all $m \in M$. The standard Riemannian metric on $\mathbb{R}^N$ is determined by

$$ds^2 = \sum_i (dx^i)^2,$$

i.e. g^x is the identity matrix. The general Riemannian metric on $\mathbb{R}^N$ is determined by $ds^2 = \sum g_{ij} dx^i dx^j$, where $g = (g_{ij})$ is smooth $gl(N, \mathbb{R})$ valued function on $\mathbb{R}^N$, such that $g(m)$ is positive definite for all $m \in \mathbb{R}^N$.

Example 3.3. Let $M = SL(n, \mathbb{R})$, and define

$$ds^2 \langle A_g \rangle \doteq \mathrm{tr}((g^{-1}A)^* g^{-1}A) \tag{3.3}$$

for all $A_g \in TM$. This metric is invariant under left translations, i.e. $ds^2 \langle L_{k*} A_g \rangle = ds^2 \langle A_g \rangle$, for all $k \in M$ and $A_g \in TM$. While the metric

$$ds^2 \langle A_g \rangle \doteq \mathrm{tr}(A^* A) \tag{3.4}$$

is not invariant under left translations.

Let M be an embedded submanifold of a finite dimensional inner product space $(E, (\cdot, \cdot))$. The manifold M **inherits** a metric from E determined by $ds^2 \langle v_m \rangle = (v, v)$ for all $v_m \in TM$. It is a well known deep fact that **all** finite dimensional Riemannian manifolds may be constructed in this way, see Nash [108] and Moser [106, 107].

Remark 3.4. The metric in Eq. (3.4) of Example 3.3 is the inherited metric from the inner product space $E = gl(n, \mathbb{R})$ with inner product $(A, B) \doteq \mathrm{tr}(A^* B)$.

To simplify the exposition, in the sequel we will assume that $(E, (\cdot, \cdot))$ is an inner product space, $M^d \subset E$ is an embedded submanifold, and the Riemannian metric on M is determined by

$$\langle v_m, w_m \rangle = (v, w), \quad \forall v_m, w_m \in T_m M \text{ and } m \in M.$$

In this setting the components $g^x_{i,j}$ of the metric ds^2 relative to a chart x may be computed as $g^x_{i,j}(m) = (\phi_{;i}(x(m)), \phi_{;j}(x(m)))$, where $\phi \doteq x^{-1}$, $\phi_{;i}(a) \doteq \frac{d}{dt}|_0 \phi(a + te_i)$, and $\{e_i\}_{i=1}^d$ is the standard basis for $\mathbb{R}^d$.

Example 3.5. Let $M = \mathbb{R}^3$ and choose spherical coordinates (r, θ, ϕ) for the chart, see Figure 6, then

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \tag{3.5}$$

Here r, θ, and ϕ are taken to be functions on

$$\mathbb{R}^3 \setminus \{p \in \mathbb{R}^3 : p_2 = 0 \text{ and } p_1 > 0\}.$$

Explicitly $r(p) = |p|$, $\theta(p) = \cos^{-1}(p_3/|p|) \in (0, \pi)$, and $\phi(p) \in (0, 2\pi)$ is given by $\phi(p) = \tan^{-1}(p_2/p_1)$ if $p_1 > 0$ and $p_2 > 0$ with similar formulas for (p_1, p_2) in the other three quadrants of $\mathbb{R}^2$.

It would be instructive for the reader to compute components of the standard metric relative to spherical coordinates using the methods just described. Here, I will present a slightly different and perhaps more intuitive method.

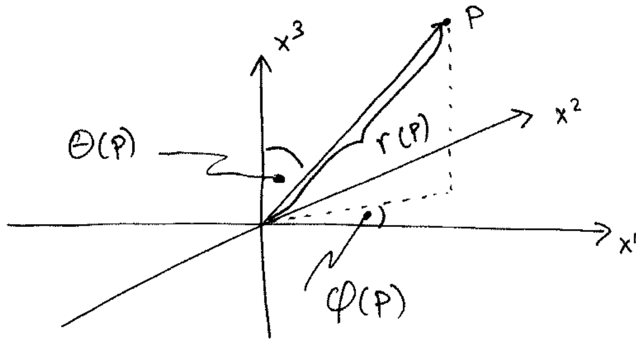

FIGURE 6. Spherical Coordinates.

Note that $x^1 = r\sin\theta\cos\phi$, $x^2 = r\sin\theta\sin\phi$, and $x^3 = r\cos\theta$. Therefore

$$\begin{aligned} dx^1 &= \partial x^1/\partial r dr + \partial x^1/\partial\theta d\theta + \partial x^1/\partial\phi d\phi \\ &= \sin\theta\cos\phi dr + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi, \end{aligned}$$

$$dx^2 = \sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi,$$

and

$$dx^3 = \cos\theta dr - r\sin\theta d\theta.$$

An elementary calculation now shows that

$$ds^2 = \sum_{i=1}^{3}(dx^i)^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2.$$

From this last equation, we see that

$$g^{(r,\theta,\phi)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}. \tag{3.6}$$

Exercise 3.6. Let $M \doteq \{x \in \mathbb{R}^3 | |x|^2 = R^2\}$, so that M is a sphere of radius R in $\mathbb{R}^3$. By a similar computation or using the results of the above example, the induced metric ds^2 on M is given by

$$ds^2 = R^2 d\theta^2 + R^2\sin^2\theta d\phi^2, \tag{3.7}$$

so that

$$g^{(\theta,\phi)} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{bmatrix}. \tag{3.8}$$

3.2. **Integration and the volume measure.**

Definition 3.7. Let $f \in C_c^\infty(M)$ (the smooth functions on M^d with compact support) and assume the support of f is contained in $\mathcal{D}(x)$, where x is some chart on M. Set

$$\int_M f dx = \int_{\mathcal{R}(x)} f \circ x^{-1}(a) da,$$

where da denotes Lebesgue measure on $\mathbb{R}^d$.

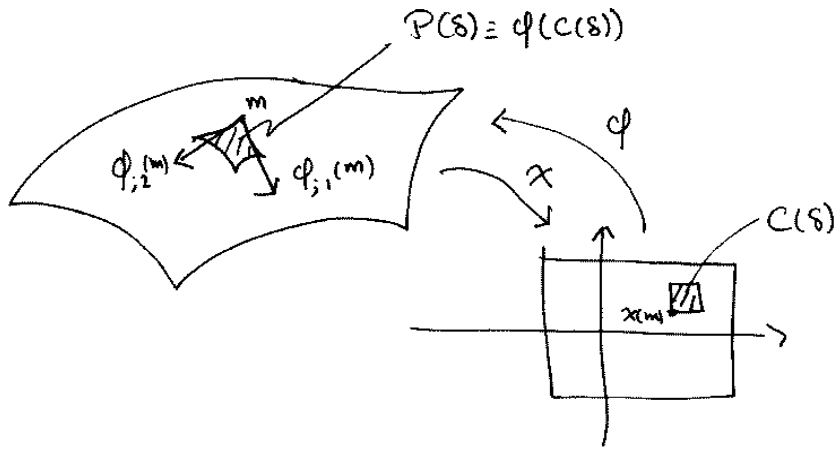

FIGURE 7. The Riemannian volume element.

The problem with this notion of integration is that (as the notation indicates) $\int_M f dx$ depends on the choice of chart x. To remedy this, consider a small cube $C(\delta)$ of side δ contained in $\mathcal{R}(x)$, see Figure 7. We wish to estimate "the volume" of $x^{-1}(C(\delta))$. Heuristically, we expect the volume of $x^{-1}(C(\delta))$ to be approximately equal to the volume of the parallelepiped $P(\delta)$ in the tangent space T_mM determined by

$$P(\delta) \equiv \{\sum_{i=1}^{d} s_i\delta \cdot \phi_{;i}(m) | 0 \le s_i \le 1, \text{ for } i = 1, 2, \ldots, d\},$$

where we are using the notation proceeding Example 3.5. Since T_mM is an inner product space, the volume of $P(\delta)$ may be defined. For example choose an isometry $\theta : T_mM \to \mathbb{R}^d$ and define the volume of $P(\delta)$ to be the volume of $\theta(P(\delta))$ in $\mathbb{R}^d$. Using this definition and the properties of the determinant, one shows that the volume of $P(\delta)$ is $\delta^d\sqrt{\det g(m)}$, where $g_{ij} \equiv \langle \phi_{;i}(x(m)), \phi_{;j}(x(m))\rangle_m = g^x_{ij}(m)$.

Because of the above computation, it is reasonable to try to define a new integral on M by

$$\int_M f \, d\mathrm{vol} \equiv \int_M f\sqrt{g^x}dx,$$

where $\sqrt{g^x} \equiv \sqrt{\det g^x}$–a smooth positive function on $\mathcal{D}(x)$.

Lemma 3.8. *Suppose that y and x are two charts on M, then*

$$g^y_{l,k} = \sum_{i,j} g^x_{i,j}(\partial x^i/\partial y^k)(\partial x^j/\partial y^l). \tag{3.9}$$

Proof. Inserting the identities

$$dx^i = \sum_k \partial x^i/\partial y^k dy^k$$

and

$$dx^j = \sum_l \partial x^j/\partial y^l dy^l$$

into the formula

$$ds^2 = \sum_{i,j} g^x_{i,j} dx^i dx^j,$$

gives

$$ds^2 = \sum_{i,j,k,l} g^x_{i,j} (\partial x^i/\partial y^k)(\partial x^j/\partial y^l) dy^l dy^k$$

from which (3.9) follows. ■

Exercise 3.9. Suppose that x and y are two charts on M and $f \in C_c^\infty(M)$ such that the support of f is contained in $\mathcal{D}(x) \cap \mathcal{D}(y)$. Using Lemma 3.8 and the change of variable formula show that

$$\int f \sqrt{g^x} dx = \int f \sqrt{g^y} dy.$$

Hence, it makes sense to define $\int f\, d\mathrm{vol}$ as $\int f \sqrt{g^x} dx$. We summarize this definition by writing

$$d\mathrm{vol} = \sqrt{g^x} dx. \tag{3.10}$$

Because of Lemma 3.8 and Exercise 3.9, we may define the integral $\int_M f\, d\mathrm{vol}$ for any continuous function f on M with compact support. To this end, choose a finite collection of charts $\{x_i\}_{i=1}^m$ such that the support of f is contained in $\cup_{i=1}^m \mathcal{D}(x_i)$. Define $U_1 \doteq \mathcal{D}(x_1)$ and $U_i \doteq \mathcal{D}(x_i) \backslash (\cup_{j=1}^{i-1} \mathcal{D}(x_j))$ for $i = 2, 3, \ldots, m$. Let $\chi_i \doteq 1_{U_i}$ be the characteristic function of the set U_i and set $f_i \doteq \chi_i f$. Then define

$$\int_M f\, d\mathrm{vol} \doteq \sum_{i=1}^m \int_M f_i \sqrt{g^{x_i}} dx_i.$$

Because of the above exercise, it is possible to check that $\int_M f\, d\mathrm{vol}$ is well defined independent of the choice of charts $\{x_i\}_{i=1}^m$.

Example 3.10. Let $M = \mathbb{R}^3$ with the standard Riemannian metric, and let x denote the standard coordinates on M determined by $x(m) = m$ for all $m \in M$. Then $d\mathrm{vol} = dx$. We may also easily express $d\mathrm{vol}$ is spherical coordinates. Using (3.6), $\sqrt{g^{(r,\theta,\phi)}} = r^2 \sin\theta$ and hence

$$d\mathrm{vol} = r^2 \sin\theta dr d\theta d\phi.$$

Similarly using Eq. (3.8), it follows that $d\mathrm{vol} = R^2 \sin\theta d\theta d\phi$ is the volume element on the sphere of radius R in $\mathbb{R}^3$.

Exercise 3.11. Compute the volume element of $\mathbb{R}^3$ in cylindrical coordinates.

3.3. **Gradients, Divergence, and Laplacians.** In the sequel, let M be a Riemannian manifold, x be a chart on M, $g_{ij} \equiv \langle \partial/\partial x^i, \partial/\partial x^j \rangle$, and $ds^2 = \sum_{i,j} g_{ij} dx^i dx^j$.

Definition 3.12. Let g^{ij} denote the i,j-matrix element of the inverse matrix to (g_{ij}).

Given $f \in C^\infty(M)$ and $m \in M$, $df_m \equiv df|_{T_m M}$ is a linear functional on $T_m M$. Hence there is a unique vector $v_m \in T_m M$ such that $df_m = \langle v_m, \cdot \rangle_m$.

Definition 3.13. The vector v_m above is called the **gradient** of f at m and will be denoted by $\mathrm{grad} f(m)$.

Exercise 3.14. Show that

$$\mathrm{grad} f(m) = \sum_{i,j=1}^{d} g^{ij}(m) \frac{\partial f(m)}{\partial x^i} \frac{\partial}{\partial x^j}|_m \quad \forall m \in \mathcal{D}(x). \tag{3.11}$$

Notice that grad$f$ is a vector field on M. Moreover, grad$f$ is smooth as can be seen from (3.11).

Remark 3.15. Suppose $M \subset \mathbb{R}^N$ is an embedded submanifold with the induced Riemannian structure. Let $F : \mathbb{R}^N \to \mathbb{R}$ be a smooth function and set $f \equiv F|_M$. Then $\mathrm{grad} f(m) = (P(m)\vec{\nabla} F(m))_m$, where $\vec{\nabla} F(m)$ denotes the usual gradient on $\mathbb{R}^N$, and $P(m)$ denotes orthogonal projection of $\mathbb{R}^N$ onto $\tau_m M$.

We now introduce the divergence of a vector field Y on M.

Lemma 3.16. *To every smooth vector field Y on M there is a unique smooth function div$Y$ on M such that*

$$\int Yf\, dvol = -\int divY \cdot f\, dvol, \quad \forall f \in C_c^\infty(M). \tag{3.12}$$

Moreover on $\mathcal{D}(x)$,

$$divY = \sum_i \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} Y^i)}{\partial x^i} = \sum_i \{ \frac{\partial Y^i}{\partial x^i} + \frac{\partial \log \sqrt{g}}{\partial x^i} Y^i \} \tag{3.13}$$

where $Y^i \equiv dx^i \langle Y \rangle$.

Proof. (Sketch) Suppose that $f \in C_c^\infty(M)$ such that the support of f is contained in $\mathcal{D}(x)$. Because $Yf = \sum Y^i \partial f / \partial x^i$,

$$\begin{aligned} \int Yf\, \mathrm{dvol} &= \int \sum Y^i \partial f/\partial x^i \cdot \sqrt{g} dx \\ &= -\int \sum f \frac{\partial(\sqrt{g} Y^i)}{\partial x^i} dx \\ &= -\int f \sum_i \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} Y^i)}{\partial x^i}\, \mathrm{dvol}, \end{aligned}$$

where the second equality follows by an integration by parts. This shows that if div$Y$ exists it must be given on $\mathcal{D}(x)$ by (3.13). This proves the uniqueness assertion. Using what we have already proved, it is easy to conclude that the formula for div$Y$ is chart independent. Hence we may define smooth function div$Y$ on M using (3.13) in each coordinate chart x on M. It is then possible to show (using a partition of unity argument) that this function satisfies (3.12). ∎

Remark 3.17. We may write (3.12) as

$$\int \langle Y, \mathrm{grad} f \rangle\, \mathrm{dvol} = -\int \mathrm{div} Y \cdot f\, \mathrm{dvol}, \quad \forall f \in C_c^\infty(M), \tag{3.14}$$

so that div is the negative of the formal adjoint of grad.

Lemma 3.18 (Integration by Parts)**.** *Suppose that* $Y \in \Gamma(TM)$, $f \in C_c^\infty(M)$, *and* $h \in C^\infty(M)$, *then*

$$\int_M Yf \cdot h\, dvol = \int_M f\{-Yh - h divY\}\, dvol.$$

Proof. By the definition of div$Y$ and the product rule, we have

$$\begin{aligned}\int_M fh\text{div}Y\,d\text{vol} &= -\int_M Y(fh)\,d\text{vol}\\ &= -\int_M \{hYf + fYh\}\,d\text{vol}.\end{aligned}$$

■

Definition 3.19. Let $\Delta : C^\infty(M) \to C^\infty(M)$ be the second order differential operator defined by

$$\Delta f \equiv \text{div}(\text{grad} f). \tag{3.15}$$

In a local chart x,

$$\Delta f = \frac{1}{\sqrt{g}}\sum_{i,j}\partial_i\{\sqrt{g}g^{ij}\partial_j f\}, \tag{3.16}$$

where $\partial_i = \partial/\partial x^i$, $g = g^x$, $\sqrt{g} = \sqrt{\det g}$, and $(g^{ij}) = (g_{ij})^{-1}$.

Remark 3.20. The Laplacian may be characterized by the equation:

$$\int_M \Delta f \cdot h\,d\text{vol} = -\int_M \langle \text{grad} f, \text{grad} h\rangle\,d\text{vol},$$

which is to hold for all $f \in C^\infty(M)$ and $g \in C_c^\infty(M)$.

Example 3.21. Suppose that $M = \mathbb{R}^N$ with the standard Riemannian metric $ds^2 = \sum_{i=1}^N (dx^i)^2$, then the standard formulas

$$\text{grad} f = \sum_{i=1}^N \partial f/\partial x^i \cdot \partial/\partial x^i$$

$$\text{div} Y = \sum_{i=1}^N \partial Y^i/\partial x^i$$

and

$$\Delta f = \sum_{i=1}^N \partial^2 f/(\partial x^i)^2,$$

are easily verified, where f is a smooth function on $\mathbb{R}^N$ and $Y = \sum_{i=1}^N Y^i\partial/\partial x^i$ is a smooth vector-field.

Exercise 3.22. Let $M = \mathbb{R}^3$, (r, θ, ϕ) be spherical coordinates on $\mathbb{R}^3$, $\partial_r = \partial/\partial r$, $\partial_\theta = \partial/\partial\theta$, and $\partial_\phi = \partial/\partial\phi$. Given a smooth function f and a vector-field $Y = Y_r\partial_r + Y_\theta\partial_\theta + Y_\phi\partial_\phi$ on $\mathbb{R}^3$ verify:

$$\text{grad} f = (\partial_r f)\partial_r + \frac{1}{r^2}(\partial_\theta f)\partial_\theta + \frac{1}{r^2\sin^2\theta}(\partial_\phi f)\partial_\phi,$$

$$\begin{aligned}\text{div}Y &= \frac{1}{r^2\sin\theta}\{\partial_r(r^2\sin\theta Y_r) + \partial_\theta(r^2\sin\theta Y_\theta) + r^2\sin\theta\partial_\phi Y_\phi\}\\ &= \frac{1}{r^2}\partial_r(r^2 Y_r) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta Y_\theta) + \partial_\phi Y_\phi,\end{aligned}$$

and

$$\Delta f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + +\frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 f.$$

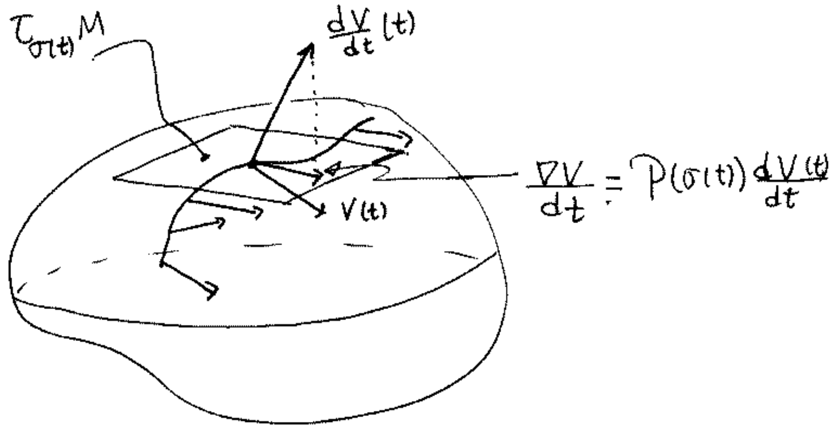

FIGURE 8. Levi-Civita covariant derivative.

3.4. **Covariant Derivatives and Curvature.** This section is motivated by the desire to have the notion of the derivative of a smooth path $W(s) \in TM$. On one hand, since TM is a manifold, we may write $W'(s)$ as an element of TTM. However, this is not what we will want for later purposes. We would like the derivative of W to be again a curve back in TM, not in TTM. In order to construct such a derivative, we will have to use more than just the manifold structure of M.

In the sequel, we assume that M^d is an embedded submanifold of an inner product space $(E, (\cdot,\cdot))$, and that M is equipped with the inherited Riemannian metric. Also let $P(m)$ denote orthogonal projection of E onto $\tau_m M$ for all $m \in M$ and $Q(m) \doteq id - P(m)$ be orthogonal projection onto $(\tau_m M)^\perp$.

Definition 3.23 (Levi-Civita Covariant Derivative)**.** Let $W(s) = (\sigma(s), w(s)) = w(s)_{\sigma(s)}$ be a smooth path in TM, define

$$\nabla W(s)/ds \doteq (\sigma(s), P(\sigma(s))\frac{d}{ds}w(s)). \tag{3.17}$$

that $\nabla W(s)/ds$ is still a smooth path in TM, see Figure 8.

Proposition 3.24 (Properties of ∇)**.** *Let $W(s) = (\sigma(s), w(s))$ and $V(s) = (\sigma(s), v(s))$ be two smooth paths in TM "over" σ in M. Then $\nabla W(s)/ds$ may be computed as:*

$$\nabla W(s)/ds \doteq (\sigma(s), \frac{d}{ds}w(s) + (dQ\langle\sigma'(s)\rangle)w(s)), \tag{3.18}$$

and ∇ is ***Metric compatible****, i.e.*

$$\frac{d}{ds}\langle W(s), V(s)\rangle = \langle\nabla W(s)/ds, V(s)\rangle + \langle W(s), \nabla V(s)/ds\rangle. \tag{3.19}$$

Now suppose that $(s,t) \to \sigma(s,t)$ is a smooth function into M and the $W(s,t) = (\sigma(s,t), w(s,t))$ is a smooth function into TM. (Notice by assumption that $w(s,t) \in T_{\sigma(s,t)}M$ for all (s,t).) Let $\sigma'(s,t) \doteq (\sigma(s,t), \frac{d}{ds}\sigma(s,t))$ and $\dot\sigma(s,t) =$

$(\sigma(s,t), \frac{d}{dt}\sigma(s,t))$. *Then:*

$$\nabla\sigma'/dt = \nabla\dot{\sigma}/ds \qquad \textbf{\textit{(Zero Torsion)}} \tag{3.20}$$

$$[\nabla/dt, \nabla/ds]W \doteq (\frac{\nabla}{dt}\frac{\nabla}{ds} - \frac{\nabla}{ds}\frac{\nabla}{dt})W = R\langle\dot{\sigma}, \sigma'\rangle W \tag{3.21}$$

where R is the ***curvature tensor*** *of ∇ given by*

$$R\langle u_m, v_m\rangle w_m = (m, [dQ\langle u_m\rangle, dQ\langle v_m\rangle]w) \tag{3.22}$$

and

$$[dQ\langle u_m\rangle, dQ\langle v_m\rangle] \doteq (dQ\langle u_m\rangle)dQ\langle v_m\rangle - (dQ\langle v_m\rangle)dQ\langle u_m\rangle.$$

Proof. To prove (3.18), differentiate the equation $P(\sigma(s))w(s) = w(s)$ relative to s to learn that

$$(dP\langle\sigma'(s)\rangle)w(s) + P(\sigma(s))\frac{d}{ds}w(s) = \frac{d}{ds}w(s),$$

so that

$$P(\sigma(s))\frac{d}{ds}w(s) = \frac{d}{ds}w(s) - (dP\langle\sigma'(s)\rangle)w(s) = \frac{d}{ds}w(s) + (dQ\langle\sigma'(s)\rangle)w(s),$$

where in the last equality we have used the fact that $Q + P = id$. The above displayed equation clearly implies (3.18).

For (3.19) just compute:

$$\begin{aligned}
\frac{d}{ds}\langle W(s), V(s)\rangle &= \frac{d}{ds}(w(s), v(s)) \\
&= (\frac{d}{ds}w(s), v(s)) + (w(s), \frac{d}{ds}v(s)) \\
&= (\frac{d}{ds}w(s), P(\sigma(s))v(s)) + (P(\sigma(s))w(s), \frac{d}{ds}v(s)) \\
&= (P(\sigma(s))\frac{d}{ds}w(s), v(s)) + (w(s), P(\sigma(s))\frac{d}{ds}v(s)) \\
&= \langle\nabla W(s)/ds, V(s)\rangle + \langle W(s), \nabla V(s)/ds\rangle,
\end{aligned}$$

where the third equality relies on $v(s)$ and $w(s)$ being in $T_{\sigma(s)}M$ and the forth equality on the orthogonality of the projection $P(\sigma(s))$.

A direct computation using the definitions shows that

$$\nabla\sigma'(s,t)/dt = (\sigma(t,s), P(\sigma(s,t))\frac{\partial^2}{\partial t\partial s}\sigma(t,s)).$$

Since mixed partial derivatives commute we have

$$\nabla\sigma'(s,t)/dt = (\sigma(t,s), P(\sigma(s,t))\frac{\partial^2}{\partial s\partial t}\sigma(t,s)) = \nabla\dot{\sigma}(s,t)/ds,$$

which proves (3.20).

For Eq. (3.21) note that,

$$\begin{aligned}
\frac{\nabla}{dt}\frac{\nabla}{ds}W(s,t) &= \frac{\nabla}{dt}(\sigma(s,t), \frac{d}{ds}w(s,t) + (dQ\langle\sigma'(s,t)\rangle)w(s,t)) \\
&= (\sigma(s,t), \eta_+(s,t))
\end{aligned}$$

where (with the arguments (s,t) suppressed from the notation)

$$\begin{aligned}\eta_+ &= \frac{d}{dt}\{\frac{d}{ds}w + (dQ\langle\sigma'\rangle)w\} + dQ\langle\dot\sigma\rangle\{\frac{d}{ds}w + (dQ\langle\sigma'\rangle)w\}\\ &= \frac{d}{dt}\frac{d}{ds}w + [\frac{d}{dt}(dQ\langle\sigma'\rangle)]w + dQ\langle\sigma'\rangle\frac{d}{dt}w + dQ\langle\dot\sigma\rangle\frac{d}{ds}w + dQ\langle\dot\sigma\rangle(dQ\langle\sigma'\rangle)w.\end{aligned}$$

Therefore

$$[\nabla/dt, \nabla/ds]W = (\sigma, \eta_+ - \eta_-),$$

where η_- is defined the same as η_+ with all s and t derivatives interchanged. Hence, it follows using that fact that $\frac{d}{dt}\frac{d}{ds}w = \frac{d}{ds}\frac{d}{dt}w$ that

$$[\nabla/dt, \nabla/ds]W = (\sigma, [\frac{d}{dt}(dQ\langle\sigma'\rangle)]w - [\frac{d}{ds}(dQ\langle\dot\sigma\rangle)]w + [dQ\langle\dot\sigma\rangle, dQ\langle\sigma'\rangle]w).$$

The proof is finished because

$$[\frac{d}{dt}(dQ\langle\sigma'\rangle)]w - [\frac{d}{ds}(dQ\langle\dot\sigma\rangle)]w = [\frac{d}{dt}\frac{d}{ds}(Q\circ\sigma)]w - [\frac{d}{ds}\frac{d}{dt}(Q\circ\sigma)]w = 0.$$

■

Example 3.25. Let $M = \{x \in \mathbb{R}^N : |x| = \rho\}$ be the sphere of radius ρ. In this case $Q(m) = \frac{1}{\rho^2}mm^t$ for all $m \in M$. Therefore

$$dQ\langle v_m\rangle = \frac{1}{\rho^2}\{vm^t + mv^t\},$$

for all $v_m \in T_mM$. Thus

$$\begin{aligned}dQ\langle u_m\rangle dQ\langle v_m\rangle &= \frac{1}{\rho^4}\{um^t + mu^t\}\{vm^t + mv^t\}\\ &= \frac{1}{\rho^4}\{\rho^2 uv^t + (u\cdot v)Q(m)\}.\end{aligned}$$

Therefore for the sphere of Radius ρ the curvature tensor is given by

$$R\langle u_m, v_m\rangle w_m = (m, \frac{1}{\rho^2}\{uv^t - vu^t\}w) = (m, \frac{1}{\rho^2}\{(v\cdot w)u - (u\cdot w)v\}).$$

Exercise 3.26. Show the curvature tensor of a cylinder ($M = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$) is zero.

Definition 3.27 (Covariant Derivative on $\Gamma(TM)$)**.** Suppose that Y is a vector field on M and $v_m \in T_mM$. Define $\nabla_{v_m}Y \in T_mM$ by

$$\nabla_{v_m}Y \doteq \nabla Y(\sigma(s))/ds|_{s=0},$$

where σ is any smooth curve in M such that $\sigma'(0) = v_m$. Notice that if $Y(m) = (m, y(m))$, then

$$\nabla_{v_m}Y = (m, P(m)dy\langle v_m\rangle) = (m, dy\langle v_m\rangle + dQ\langle v_m\rangle y(m)),$$

so that $\nabla_{v_m}Y$ is well defined.

The following proposition relates curvature and torsion to the covariant derivative ∇ on vector fields.

Proposition 3.28. *Let $m \in M$, $v \in T_mM$, $X, Y, Z \in \Gamma(TM)$, and $f \in C^\infty(M)$, then*

1. **Product Rule::** $\nabla_v(fX) = df\langle v\rangle \cdot X(m) + f(m)\nabla_v X$,
2. **Zero Torsion::** $\nabla_X Y - \nabla_Y X - [X,Y] = 0$,

3. Zero Torsion:: *For all* $v_m, w_m \in T_m M$, $dQ\langle v_m\rangle w_m = dQ\langle w_m\rangle v_m$, *and*

4. Curvature Tensor:: $R\langle X,Y\rangle Z = [\nabla_X,\nabla_Y]Z - \nabla_{[X,Y]}Z$, *where* $[\nabla_X,\nabla_Y]Z \equiv \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z)$.

Proof. The product rule is easily checked and may be left to the reader. For the second and third items, write $X(m) = (m, x(m))$, $Y(m) = (m, y(m))$, and $Z(m) = (m, z(m))$ where $x, y, z : M \to \mathbb{R}^N$ are smooth functions such that $x(m)$, $y(m)$, and $z(m)$ are in $\tau_m M$ for all $m \in M$. Then using Eq. (2.12), we have

$$\begin{aligned}(\nabla_X Y - \nabla_Y X)(m) &= (m, P(m)(dy\langle X(m)\rangle - dx\langle Y(m)\rangle))\\ &= (m, (dy\langle X(m)\rangle - dx\langle Y(m)\rangle)) = [X,Y](m),\end{aligned}$$

which proves the second item. Noting that $(\nabla_X Y)(m)$ is also given by $(\nabla_X Y)(m) = (m, dy\langle X(m)\rangle + dQ\langle X(m)\rangle y(m))$, this last equation may be expressed as $dQ\langle X(m)\rangle y(m) = dQ\langle Y(m)\rangle x(m)$ which implies the third item.

Similarly for the last item:

$$\begin{aligned}\nabla_X \nabla_Y Z &= \nabla_X(\cdot, Yz + (YQ)z)\\ &= (\cdot, XYz + (XYQ)z + (YQ)Xz + (XQ)(Yz + (YQ)z)),\end{aligned}$$

where $YQ \equiv dQ\langle Y\rangle$ and $Yz \equiv dz\langle Y\rangle$. Interchanging X and Y in this last expression and then subtracting gives:

$$\begin{aligned}[][\nabla_X,\nabla_Y]Z &= (\cdot, [X,Y]z + ([X,Y]Q)z + [XQ, YQ]z)\\ &= \nabla_{[X,Y]}Z + R\langle X,Y\rangle Z.\end{aligned}$$

■

3.5. Formulas for the Divergence and the Laplacian.

Theorem 3.29. *Let Y be a vector field on M, then*

$$\mathrm{div} Y = \mathrm{tr}(\nabla Y). \tag{3.23}$$

(Note: $(v_m \to \nabla_{v_m} Y) \in End(T_m M)$ for each $m \in M$, so it makes sense to take the trace.) Consequently, if f is a smooth function on M, then

$$\Delta f = \mathrm{tr}(\nabla \mathrm{grad} f). \tag{3.24}$$

Proof. Let x be a chart on M, $\partial_i \doteq \partial/\partial x^i$, $\nabla_i \doteq \nabla_{\partial_i}$, and $Y^i \doteq dx^i\langle Y\rangle$. Then by the product rule and the fact that ∇ is Torsion free (item 2. of the Proposition 3.28),

$$\nabla_i Y = \sum_j \nabla_i (Y^j \partial_j) = \sum_j (\partial_i Y^j \partial_j + Y^j \nabla_i \partial_j),$$

and $\nabla_i \partial_j = \nabla_j \partial_i$. Hence,

$$\begin{aligned}\mathrm{tr}(\nabla Y) = \sum_{i=1}^d dx^i\langle \nabla_i Y\rangle &= \sum_i \partial_i Y^i + \sum_{i,j} dx^i\langle Y^j \nabla_i \partial_j\rangle\\ &= \sum_i \partial_i Y^i + \sum_{i,j} dx^i\langle Y^j \nabla_j \partial_i\rangle.\end{aligned}$$

Therefore, according to Eq. (3.13), to finish the proof it suffices to show that

$$\sum_i dx^i\langle \nabla_j \partial_i\rangle = \partial_j \log\sqrt{g}.$$

Now

$$\partial_j \log \sqrt{g} = \frac{1}{2}\partial_j \log(\det g) = \frac{1}{2}\mathrm{tr}(g^{-1}\partial_j g)$$
$$= \frac{1}{2}\sum_{k,l} g^{kl}\partial_j g_{kl},$$

and using (3.19),

$$\partial_j g_{kl} = \partial_j \langle \partial_k, \partial_l\rangle = \langle \nabla_j \partial_k, \partial_l\rangle + \langle \partial_k, \nabla_j \partial_l\rangle.$$

Combining the two above equations along with the symmetry of g^{kl},

$$\partial_j \log \sqrt{g} = \sum_{k,l} g^{kl}\langle \nabla_j \partial_k, \partial_l\rangle = \sum_k dx^k \langle \nabla_j \partial_k\rangle,$$

where we have used

$$\sum_k g^{kl}\langle \cdot, \partial_l\rangle = dx^k.$$

This last equality is easily verified by applying both sides of this equation to ∂_i for $i = 1, 2, \ldots, n$. ■

Definition 3.30 (One forms)**.** A one form ω on M is a smooth function $\omega : TM \to \mathbb{R}$ such that $\omega_m \equiv \omega|_{T_m M}$ is linear for all $m \in M$. Note: if x is a chart of M with $m \in \mathcal{D}(x)$, then

$$\omega_m = \sum \omega_i(m) dx^i|_{T_m M},$$

where $\omega_i \equiv \omega\langle \partial/\partial x^i\rangle$. The condition that ω be smooth is equivalent to the condition that each of the functions ω_i is smooth on M. Let $\Omega^1(M)$ denote the smooth one-forms on M.

Given a $\omega \in \Omega^1(M)$, there is a unique vector field X on M such that $\omega_m = \langle X(m), \cdot\rangle_m$ for all $m \in M$. Using this observation, we may extend the definition of ∇ to one forms by requiring

$$\nabla_{v_m}\omega \equiv (\nabla_{v_m} X, \cdot) \in T_m^* M \equiv (T_m M)^*. \tag{3.25}$$

Lemma 3.31 (Product Rule)**.** *Keep the notation of the above paragraph. Let $Y \in \Gamma(TM)$, then*

$$v_m(\omega\langle Y\rangle) = (\nabla_{v_m}\omega)\langle Y(m)\rangle + \omega\langle \nabla_{v_m} Y\rangle. \tag{3.26}$$

Moreover, if $\theta : M \to (\mathbb{R}^N)^*$ is a smooth function and

$$\omega\langle v_m\rangle \equiv \theta(m)v$$

for all $v_m \in TM$, then

$$(\nabla_{v_m}\omega)\langle w_m\rangle = d\theta\langle v_m\rangle w - \theta(m) dQ\langle v_m\rangle w = (d(\theta P)\langle v_m\rangle)w, \tag{3.27}$$

where $(\theta P)(m) \equiv \theta(m)P(m) \in (\mathbb{R}^N)^*$.

Proof. Using the metric compatibility of ∇,

$$v_m(\omega\langle Y\rangle) = v_m(\langle X, Y\rangle) = \langle \nabla_{v_m} X, Y(m)\rangle + \langle X(m), \nabla_{v_m} Y\rangle$$
$$= (\nabla_{v_m}\omega)\langle Y(m)\rangle + \omega\langle \nabla_{v_m} Y\rangle.$$

Writing $Y(m) = (m, y(m)) = y(m)_m$ and using (3.26), it follows that

$$\begin{aligned}(\nabla_{v_m}\omega)\langle Y(m)\rangle &= v_m(\omega\langle Y\rangle) - \omega\langle\nabla_{v_m}Y\rangle \\ &= v_m(\theta(\cdot)y(\cdot)) - \theta(m)(dy\langle v_m\rangle + dQ\langle v_m\rangle y(m)) \\ &= (d\theta\langle v_m\rangle)y(m) - \theta(m)(dQ\langle v_m\rangle)y(m).\end{aligned}$$

Choosing Y such that $Y(m) = w_m$ proves the first equality in(3.27). The second equality in (3.27) is a simple consequence of the formula

$$d(\theta P) = d\theta\langle\cdot\rangle P + \theta dP = d\theta\langle\cdot\rangle P - \theta dQ.$$

■

Definition 3.32. For $f \in C^\infty(M)$ and v_m, w_m in T_mM, let

$$\nabla df\langle v_m, w_m\rangle \equiv (\nabla_{v_m}df)\langle w_m\rangle,$$

so that

$$\nabla df : \cup_{m\in M}(T_mM \times T_mM) \to \mathbb{R}.$$

We call ∇df the **Hessian** of f.

In the next lemma, ∂_v will denote the vector field on $\mathbb{R}^N$ defined by $\partial_v(x) = v_x = \frac{d}{dt}|_0(x + tv)$. So if $F \in C^\infty(\mathbb{R}^N)$, then $(\partial_v F)(x) \equiv \frac{d}{dt}|_0 F(x + tv)$.

Lemma 3.33. *Let $f \in C^\infty(M)$ and $F \in C^\infty(\mathbb{R}^N)$ such that $f = F|_M$.*

1. *If $X, Y \in \Gamma(TM)$, then $\nabla df\langle X, Y\rangle = XYf - df\langle\nabla_X Y\rangle$.*
2. *If $v_m, w_m \in T_mM$ then*
$$\nabla df\langle v_m, w_m\rangle = F''(m)\langle v, w\rangle - F'(m)dQ\langle v_m\rangle w,$$
where $F''(m)\langle v, w\rangle \equiv (\partial_v\partial_w F)(m)$ for all $v, w \in \mathbb{R}^N$.
3. *If $v_m, w_m \in T_mM$ then*
$$\nabla df\langle v_m, w_m\rangle = \nabla df\langle w_m, v_m\rangle.$$

Proof. Using the product rule (Eq. (3.26)):

$$XYf = X(df\langle Y\rangle) = (\nabla_X df)\langle Y\rangle + df\langle\nabla_X Y\rangle,$$

so that

$$\nabla df\langle X, Y\rangle = (\nabla_X df)\langle Y\rangle = XYf - df\langle\nabla_X Y\rangle.$$

This proves item 1. From this last equation and Proposition 3.28 (∇ has zero torsion), it follows that

$$\nabla df\langle X, Y\rangle - \nabla df\langle Y, X\rangle = [X, Y]f - df\langle\nabla_X Y - \nabla_Y X\rangle = 0.$$

This proves the third item upon choosing X and Y such that $X(m) = v_m$ and $Y(m) = w_m$. Item 2 follows easily from Lemma 3.31 applied to $\theta = F'$. ■

Corollary 3.34. *Suppose that $F \in C^\infty(\mathbb{R}^N)$, $f \equiv F|_M$, and $m \in M$. Let $\{e_i\}_{i=1}^d$ be an orthonormal basis for τ_mM and let $\{E_i\}_{i=1}^d$ be an orthonormal frame near $m \in M$. That is each E_i is a smooth local vector field on M defined in a neighborhood $\mathcal{N}$ of m such that $\{E_i(p)\}_{i=1}^d$ is an orthonormal basis for T_pM for $p \in \mathcal{N}$. Then*

$$\Delta f(m) = \sum_{i=1}^d \nabla df\langle E_i(m), E_i(m)\rangle \tag{3.28}$$

or equivalently

$$\Delta f(m)=\sum_{i=1}^{d}\{E_iE_if)(m)-df\langle\nabla_{E_i(m)}E_i\rangle\}, \tag{3.29}$$

and

$$\Delta f(m)=\sum_{i=1}^{d}F''(m)\langle e_i,e_i\rangle-F'(m)\langle dQ\langle e_i\rangle e_i\rangle. \tag{3.30}$$

Proof. By Theorem 3.29, $\Delta f=\sum_{i=1}^{d}(\nabla_{E_i}\mathrm{grad}f,E_i)$ and by Eq. (3.25), $\nabla_{E_i}df=(\nabla_{E_i}\mathrm{grad}f,\cdot)$. Therefore

$$\Delta f=\sum_{i=1}^{d}(\nabla_{E_i}df)\langle E_i\rangle=\sum_{i=1}^{d}\nabla df\langle E_i,E_i\rangle,$$

which proves (3.28). Eqs. (3.29) and (3.30) follow form (3.28) and Lemma 3.33. ■

3.6. **Parallel Translation.** Let $\pi:TM\to M$ denote the **projection** defined by $\pi(v_m)=m$ for all $v_m=(m,v)\in TM$. We say a smooth curve $s\to V(s)$ in TM is a **vector-field along** a smooth curve $s\to\sigma(s)$ in M if $\pi\circ V(s)=\sigma(s)$ for all s, i.e. $V(s)\in T_{\sigma(s)}M$ for all s. Note that if V is a smooth curve in TM then V is a vector-field along $\sigma\equiv\pi\circ V$.

Definition 3.35. Let V be a smooth curve in TM. V is said to **parallel** or **covariantly constant** iff $\nabla V(s)/ds\equiv 0$.

Theorem 3.36. *Let σ be a smooth curve in M and $(v_0)_{\sigma(0)}\in T_{\sigma(0)}M$. Then there exists a unique smooth vector field V along σ such that V is parallel and $V(0)=(v_0)_{\sigma(0)}$. Moreover $\langle V(s),V(s)\rangle=\langle(v_0)_{\sigma(0)},(v_0)_{\sigma(0)}\rangle$ for all s.*

Proof. First note that if V is parallel then

$$\frac{d}{ds}\langle V(s),V(s)\rangle=2\langle\nabla V(s)/ds,V(s)\rangle=0,$$

so the last assertion of the theorem is true.

If a parallel vector field $V(s)=(\sigma(s),v(s))$ along $\sigma(s)$ is to exist, then

$$dv(s)/ds+dQ\langle\sigma'(s)\rangle v(s)=0\quad\text{and}\quad v(0)=v_0. \tag{3.31}$$

By existence and uniqueness of solutions to ordinary differential equations, there is exactly one solution to (3.31). Hence, if V exists it is unique.

Now let v be the unique solution to (3.31) and set $V(s)\equiv(\sigma(s),v(s))$. To finish the proof it suffices to show that $v(s)\in\tau_{\sigma(s)}M$. Equivalently, we must show that $w(s)\equiv q(s)v(s)$ is identically zero, where $q(s)\equiv Q(\sigma(s))$. To simplify notation, I will write $v'(s)$ for $dv(s)/ds$ and $p(s)$ for $P(\sigma(s))$. Notice that $w(0)=0$ and that

$$w'=q'v+qv'=q'v-qq'v=pq'v,$$

where we have used the differential equation for v and the fact that $q'=dQ\langle\sigma'\rangle$. Now differentiating the equation $0=pq$ implies that $pq'=-p'q=q'q$. Therefore w solves the linear differential equation

$$w'=q'w=dQ\langle\sigma'\rangle w\quad\text{with}\quad w(0)=0,$$

and hence by uniqueness of solutions $w\equiv 0$. ■

Definition 3.37. Given a smooth curve σ, let $//_s(\sigma) : T_{\sigma(0)}M \to T_{\sigma(s)}M$ be defined by $//_s(\sigma)(v_0)_{\sigma(0)} = V(s)$, where V is the unique vector parallel vector field along σ such that $V(0) = (v_0)_{\sigma(0)}$. We call $//_s(\sigma)$ **parallel translation** along σ up to s.

Remark 3.38. Notice that $//_s(\sigma)v_{\sigma(0)} = (u(s)v)_{\sigma(0)}$, where $s \to u(s) \in End(\tau_{\sigma(0)}M, \mathbb{R}^N)$ is the unique solution to the differential equation

$$u'(s) + dQ\langle\sigma'(s)\rangle u(s) = 0 \quad \text{with} \quad u(0) = u_0, \tag{3.32}$$

where $u_0 v \equiv v$ for all $v \in \tau_{\sigma(0)}M$. Because of Theorem 3.36, $u(s) : \tau_{\sigma(0)}M \to \mathbb{R}^N$ is an isometry for all s and the range of $u(s)$ is $\tau_{\sigma(s)}M$.

The remainder of this section discusses a covariant derivative on $M \times \mathbb{R}^N$ which "extends" ∇ defined above. This will be needed in Section 4, where it will be convenient to have a covariant derivative on the "normal bundle"

$$N(M) \equiv \cup_{m\in M}(\{m\} \times \tau_m M^{\perp}) \subset M \times \mathbb{R}^N.$$

Analogous to the definition of ∇ on TM, it is reasonable to extend ∇ to the normal bundle $N(M)$ by setting

$$\nabla V(s)/ds = (\sigma(s), Q(\sigma(s))v'(s)) = (\sigma(s), v'(s) + dP\langle\sigma'(s)\rangle v(s)),$$

for all smooth curves $s \to V(s) = (\sigma(s), v(s))$ in $N(M)$. Then this covariant derivative on the normal bundle satisfies analogous properties to ∇ on the tangent bundle TM. These two covariant derivatives can be put together to make a covariant derivative on $M \times \mathbb{R}^N$. Explicitly, if $V(s) = (\sigma(s), v(s))$ is a smooth curve in $M \times \mathbb{R}^N$, let $p(s) \equiv P(\sigma(s))$, $q(s) \equiv Q(\sigma(s))$, and

$$\begin{aligned}
\nabla V(s)/ds &\equiv (\sigma(s), p(s)\frac{d}{ds}\{p(s)v(s)\} + q(s)\frac{d}{ds}\{q(s)v(s)\}) \\
&= (\sigma(s), \frac{d}{ds}\{p(s)v(s)\} + q'(s)p(s)v(s) \\
&\qquad + \frac{d}{ds}\{q(s)v(s)\} + p'(s)q(s)v(s)) \\
&= (\sigma(s), v'(s) + q'(s)p(s)v(s) + p'(s)q(s)v(s)) \\
&= (\sigma(s), v'(s) + dQ\langle\sigma'(s)\rangle P(\sigma(s))v(s) + dP\langle\sigma'(s)\rangle Q(\sigma(s))v(s)).
\end{aligned}$$

This may be written as

$$\nabla V(s)/ds = (\sigma(s), v'(s) + \Gamma\langle\sigma'(s)\rangle v(s)) \tag{3.33}$$

where

$$\Gamma\langle w_m\rangle v \equiv dQ\langle w_m\rangle P(m)v + dP\langle w_m\rangle Q(m)v \tag{3.34}$$

for all $w_m \in TM$ and $v \in \mathbb{R}^N$.

It should be clear from the above computation that the covariant derivative defined in (3.33) agrees with those already defined on on TM and $N(M)$. Many of the properties of the covariant derivative on TM follow quite naturally from this fact and Eq. (3.33).

Lemma 3.39. *For each $w_m \in TM$, $\Gamma\langle w_m\rangle$ is a skew symmetric $N \times N$-matrix. Hence, if $u(s)$ is the solution to the differential equation*

$$u'(s) + \Gamma\langle\sigma'(s)\rangle u(s) = 0 \quad \text{with} \quad u(0) = I, \tag{3.35}$$

then u is an orthogonal matrix for all s.

Proof. Since $\Gamma = dQP + dPQ$ and P and Q are orthogonal projections and hence symmetric, the adjoint Γ^{tr} of Γ is given by $\Gamma^{tr} = PdQ + QdP$. Thus $\Gamma^{tr} = -\Gamma$ because $PdQ = -dPQ$ and $QdP = -dQP$. Hence Γ is a skew-symmetric valued one form. Now let u denote the solution to (3.35) and $A(s) \equiv \Gamma\langle\sigma'(s)\rangle$. Then

$$\frac{d}{ds}u^{tr}u = (-Au)^{tr}u + u^{tr}(-Au) = u^{tr}(A - A)u = 0,$$

which shows that $u^{tr}(s)u(s) = u^{tr}(0)u(0) = I$ for all s. ■

Lemma 3.40. *Let u be the solution to (3.35). Then*

$$u(s)(\tau_{\sigma(0)}M) = \tau_{\sigma(s)}M \tag{3.36}$$

and

$$u(s)(\tau_{\sigma(0)}M)^{\perp} = \tau_{\sigma(s)}M^{\perp}. \tag{3.37}$$

In particular, if $v \in \tau_{\sigma(0)}M$ ($v \in \tau_{\sigma(0)}M^{\perp}$) then $V(s) \equiv (\sigma(s), u(s)v)$ is the parallel vector field along σ in TM ($N(M)$) such that $V(0) = v_{\sigma(0)}$.

Proof. Let $p(s) = P(\sigma(s))$ and $q(s) \equiv Q(\sigma(s))$, so that $\Gamma\langle\sigma'\rangle = q'p + p'q$. Then making use of the identities $pq' = -p'q$ and $q'p = -qp'$, it follows that

$$\begin{aligned}\frac{d}{ds}\{u^{tr}pu\} &= u^{tr}\{(q'p + p'q)p + p' - p(q'p + p'q)\}u \\ &= u^{tr}\{q'p + p' + pq'\}u \\ &= u^{tr}\{-p'p + p' - pp'\}u \\ &= u^{tr}\{(p - p^2)'\}u = 0.\end{aligned}$$

Therefore, $u^{tr}(s)p(s)u(s) = p(0)$ for all s. By Lemma 3.39, $u^{tr} = u^{-1}$, so

$$p(s)u(s) = u(s)p(0) \quad \forall s.$$

This last equation is equivalent to (3.36). Eq. (3.37) has completely analogous proof or can be seen easily from the fact that $p + q = I$. ■

3.7. **Smooth Development Map.** To avoid technical complications of possible explosions to certain differential equations, we will assume for the remainder of this chapter that M is a **compact** manifold. Let $o \in M$ be a fixed base point.

Theorem 3.41 (Development Map)**.** *Suppose that b is a smooth curve in T_0M such that $b(0) = 0_o \in T_oM$. Then there exists a unique smooth curve σ in M such that*

$$\sigma'(s) \equiv (\sigma(s), d\sigma(s)/ds) = //_s(\sigma)b'(s) \quad \textit{and} \quad \sigma(0) = o, \tag{3.38}$$

where $//_s(\sigma)$ denotes parallel translation along σ and $b'(s) = (o, db(s)/ds) \in T_oM$.

Proof. In the proof, I will not distinguish between $b'(s)$ and $db(s)/ds$. The meaning should be clear from the context. Suppose that σ is a solution to (3.38) and $//_s(\sigma)v_o = (o, u(s)v)$, where $u(s) : \tau_oM \to \mathbb{R}^N$. Then u satisfies the differential equation

$$du(s)/ds + dQ\langle\sigma'(s)\rangle u(s) = 0 \quad \text{with} \quad u(0) = u_0, \tag{3.39}$$

where $u_0v \equiv v$ for all $v \in \tau_0M$. Hence (3.38) is equivalent to the following pair of coupled ordinary differential equations:

$$d\sigma(s)/ds = u(s)b'(s) \quad \text{with} \quad \sigma(0) = o, \tag{3.40}$$

and

$$du(s)/ds + dQ\langle(\sigma(s), u(s)b'(s)\rangle u(s) = 0 \quad \text{with} \quad u(0) = u_0. \tag{3.41}$$

Therefore the uniqueness assertion follows from standard uniqueness theorems for ordinary differential equations.

For existence, first notice that by looking at the proof of Lemma 2.24, that Q has an extension to a neighborhood in $\mathbb{R}^N$ of $m \in M$ in such a way that $Q(x)$ is still an orthogonal projection onto $\mathrm{nul}(F'(x))$, where $F(x) = z_>(x)$ is as in Lemma 2.24. Hence for small s, we may define σ and u to be the unique solutions to (3.40) and (3.41) with values in $\mathbb{R}^N$ and $End(\tau_0 M, \mathbb{R}^N)$ respectively. The key point now is to show that $\sigma(s) \in M$ and that the range of $u(s)$ is $\tau_{\sigma(s)}M$.

Using the same proof as in Theorem 3.36, it is easy to show that $w(s) \equiv Q(\sigma(s))u(s)$ solves the differential equation

$$dw(s)/ds = dQ\langle\sigma'(s)\rangle w(s) \quad \text{with} \quad w(0) = 0,$$

so that $w \equiv 0$. Thus

$$\operatorname{ran} u(s) \subset \operatorname{nul} Q(\sigma(s)) = \operatorname{nul} F'(\sigma(s)),$$

and hence

$$dF(\sigma(s))/ds = F'(\sigma(s))d\sigma(s)/ds = F'(\sigma(s))u(s)b'(s) = 0$$

for small s. Since $F(\sigma(0)) = F(o) = 0$, it follows that $F(\sigma(s)) = 0$ and that $\sigma(s) \in M$. So we have shown that there is a solution (σ, u) to (3.40) and (3.41) for small s such that σ stays in M and $u(s)$ is parallel translation along s. By standard methods, there is a maximal solution (σ, u) with these properties. Notice that (σ, u) is a path in $M \times \mathrm{Iso}(T_0 M, \mathbb{R}^N)$, where $\mathrm{Iso}(T_0 M, \mathbb{R}^N)$ is the set of isometries from $T_0 M$ to $\mathbb{R}^N$. Since $M \times \mathrm{Iso}(T_0 M, \mathbb{R}^N)$ is a compact space, (σ, u) can not exploded. Therefore (σ, u) is defined on the same interval where b is defined. ■

3.8. **The Differential of Development Map and Its Inverse.** Let

$$W_o \equiv \{b \in C([0,1] \to T_o M) | b(0) = 0_o \in T_o M\},$$
$$W_o^\infty \equiv W_o \cap C^\infty([0,1] \to T_o M),$$
$$W_o(M) \equiv \{\sigma \in C([0,1] \to M) | \sigma(0) = o\},$$

and

$$W_o^\infty(M) \equiv W_0(M) \cap C^\infty([0,1] \to M).$$

Let $\phi : W_o^\infty \to W_o^\infty(M)$ be the map $b \to \sigma$, where σ is the solution to (3.38). It is easy to construct the inverse map $\Psi \equiv \phi^{-1}$. Namely, $\Psi(\sigma) = b$, where

$$b(s) \equiv \int_0^s //_{\tilde{s}}(\sigma)^{-1}\sigma'(\tilde{s})d\tilde{s}.$$

We now conclude this section with the important computation of the differential of Ψ.

Theorem 3.42 (Differential of Ψ)**.** *Let $(t, s) \to \Sigma(t, s)$ be a smooth map into M such that $\Sigma(t, \cdot) \in W_o^\infty(M)$ for all t. Let*

$$H(s) \equiv \dot{\Sigma}(0, s) \equiv (\Sigma(0, s), d\Sigma(t, s)/dt|_{t=0}),$$

so that H is a vector-field along $\sigma \equiv \Sigma(0, \cdot)$. One should view H as an element of the "tangent space" to $W_o^\infty(M)$ at σ, see Figure 9. Let $u(s) \equiv //_s(\sigma)$, $(\Omega_u\langle a, c\rangle)(s) \equiv$

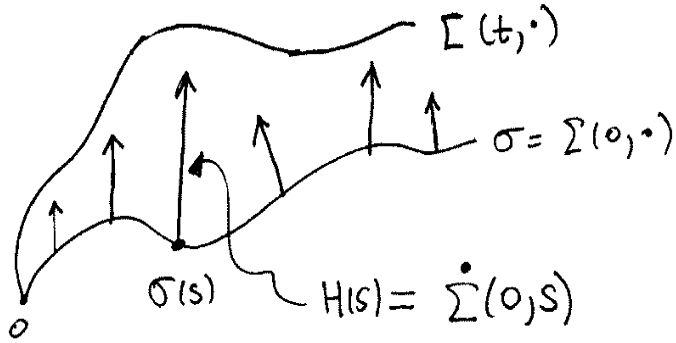

FIGURE 9. Variation of σ.

$u(s)^{-1}R\langle u(s)a, u(s)c\rangle u(s)$ for all $a, c \in T_oM$, $h(s) \equiv //_s(\sigma)^{-1}H(s)$ and $b \equiv \Psi(\sigma)$. Then

$$d\Psi\langle H\rangle = d\Psi(\Sigma(t,\cdot))/dt|_{t=0} = h + \int_0 (\int_0 \Omega_u\langle h, \delta b\rangle)\delta b, \tag{3.42}$$

where $\delta b(s)$ is short hand notation for $b(s)ds$, and $\int_0 f\delta b$ denotes the function $s \to \int_0^s f(\tilde{s})b'(\tilde{s})d\tilde{s}$ when f is a path of matrices.

Proof. To simplify notation let $\dot{} = \frac{d}{dt}|_0$, $' = \frac{d}{ds}$, $B(t,s) \equiv \Psi(\Sigma(t,\cdot))(s)$, $U(t,s) \equiv //_s(\Sigma(t,\cdot))$, $u(s) \equiv //_s(\sigma) = U(0,s)$ and

$$\dot{b}(s) \equiv (d\Psi\langle H\rangle)(s) \equiv dB(t,s)/dt|_{t=0}.$$

I will also suppress (t,s) from the notation when possible. With this notation

$$\Sigma' = UB', \quad \dot{\Sigma} = H = uh, \tag{3.43}$$

and

$$\nabla U/ds = 0, \tag{3.44}$$

where Σ' and $\dot{\Sigma}$ mean $(\Sigma, d\Sigma/ds)$ and $(\Sigma(0,\cdot), d\Sigma(t,\cdot)|_{t=0})$ respectively. Taking ∇/dt of (3.43) at $t = 0$ gives, with the aid of Proposition 3.24,

$$(\nabla U/dt)|_{t=0}b' + u\dot{b}' = \nabla\Sigma'/dt|_{t=0} = \nabla\dot{\Sigma}/ds = uh'.$$

Therefore,

$$\dot{b}' = h' + Ab', \tag{3.45}$$

where $A \equiv -U^{-1}\nabla U/dt|_{t=0}$, i.e.

$$\nabla U/dt(0,\cdot) = -uA.$$

Taking ∇/ds of this last equation and using again Proposition 3.24 and $\nabla u/ds = 0$, one shows

$$-uA' = \frac{\nabla}{ds}\frac{\nabla}{dt}U|_{t=0} = [\frac{\nabla}{ds}, \frac{\nabla}{dt}]U|_{t=0} = R\langle\sigma', H\rangle u$$

and hence

$$A' = \Omega_u\langle h, b'\rangle.$$

Since $A(0) = 0$ because

$$\nabla U(t,0)/dt|_{t=0} = \nabla //_0(\Sigma(t,\cdot))/dt|_{t=0} = \nabla(I)/dt|_{t=0},$$

it follows that

$$A = \int_0 \Omega_u \langle h, \delta b \rangle. \tag{3.46}$$

The theorem now follows, using (3.46) and the fact that $\dot{b}(0) = 0$, by integrating (3.45) relative to s. ■

Theorem 3.43 (Differential of ϕ)**.** *Let $b, k \in W_o^\infty$ and $(t,s) \to B(t,s)$ be a smooth map into T_oM such that $B(t,\cdot) \in W_o^\infty$, $B(0,s) = b(s)$, and $\dot{B}(0,s) = k(s)$. (For example take $B(t,s) = b(s) + tk(s)$.) Then*

$$\phi_* \langle k_b \rangle \equiv \frac{d}{dt}|_0 \phi(B(t,\cdot)) = //_\cdot(\sigma) h,$$

where $\sigma \equiv \phi(b)$ and h is the first component in the solution (h, A) to the pair of coupled differential equations:

$$k' = h' + Ab', \quad \textit{with} \quad h(0) = 0 \tag{3.47}$$

and

$$A' = \Omega_u \langle h, b' \rangle \quad \textit{with} \quad A(0) = 0. \tag{3.48}$$

Proof. This theorem has an analogous proof to that of Theorem 3.42. We can also deduce the result from Theorem 3.42 by defining Σ by $\Sigma(t,s) \equiv \phi_s(B(t,\cdot))$. We now assume the same notation used in Theorem 3.42 and its proof. Then $B(t,\cdot) = \Psi(\Sigma(t,\cdot))$ and hence by Theorem 3.43

$$k = \frac{d}{dt}|_0 \Psi(\Sigma(t,\cdot)) = d\Psi \langle H \rangle = h + \int_0 (\int_0 \Omega_u \langle h, \delta b \rangle) \delta b.$$

Therefore, defining $A \equiv \int_0 \Omega_u \langle h, \delta b \rangle$ and differentiating this last equation relative to s, it follows that A solves (3.48) and that h solves (3.47).

The following theorem is a mild extension of Theorem 3.42 to include the possibility that $\Sigma(t,\cdot) \notin W_o^\infty(M)$ when $t \neq 0$, i.e. the base point may change. The proof of the next theorem is identical to the proof of Theorem 3.42 and hence will be left to the reader. ■

Theorem 3.44. *Let $(t,s) \to \Sigma(t,s)$ be a smooth map into M such that $\sigma \equiv \Sigma(0,\cdot) \in W_o^\infty(M)$. Define $H(s) \equiv d\Sigma(t,s)/dt|_{t=0}$, $\sigma \equiv \Sigma(0,\cdot)$, and $h(s) \equiv //_s(\sigma)^{-1} H(s)$. (**Note:** $H(0)$ and $h(0)$ are no longer necessarily equal to zero.) Let*

$$U(t,s) \equiv //_s(\Sigma(t,\cdot)) //_t(\Sigma(\cdot,0)) : T_oM \to T_{\Sigma(t,s)}M,$$

so that $\nabla U(t,0)/dt = 0$ and $\nabla U(t,s)/ds \equiv 0$. Set $B(t,s) \equiv \int_0^s U(t,\tilde{s})^{-1} \Sigma'(t,\tilde{s}) d\tilde{s}$, then

$$\dot{b}(s) \equiv \frac{d}{dt}|_0 B(t,s) = h + \int_0 (\int_0 \Omega_u \langle h, \delta b \rangle) \delta b, \tag{3.49}$$

where as before $b \equiv \Psi(\sigma)$.

4. Stochastic Calculus on Manifolds

In this section, let $(\Omega, \{\mathcal{F}_s\}_{s\geq 0}, \mathcal{F}, \mu)$ be a filtered probability space satisfying the "usual hypothesis." Namely, $\mathcal{F}$ is $\mu-$complete, $\mathcal{F}_s$ contains all of the null sets in $\mathcal{F}$, and $\mathcal{F}_s$ is right continuous. For simplicity, we will call a function $X : \mathbb{R}_+ \times \Omega \to V$ (V a vector space) a **process** if $X_s = X(s) \equiv X(s, \cdot)$ is $\mathcal{F}_s$-measurable for all $s \in \mathbb{R}_+ \equiv [0, \infty)$, i.e. a process will mean an adapted process. As above, we will always assume that M is an embedded submanifold of $\mathbb{R}^N$ with the induced Riemannian structure.

Definition 4.1. An $M-$**valued semi-martingale** is a **continuous** $\mathbb{R}^N$-valued semi-martingale (σ) such that $\sigma(s,\omega) \in M$ for all $(s,\omega) \in \mathbb{R}_+ \times \Omega$.

Since $f \in C^\infty(M)$ is the restriction of a smooth function F on $\mathbb{R}^N$, it follows by Itô's lemma that $f \circ \sigma$ is a real-valued semi-martingale if σ is an M-valued semi-martingale. Conversely, if σ is an M-valued process and $f \circ \sigma$ is a real-valued semi-martingale for all $f \in C^\infty(M)$ then σ is an M-valued semi-martingale. Indeed, let $x = (x^1, \ldots, x^N)$ be the standard coordinates on $\mathbb{R}^N$, then $\sigma^i \equiv x^i \circ \sigma$ is a real semi-martingale for each i, which implies that σ is a $\mathbb{R}^N$- valued semi-martingale.

4.1. Line Integrals.

For $a, b \in \mathbb{R}^N$, let $a \cdot b \equiv \sum_{i=1}^N a_i b_i$ denote the standard inner product on $\mathbb{R}^N$. Also let $\mathfrak{g}l(N)$ be the set of $N \times N$ real matrices.

Theorem 4.2. *Let $Q : \mathbb{R}^N \to \mathfrak{g}l(N)$ be a smooth function such that $Q(m)$ is orthogonal projection onto $\tau_m M^\perp$ for all $m \in M$. Then for any M-valued semi-martingale σ, $Q(\sigma)\delta\sigma = \delta\sigma$ where $\delta\sigma$ denotes the Stratonovich differential of σ, i.e.*

$$\sigma_s - \sigma_0 = \int_0^s Q(\sigma_{s'})\delta\sigma_{s'}.$$

Remark 4.3. Let $f \in C^\infty(M)$, we will define

$$\int_0^s f(\sigma)\delta\sigma = \lim_{|\pi|\to 0} \sum \frac{1}{2}\{f(\sigma_{s\wedge s_i}) + f(\sigma_{s\wedge s_{i+1}})\}(\sigma_{s\wedge s_{i+1}} - \sigma_{s\wedge s_i}) \in \mathbb{R}^N,$$

where $s \wedge t \equiv \min\{s,t\}$ and the limit is taken in probability. Here $\pi = \{0 = s_0 < s_1 < s_2 < \cdots\}$ is a partition of $\mathbb{R}_+$ and $|\pi| \equiv \sup_i |s_{i+1} - s_i|$ is the mesh size of π. Notice that this limit exists since $f \circ \sigma$ is a real valued semi-martingale and the limit is equal to $\int_0^s F(\sigma)\delta\sigma$ where F is any smooth function on $\mathbb{R}^N$ such $f = F|_M$. We may similarly define $\int_0^s f(\sigma)\delta\sigma \in V$ whenever V is a finite dimensional vector space and f is a smooth map on M with values in the linear transformations from $\mathbb{R}^N$ to V.

Proof of Theorem 4.2. First assume that M is the level set of a function F as in Theorem 2.5. Then we may assume that

$$Q(x) = \phi(x)F'(x)^*(F'(x)F'(x)^*)^{-1}F'(x),$$

where ϕ is smooth function on $\mathbb{R}^N$ such that $\phi \equiv 1$ in a neighborhood of M and the support of ϕ is contained in the set: $\{x \in \mathbb{R}^N | F'(x) \text{ is surjective}\}$. By Itô 's lemma

$$0 = \delta 0 = \delta(F(\sigma)) = F'(\sigma)\delta\sigma.$$

The lemma follows in this special case by multiplying the above equation through by $\phi(\sigma)F'(\sigma)^*(F'(\sigma)F'(\sigma)^*)^{-1}$.

For the general case, choose two open covers $\{V_i\}$ and $\{U_i\}$ of M such that each $\bar{V}_i$ is compactly contained in U_i, there is a smooth function $F_i \in C_c^\infty(U_i \to \mathbb{R}^{N-d})$ such that $V_i \cap M = V_i \cap \{F_i^{-1}(\{0\})\}$ and F_i has a surjective differential on $V_i \cap M$. Choose $\phi_i \in C_c^\infty(\mathbb{R}^N)$ such that the support of ϕ_i is contained in V_i and $\sum \phi_i = 1$ on M, with the sum being locally finite. (For the existence of such covers and functions, see the discussion of partitions of unity in any reasonable book about manifolds.) Notice that $\phi_i F_i \equiv 0$ and that $F_i \phi_i' \equiv 0$ on M so that

$$\begin{aligned} 0 = \delta\{\phi_i(\sigma)F_i(\sigma)\} &= (\phi_i'(\sigma)\delta\sigma)F_i(\sigma) + \phi_i(\sigma)F_i'(\sigma)\delta\sigma \\ &= \phi_i(\sigma)F_i'(\sigma)\delta\sigma. \end{aligned}$$

Multiplying this equation by $\Psi_i(\sigma)F_i'(\sigma)^*(F_i'(\sigma)F_i'(\sigma)^*)^{-1}$, where each Ψ_i is a smooth function on $\mathbb{R}^N$ such that $\Psi_i \equiv 1$ on the support of ϕ_i and the support of Ψ_i is contained in the set where F_i' is surjective, we learn that

$$0 = \phi_i(\sigma)F_i'(\sigma)^*(F_i'(\sigma)F_i'(\sigma)^*)^{-1}F_i'(\sigma)\delta\sigma = \phi_i(\sigma)Q(\sigma)\delta\sigma \tag{4.1}$$

for all i. By a stopping time argument we may assume that σ never leaves a compact set, and therefore we may choose a finite subset $\mathcal{I}$ of the indices $\{i\}$ such that $\sum_{i\in\mathcal{I}} \phi_i(\sigma)Q(\sigma) = Q(\sigma)$. Hence summing over $i \in \mathcal{I}$ in equation (4.1) shows that $0 = Q(\sigma)\delta\sigma$.

Corollary 4.4. *If σ is an M valued semi-martingale, then $P(\sigma)\delta\sigma = \delta\sigma$.*

We now would like to define line integrals along a semi-martingale σ. For this we need a little notation. Given a one-form α on M let $\tilde{\alpha} : M \to (\mathbb{R}^N)^*$ be defined by

$$\tilde{\alpha}(m)v \equiv \alpha\langle(P(m)v)_m\rangle \tag{4.2}$$

for all $m \in M$ and $v \in \mathbb{R}^N$. Let $\Gamma(T^*M \otimes T^*M)$ denote the set of functions $\rho : \cup_{m\in M} T_mM \otimes T_mM \to \mathbb{R}$ such that $\rho_m \equiv \rho|_{T_mM\otimes T_mM}$ is linear, and $m \to \rho\langle X(m) \otimes Y(m)\rangle$ is a smooth function on M for all smooth vector-fields $X, Y \in \Gamma(TM)$. Riemannian metrics and Hessians of smooth functions are examples of elements of $\Gamma(T^*M \otimes T^*M)$. For $\rho \in \Gamma(T^*M \otimes T^*M)$, let $\tilde{\rho} : M \to (\mathbb{R}^N \otimes \mathbb{R}^N)^*$ be defined by

$$\tilde{\rho}(m)\langle v \otimes w\rangle \equiv \rho\langle(P(m)v)_m \otimes (P(m)w)_m\rangle. \tag{4.3}$$

Definition 4.5. Let α be a one form on M, $\rho \in \Gamma(T^*M \otimes T^*M)$, and σ be an M-valued semi-martingale. Then the **Stratonovich** integral of α along σ is:

$$\int \alpha\langle\delta\sigma\rangle \equiv \int \tilde{\alpha}(\sigma)\delta\sigma, \tag{4.4}$$

the **Itô** integral is given by:

$$\int \alpha\langle\bar{d}\sigma\rangle \equiv \int \tilde{\alpha}(\sigma)d\sigma, \tag{4.5}$$

where the stochastic integrals on the right hand sides of Eqs. (4.4) and (4.5) are Stratonovich and Itô integrals respectively. Formally, $\bar{d}\sigma \equiv P(\sigma)d\sigma$. We also define **quadratic integral**:

$$\int \rho\langle d\sigma \otimes d\sigma\rangle \equiv \int \tilde{\rho}(\sigma)\langle d\sigma \otimes d\sigma\rangle \equiv \sum_{i,j=1}^{N} \int \tilde{\rho}(\sigma)\langle e_i \otimes e_j\rangle d[\sigma^i, \sigma^j], \tag{4.6}$$

where $\{e_i\}_{i=1}^N$ is an orthonormal basis for $\mathbb{R}^N$, $\sigma^i \equiv e_i \cdot \sigma$, and $[\sigma^i, \sigma^j]$ is the mutual quadratic variation of σ^i and σ^j.

Remark 4.6. The above definitions may be generalized as follows. Suppose that α is now a T^*M-valued semi-martingale and σ is the M valued semi-martingale such that $\alpha(s) \in T^*_{\sigma(s)}M$ for all s. Then we may define

$$\tilde{\alpha}(s)v \equiv \alpha(s)\langle (P(\sigma(s))v)_{\sigma(s)}\rangle,$$

$$\int \alpha\langle \delta\sigma\rangle \equiv \int \tilde{\alpha}\delta\sigma, \tag{4.7}$$

and

$$\int \alpha\langle \bar{d}\sigma\rangle \equiv \int \tilde{\alpha}d\sigma. \tag{4.8}$$

Similarly, if ρ is a process in $T^*M \otimes T^*M$ such that $\rho(s) \in T^*_{\sigma(s)}M \otimes T^*_{\sigma(s)}M$, let

$$\int \rho\langle d\sigma \otimes d\sigma\rangle = \int \tilde{\rho}\langle d\sigma \otimes d\sigma\rangle, \tag{4.9}$$

where

$$\tilde{\rho}(s)\langle v \otimes w\rangle \equiv \rho(s)\langle (P(\sigma(s))v)_{\sigma(s)} \otimes (P(\sigma(s))v)_{\sigma(s)}\rangle$$

and

$$d\sigma \otimes d\sigma = \sum_{i,j=1}^{N} e_i \otimes e_j d[\sigma^i, \sigma^j]$$

as in Eq. (4.6).

Lemma 4.7. *Suppose that $\alpha = fdg$ for some $f, g \in C^\infty(M)$, then*

$$\int \alpha\langle \delta\sigma\rangle = \int f(\sigma)\delta[g(\sigma)].$$

Since any one form α on M may be written as a finite linear combination $\alpha = \sum_i f_i dg_i$, it follows that the Stratonovich integral is intrinsically defined independent of how M is embedded in $\mathbb{R}^N$.

Proof. Let G be a smooth function on $\mathbb{R}^N$ such that $g = G|_M$. Then $\tilde{\alpha}(m) = f(m)G'(m)P(m)$, so that

$$\begin{aligned}
\int \alpha\langle \delta\sigma\rangle &= \int f(\sigma)G'(\sigma)P(\sigma)\delta\sigma \\
&= \int f(\sigma)G'(\sigma)\delta\sigma \qquad \text{(by Corollary 4.4)} \\
&= \int f(\sigma)\delta[G(\sigma)] \qquad \text{(by Itô's Lemma)} \\
&= \int f(\sigma)\delta[g(\sigma)]. \qquad (g(\sigma) = G(\sigma))
\end{aligned}$$

■

Lemma 4.8. *Suppose that $\rho = fdh \otimes dg$, where $f, g, h \in C^\infty(M)$, then*

$$\int \rho\langle d\sigma \otimes d\sigma\rangle = \int f(\sigma)d[h(\sigma), g(\sigma)].$$

Since any $\rho \in \Gamma(T^*M \otimes T^*M)$ may be written as a finite linear combination $\rho = \sum_i f_i dh_i \otimes dg_i$, it follows that the quadratic integral is intrinsically defined independent of the embedding.

Proof. By Corollary 4.4 $\delta\sigma = P(\sigma)\delta\sigma$, so that

$$\begin{aligned}\sigma_s^i &= \sigma_0^i + \int (e_i, P(\sigma)d\sigma) + B.V. \\ &= \sigma_0^i + \sum_k \int (e_i, P(\sigma)e_k)d\sigma^k + B.V.,\end{aligned}$$

where $B.V.$ above stands for a process of bounded variation. Therefore

$$d[\sigma^i, \sigma^j] = \sum_{k,l} (e_i, P(\sigma)e_k)(e_i, P(\sigma)e_l)d[\sigma^k, \sigma^l]. \tag{4.10}$$

Now let H and G be in $C^\infty(\mathbb{R}^N)$ such that $h = H|_M$ and $g = G|_M$. By Itô's lemma and the above equation,

$$\begin{aligned}d[h(\sigma), g(\sigma)] &= \sum_{i,j,k,l} (H'(\sigma)e_i)(G'(\sigma)e_j)(e_i, P(\sigma)e_k)(e_i, P(\sigma)e_l)d[\sigma^k, \sigma^l] \\ &= \sum_{k,l} (H'(\sigma)P(\sigma)e_k)(G'(\sigma)P(\sigma)e_l)d[\sigma^k, \sigma^l].\end{aligned}$$

Since

$$\tilde{\rho}(m) = f(m) \cdot (H'(m)P(m)) \otimes (G'(m)P(m)),$$

it follows from Eq. (4.6) and the two above displayed equations that

$$\begin{aligned}\int f(\sigma)d[h(\sigma), g(\sigma)] &\equiv \int \sum_{k,l} f(\sigma)(H'(\sigma)P(\sigma)e_k)(G'(\sigma)P(\sigma)e_l)d[\sigma^k, \sigma^l] \\ &= \int \tilde{\rho}(\sigma)\langle d\sigma \otimes d\sigma\rangle \\ &\equiv \int \rho\langle d\sigma \otimes d\sigma\rangle.\end{aligned}$$

■

Theorem 4.9. *Let α be a one form on M, and σ be a M-valued semi-martingale. Then*

$$\int \alpha\langle\delta\sigma\rangle = \int \alpha\langle\bar{d}\sigma\rangle + \frac{1}{2}\int \nabla\alpha\langle d\sigma \otimes d\sigma\rangle, \tag{4.11}$$

where $\nabla\alpha\langle v_m \otimes w_m\rangle \equiv (\nabla_{v_m}\alpha)\langle w_m\rangle$. (This show that the Itô integral depends not only on the manifold structure of M but on the geometry of M as reflected in the covariant derivative ∇.)

Proof. Let $\tilde{\alpha}$ be as in Eq. (4.2). For the purposes of the proof, suppose that $\tilde{\alpha} : M \to (\mathbb{R}^N)^*$ has been extended to a smooth function from $\mathbb{R}^N \to (\mathbb{R}^N)^*$. We still denote this extension by $\tilde{\alpha}$. Then using Eq. (4.10),

$$\begin{aligned}\int \alpha\langle\delta\sigma\rangle &\equiv \int \tilde{\alpha}(\sigma)\delta\sigma \\ &= \int \tilde{\alpha}(\sigma)d\sigma + \frac{1}{2}\int \tilde{\alpha}'(\sigma)\langle d\sigma\rangle d\sigma \\ &= \int \alpha\langle\bar{d}\sigma\rangle \\ &+ \frac{1}{2}\sum_{i,j,k,l}\int \tilde{\alpha}'(\sigma)\langle e_i\rangle e_j(e_i, P(\sigma)e_k)(e_i, P(\sigma)e_l)d[\sigma^k,\sigma^l] \\ &= \int \alpha\langle\bar{d}\sigma\rangle + \frac{1}{2}\sum_{k,l}\int \tilde{\alpha}'(\sigma)\langle P(\sigma)e_k\rangle P(\sigma)e_l d[\sigma^k,\sigma^l] \\ &= \int \alpha\langle\bar{d}\sigma\rangle + \frac{1}{2}\sum_{k,l}\int d\tilde{\alpha}\langle (P(\sigma)e_k)_\sigma\rangle P(\sigma)e_l d[\sigma^k,\sigma^l].\end{aligned}$$

But by Eq. (3.27), we know for all $v_m, w_m \in TM$ that

$$\nabla\alpha\langle v_m\otimes w_m\rangle = d\tilde{\alpha}\langle v_m\rangle w - \tilde{\alpha}(m)dQ\langle v_m\rangle w.$$

Since $\tilde{\alpha}(m) = \tilde{\alpha}(m)P(m)$ and $PdQ = dQQ$, we find

$$\tilde{\alpha}(m)dQ\langle v_m\rangle w = \tilde{\alpha}(m)dQ\langle v_m\rangle Q(m)w = 0 \quad \forall v_m, w_m\in TM.$$

Hence combining the three above displayed equations shows that

$$\begin{aligned}\int \alpha\langle\delta\sigma\rangle &= \int \alpha\langle\bar{d}\sigma\rangle + \frac{1}{2}\sum_{k,l}\int \nabla\alpha\langle (P(\sigma)e_k)_\sigma\otimes(P(\sigma)e_l)_\sigma\rangle d[\sigma^k,\sigma^l] \\ &= \int \alpha\langle\bar{d}\sigma\rangle + \frac{1}{2}\sum_{k,l}\int \nabla\alpha\langle d\sigma\otimes d\sigma\rangle.\end{aligned}$$

■

Corollary 4.10. *Suppose that $f \in C^\infty(M)$ and σ is an $M-$valued semi-martingale, then*

$$d[f(\sigma)] = df\langle\delta\sigma\rangle = df\langle\bar{d}\sigma\rangle + \frac{1}{2}\nabla df\langle d\sigma\otimes d\sigma\rangle. \tag{4.12}$$

Proof. Let $F\in C^\infty(\mathbb{R}^N)$ such that $f = F|_M$. Then by Itô's lemma and Corollary 4.4,

$$d[F(\sigma)] = F'(\sigma)\delta\sigma = F'(\sigma)P(\sigma)\delta\sigma = df\langle\delta\sigma\rangle,$$

which proves the first equality in (4.12). The second equality follows directly from Theorem 4.9. ■

4.2. **Martingales and Brownian Motions.**

Definition 4.11. An M-valued semi-martingale σ is said to be a **martingale** (more precisely a ∇-martingale) if

$$\int df\langle\bar{d}\sigma\rangle = f(\sigma) - f(\sigma_0) - \frac{1}{2}\int \nabla df\langle d\sigma\otimes d\sigma\rangle \tag{4.13}$$

is a local martingale for all $f \in C^\infty(M)$. The process σ is said to be a **Brownian motion** if

$$f(\sigma) - f(\sigma_0) - \frac{1}{2}\int \Delta f(\sigma) d\lambda \tag{4.14}$$

is a local martingale for all $f \in C^\infty(M)$, where $\lambda(s) \equiv s$ and $\int \Delta f(\sigma) d\lambda$ denotes the process $s \to \int_0^s \Delta f(\sigma) d\lambda$.

Lemma 4.12 (Lévy-criteria)**.** *For each $m \in M$, let $\mathcal{I}(m) \equiv \sum_{i=1}^d E_i \otimes E_i$, where $\{E_i\}_{i=1}^d$ is an orthonormal basis for $T_m M$. An M-valued semi-martingale (σ) is a Brownian motion iff σ is a Martingale and*

$$d\sigma \otimes d\sigma = \mathcal{I}(\sigma) d\lambda. \tag{4.15}$$

More precisely, this last condition is to be interpreted as:

$$\int \rho\langle d\sigma \otimes d\sigma\rangle = \int \rho\langle \mathcal{I}(\sigma)\rangle d\lambda \quad \forall \rho \in \Gamma(T^*M \otimes T^*M). \tag{4.16}$$

Proof. ($\Rightarrow$) Suppose that σ is a Brownian motion on M. Let $f, g \in C^\infty(M)$. Then on one hand

$$\begin{aligned} d(f(\sigma)g(\sigma)) &= df(\sigma)\cdot g(\sigma) + f(\sigma)dg(\sigma) + d[f(\sigma), g(\sigma)] \\ &\cong \frac{1}{2}\{\Delta f(\sigma)g(\sigma) + f(\sigma)\Delta g(\sigma)\}d\lambda + d[f(\sigma), g(\sigma)], \end{aligned}$$

where " $\cong$" denotes equality up to the differential of a local martingale. While on the other hand,

$$\begin{aligned} d(f(\sigma)g(\sigma)) &\cong \frac{1}{2}\Delta(fg)(\sigma)d\lambda \\ &= \frac{1}{2}\{\Delta f(\sigma)g(\sigma) + f(\sigma)\Delta g(\sigma) + 2\langle \mathrm{grad} f, \mathrm{grad} g\rangle(\sigma)\}d\lambda. \end{aligned}$$

Comparing the above two equations implies that

$$d[f(\sigma), g(\sigma)] = \langle \mathrm{grad} f, \mathrm{grad} g\rangle(\sigma)d\lambda = df \otimes dg\langle \mathcal{I}(\sigma)\rangle d\lambda.$$

Therefore by Lemma 4.8, if $\rho = h df \otimes dg$ then

$$\begin{aligned} \int \rho\langle d\sigma \otimes d\sigma\rangle &= \int h(\sigma) d[f(\sigma), g(\sigma)] \\ &= \int h(\sigma)(df \otimes dg)\langle \mathcal{I}(\sigma)\rangle d\lambda \\ &= \int \rho\langle \mathcal{I}(\sigma)\rangle d\lambda. \end{aligned}$$

Since the general element ρ of $\Gamma(T^*M \otimes T^*M)$ is a finite linear combination of expressions of the form $h df \otimes dg$, it follows that (4.19) holds. In particular, we have that

$$\nabla df\langle d\sigma \otimes d\sigma\rangle = \nabla df\langle \mathcal{I}(\sigma)\rangle d\lambda = \Delta f(\sigma) d\lambda.$$

Hence (4.13) is also a consequence of (4.14). Conversely assuming (4.15), then $\nabla df\langle d\sigma \otimes d\sigma\rangle = \Delta f(\sigma) d\lambda$ and hence (4.14) now follows from (4.13). ∎

Definition 4.13. Suppose α is a one form on M and V is a $TM-$valued semi-martingale, i.e. $V(s) = (\sigma(s), v(s))$, where σ is an M-valued semi-martingale and v is a $\mathbb{R}^N$-valued semi-martingale such that $v(s) \in \tau_{\sigma(s)}M$ for all s. Then we define:

$$\int \alpha\langle\nabla V\rangle \equiv \int \tilde{\alpha}(\sigma)\delta v. \tag{4.17}$$

Remark 4.14. Suppose that $\alpha\langle v_m\rangle = \theta(m)v$, where $\theta : M \to (\mathbb{R}^N)^*$ is a smooth function. Then

$$\int \alpha\langle\nabla V\rangle \equiv \int \theta(\sigma)P(\sigma)\delta v = \int \theta(\sigma)\{\delta v + dQ\langle\delta\sigma\rangle v\},$$

where we have used the identity:

$$P(\sigma)\delta v = \delta v + dQ\langle\delta\sigma\rangle v.$$

This is derived by taking the differential of the equation $v = P(\sigma)v$ as in the proof of Proposition 3.24.

Proposition 4.15 (Product Rule)**.** *Keeping the notation of above, we have*

$$\delta(\alpha\langle V\rangle) = \nabla\alpha\langle\delta\sigma \otimes V\rangle + \alpha\langle\nabla V\rangle, \tag{4.18}$$

where $\nabla\alpha\langle\delta\sigma \otimes V\rangle \equiv \gamma\langle\delta\sigma\rangle$ *and* γ *is the* T^*M*-valued semi-martingale defined by* $\gamma\langle\cdot\rangle \equiv \nabla\alpha\langle(\cdot) \otimes V\rangle$.

Proof. Let $\theta : \mathbb{R}^N \to (\mathbb{R}^N)^*$ be a smooth map such that $\tilde{\alpha}(m) = \theta(m)|_{\tau_m M}$ for all $m \in M$. By Lemma 4.7 $\delta(\theta(\sigma)P(\sigma)) = d(\theta P)\langle\delta\sigma\rangle$ and hence by Lemma 3.31 $\delta(\theta(\sigma)P(\sigma))v = \nabla\alpha\langle\delta\sigma \otimes V\rangle$, where $\nabla\alpha\langle v_m \otimes w_m\rangle \equiv (\nabla_{v_m}\alpha)\langle w_m\rangle$ for all $v_m, w_m \in TM$. Therefore:

$$\begin{aligned}\delta(\alpha\langle V\rangle) &= \delta(\theta(\sigma)v) = \delta(\theta(\sigma)P(\sigma)v)\\ &= (d(\theta P)\langle\delta\sigma\rangle)v + \theta(\sigma)P(\sigma)\delta v\\ &= (d(\theta P)\langle\delta\sigma\rangle)v + \tilde{\alpha}(\sigma)\delta v\\ &= \nabla\alpha\langle\delta\sigma \otimes V\rangle + \alpha\langle\nabla V\rangle.\end{aligned}$$

■

4.3. Parallel Translation and the Development Map.

Definition 4.16. A TM-valued semi-martingale V is said to be parallel if $\nabla V \equiv 0$, i.e.

$$\int \alpha\langle\nabla V\rangle \equiv 0$$

for all one forms α on M.

Proposition 4.17. *A* TM *valued semi-martingale* $V = (\sigma, v)$ *is parallel iff*

$$\int P(\sigma)\delta v = \int \{\delta v + dQ\langle\delta\sigma\rangle v\} \equiv 0. \tag{4.19}$$

Proof. Let $x = (x^1, \dots, x^N)$ denote the standard coordinates on $\mathbb{R}^N$. Then if V is parallel,

$$0 \equiv \int dx^i\langle\nabla V\rangle = \int (e_i, P(\sigma)\delta v)$$

for each i. This implies (4.19). The converse follows from Remark 4.14. ■

Theorems 4.18 and 4.20 are stochastic analogs of Lemma 3.39 and Theorem 3.41 above. The proofs of Theorems 4.18 and 4.20 are quite analogous to their smooth cousins and hence will be omitted. The reader is referred to Section 3 of Driver [39] for a detailed exposition written in the setting of these notes. In the following theorem, V_0 is said to be a measurable vector-field on M if $V_0(m) = (m, v(m))$ with $v : M \to \mathbb{R}^N$ being a measurable function such that $v(m) \in \tau_m M$ for all $m \in M$.

Theorem 4.18 (Stochastic Parallel Translation on $M \times \mathbb{R}^N$)**.** *Let σ be an M-valued semi-martingale, and $V_0(m) = (m, v(m))$ be a measurable vector-field on M, then there is a unique parallel TM valued semi-martingale V such that $V(0) = V_0(\sigma(0))$ and $V(s) \in T_{\sigma(s)}M$ for all s. Moreover, if u denotes the solution to the stochastic differential equation:*

$$\delta u + \Gamma\langle\delta\sigma\rangle u = 0 \quad \text{with} \quad u(0) = I \in End(\mathbb{R}^N), \tag{4.20}$$

then $V(s) = (\sigma(s), u(s)v(\sigma(0))$. The process u defined in (4.20) is orthogonal for all s and satisfies $P(\sigma(s))u(s) = u(s)P(\sigma(0))$.

Definition 4.19 (Stochastic Parallel Translation)**.** Given $v \in \mathbb{R}^N$ and M valued semi-martingale σ, let $//_s(\sigma)v_{\sigma(0)} = (\sigma(s), u(s)v)$, where u solves (4.20). (Note: $V(s) = //_s(\sigma)V(0)$.)

In the remainder of these notes, I will often abuse notation and write $u(s)$ instead of $//_s(\sigma)$ and $v(s)$ rather than $V(s) = (\sigma(s), v(s))$. For example, the reader should sometimes interpret $u(s)v$ as $//_s(\sigma)v_{\sigma(0)}$ depending on the context. Essentially, we will be identifying $\tau_m M$ with $T_m M$ when no particular confusion will arise. To avoid technical problems with possible explosions of stochastic differential equations in the sequel, we make the following assumption.

Standing Assumption Unless otherwise stated, in the remainder of these notes, M will be a compact manifold embedded in $\mathbb{R}^N$.

We also fix a base point $o \in M$ and unless otherwise noted, all M-valued semi-martingales (σ) are now assumed to satisfy $\sigma(0) = o$ (a.s.). Now suppose σ is a M-valued semi-martingale, let $\Psi(\sigma) \equiv b$ where

$$b \equiv \int u^{-1}\delta\sigma = \int u^{tr}\delta\sigma.$$

Then $b = \Psi(\sigma)$ is T_oM-valued semi-martingale such that $b(0) = 0_o$. Conversely we have,

Theorem 4.20 (Stochastic Development Map)**.** *Suppose that $o \in M$ is given and b is a T_oM-valued semi-martingale. Then there exists a unique M-valued semi-martingale σ such that*

$$\delta\sigma = u\delta b \quad \text{with} \quad \sigma(0) = o \tag{4.21}$$

and u solves (4.20). As in the smooth case, we will write $\sigma = \phi(b)$.

In what follows, we will assume that b, u ($//_s(\sigma)$), and σ are related by Equations (4.21) and (4.20). Recall that $\bar{d}\sigma$ is the Itô differential of σ defined in Definition 4.5.

Proposition 4.21. *The relation between $\bar{d}\sigma$ and db is*

$$\bar{d}\sigma = P(\sigma)d\sigma = udb. \tag{4.22}$$

Also

$$d\sigma \otimes d\sigma = udb \otimes udb \equiv \sum_{i=1}^{d} ue_i \otimes ue_i d[b^i, b^j], \tag{4.23}$$

where $\{e_i\}_{i=1}^d$ *is an orthonormal basis for* T_oM *and*

$$b = \sum_i b^i e_i.$$

More precisely

$$\int \rho\langle d\sigma \otimes d\sigma\rangle = \int \sum_{i=1}^{d} \rho\langle ue_i \otimes ue_i\rangle d[b^i, b^j],$$

for all $\rho \in \Gamma(T^*M \otimes T^*M)$.

Proof. Consider the identity:

$$\begin{aligned} d\sigma = u\delta b &= udb + \frac{1}{2} du db \\ &= udb - \frac{1}{2}\Gamma\langle\delta\sigma\rangle udb \\ &= udb - \frac{1}{2}\Gamma\langle udb\rangle udb. \end{aligned}$$

Hence

$$\bar{d}\sigma = P(\sigma)d\sigma = udb - \frac{1}{2}\sum_{i=1}^{d} P(\sigma)\Gamma\langle(ue_i)_\sigma\rangle ue_j d[b^i, b^j].$$

The proof of (4.22) is finished upon noting that

$$P\Gamma P = P\{dQP + dPQ\}P = PdQP = -PQdP = 0.$$

The proof of (4.23) is easy and will be left for the reader. ■

Theorem 4.22. *Let* σ, u, *and* b *be as above, then:*

1. (1) σ *is a martingale iff* b *is a* T_oM*-valued local martingale, and*
2. (2) σ *is a Brownian motion iff* b *is a* T_oM*-valued Brownian motion.*

Proof. Keep the same notation as in Proposition 4.21. Let $f \in C^\infty(M)$, then by Proposition 4.21, if b is a local martingale, then $\int df\langle\bar{d}\sigma\rangle = \int df\langle udb\rangle$ is also a local martingale and hence σ is a martingale. Also by Proposition 4.21,

$$\begin{aligned} d[f(\sigma)] &= df\langle\bar{d}\sigma\rangle + \frac{1}{2}\nabla df\langle d\sigma \otimes d\sigma\rangle \\ &= df\langle udb\rangle + \frac{1}{2}\nabla df\langle udb \otimes udb\rangle. \end{aligned}$$

If b is a Brownian motion, $udb \otimes udb = \mathcal{I}(\sigma)d\lambda$ (u is an isometry). Hence $d[f(\sigma)] = df\langle udb\rangle + \frac{1}{2}\Delta f(\sigma)d\lambda$ from which it follows that σ is a Brownian motion.

Conversely, if σ is a M-valued martingale, then

$$N \equiv \sum_{i=1}^{N}(\int dx^i\langle\bar{d}\sigma\rangle)e_i = \sum_{i=1}^{N}(\int (e_i, udb)e_i = \int udb$$

is a local martingale, where $x = (x^1, \ldots, x^N)$ are standard coordinates on $\mathbb{R}^N$ and $\{e_i\}$ is the standard basis for $\mathbb{R}^N$. From the above equation it follows that $b = \int u^{-1} dN$ is also a local martingale.

Now suppose that σ is an M-valued Brownian motion, then we have already proved that b is a local martingale. To finish the proof is suffices by Lévy's theorem to show that $db \otimes db = \mathcal{I}(o) d\lambda$, where for $m \in M$, $\mathcal{I}(m) = \sum_{i=1}^n v_i \otimes v_i$ provided that $\{v_i\}_{i=1}^n$ is an orthonormal basis of $\tau_m M$. Now using the fact that $\sigma = N +$ (bounded variation), it follows that

$$\begin{aligned} db \otimes db &= u^{-1} dN \otimes u^{-1} dN \\ &= (u^{-1} \otimes u^{-1})(d\sigma \otimes d\sigma) \\ &= (u^{-1} \otimes u^{-1})\, \mathcal{I}(\sigma) d\lambda \qquad \text{(by (4.15))} \\ &= \mathcal{I}(o) d\lambda \qquad \text{(because } u \text{ is orthogonal.)} \end{aligned}$$

■

4.4. **Projection Construction of Brownian Motion.** In the last theorem, we saw how to construct a Brownian motion on M starting with a Brownian motion on $T_o M$. In this section, we will show how to construct an M-valued Brownian motion starting with a Brownian motion on $\mathbb{R}^N$. As in Section 3, for $m \in M$, let $P(m)$ be the orthogonal projection of $\mathbb{R}^N$ onto $\tau_m M$ and $Q(m) \equiv I - P(m)$.

Theorem 4.23. *Suppose that B is a semi-martingale on $\mathbb{R}^N$, then there exists a unique M-valued semi-martingale satisfying the Stratonovich stochastic differential equation*

$$\delta\sigma = P(\sigma)\delta B \quad \text{with} \quad \sigma(0) = o \in M, \tag{4.24}$$

see Figure 10. Moreover, σ is an M-valued martingale if B is a local martingale and σ is a Brownian motion on M if B is a Brownian motion on $\mathbb{R}^N$.

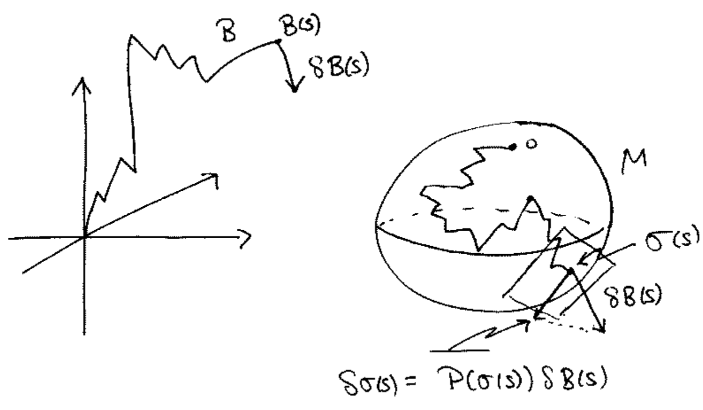

FIGURE 10. Projection construction of Brownian motion on M.

For the proof this theorem we will need the following lemma. First some more notation. Let Γ be the one form on M with values in the skew symmetric $N \times N$ matrices defined by $\Gamma = dQP + dPQ$ as in (3.34). Given an $M-$valued semi-martingale σ, let u denote parallel translation along σ as defined in Eq. (4.20) of Theorem 4.18.

Lemma 4.24. *Suppose that B is as in Theorem 4.23 and σ is the solution to (4.24), then*

$$P(\sigma)dB \otimes Q(\sigma)dB = 0.$$

The explicit meaning of this statement should become clear from the proof.

Proof. Let $\{e_i\}_{i=1}^N$ be an orthonormal basis for $\mathbb{R}^N$ and set

$$\beta \equiv \int u^{-1}dB,$$

$$\beta^i \equiv (e_i, \beta) = \int (ue_i, dB),$$

and $B^i \equiv (e_i, B)$. Then

$$\begin{aligned}\sum_{i,j} ue_i \otimes ue_j d[\beta^i, \beta^j] &= \sum_{i,j,k,l} ue_i \otimes ue_j (ue_i, e_k)(ue_j, e_l) d[B^k, B^l] \\ &= \sum_{k,l} e_k \otimes e_l d[B^k, B^l] \\ &= dB \otimes dB.\end{aligned}$$

Therefore

$$\begin{aligned}P(\sigma)dB \otimes Q(\sigma)dB &= (P(\sigma) \otimes Q(\sigma))(dB \otimes dB) \\ &= \sum_{i,j} P(\sigma)ue_i \otimes Q(\sigma)ue_j \cdot d[\beta^i, \beta^j]. \\ &= \sum_{i,j} uP(o)e_i \otimes uQ(o)e_j \cdot d[\beta^i, \beta^j],\end{aligned}$$

wherein we have used $P(\sigma)u = uP(o)$ and $Q(\sigma)u = uQ(o)$, see Theorem 4.18. This last expression is easily seen to be zero by choosing $\{e_i\}$ such that $P(o)e_i = e_i$ for $i = 1, 2, \ldots, d$. ∎

Proof. (Proof of Theorem 4.23.) For the existence and uniqueness of solutions to (4.24) we refer the reader to Theorem 3.1. of Section 3 in [39]. Now let σ be the unique solution to (4.24) and note by Theorem 4.9 that

$$\begin{aligned}d(P(\sigma)) &= dP\langle \bar{d}\sigma\rangle + (BV) \\ &= dP\langle P(\sigma)P(\sigma)dB + d(BV))\rangle + (BV) \\ &= dP\langle P(\sigma)dB\rangle + (BV),\end{aligned}$$

where (BV) denotes a process of bounded variation. Therefore, by definition of σ,

$$\begin{aligned}d\sigma = P(\sigma)\delta B &= P(\sigma)dB + \frac{1}{2}dP\langle P(\sigma)dB\rangle dB \\ &= P(\sigma)dB + \frac{1}{2}dP\langle P(\sigma)dB\rangle P(\sigma)dB + \frac{1}{2}dP\langle P(\sigma)dB\rangle Q(\sigma)dB \\ &= P(\sigma)dB + \frac{1}{2}dP\langle P(\sigma)dB\rangle P(\sigma)dB,\end{aligned}$$

where in the last equality we have used Lemma 4.24 to concluded that $dP\langle P(\sigma)dB\rangle Q(\sigma)dB = 0$. Since

$$P(dP)P = -P(dQ)P = PQdP = 0,$$

it follows that

$$\bar{d}\sigma = P(\sigma)d\sigma = P(\sigma)dB. \tag{4.25}$$

From this identity it clearly follows that if B is a local martingale, then so is $\int df\langle \bar{d}\sigma\rangle$ for all $f\in C^\infty(M)$. Moreover, if B is a Brownian motion then

$$d\sigma\otimes d\sigma = P(\sigma)dB\otimes P(\sigma)dB = \sum_{i=1}^N P(\sigma)e_i\otimes P(\sigma)e_i d\lambda,$$

where $\{e_i\}$ is any orthonormal basis of $\mathbb{R}^N$. Since

$$\sum_{i=1}^N P(m)e_i\otimes P(m)e_i = (P(m)\otimes P(m))\sum_{i=1}^N e_i\otimes e_i \tag{4.26}$$

is independent of the choice of orthonormal basis for $\mathbb{R}^N$, we may choose $\{e_i\}$ such that $\{e_i\}_{i=1}^d$ is an orthonormal basis for $\tau_m M$. Then the sum in (4.26) becomes $\mathcal{I}(m)$. Therefore $d\sigma\otimes d\sigma = \mathcal{I}(\sigma)d\lambda$, and hence σ is a Brownian motion on M by the Lévy criteria in Lemma 4.12. ■

4.5. **Starting Point Differential of the Projection Brownian Motion.** Let $\Sigma(s,x)$ denote the solution to the stochastic differential equation:

$$\Sigma(\delta s,x) = P(\Sigma(s,x)B(\delta s) \quad \text{with} \quad \Sigma(0,x) = x\in M. \tag{4.27}$$

It is well known, see Kunita [91] that there is a version of Σ which is continuous in s and smooth in x, moreover the differential of Σ relative to x solves a stochastic differential equation found by differentiating (4.27). Let $\alpha(t)$ be a smooth curve in M such that $\alpha(0) = o\in M$. By abuse of notation, let $\Sigma(s,t) = \Sigma(s,\alpha(t))$, $\sigma(s)\equiv\Sigma(s,0)$, $u(s)$ denote stochastic parallel translation along σ (see Eq. 4.20), and v and V are defined by

$$V(s) = \frac{d}{dt}|_0\Sigma(s,t) =: (\sigma(s),v(s)) = v(s)_{\sigma(s)}\in T_{\sigma(s)}M.$$

We wish to derive a convenient form for the stochastic differential equation which v solves. The next two theorems play a key role in Aida's and Elworthy's proof of a Logarithmic Sobolev Inequality on the path space of a Riemannian manifold M, see [8].

Theorem 4.25. *Keeping the notation in the above paragraph, let $a\equiv u^{-1}v$. Then a solves the Itô stochastic differential equation*

$$\begin{aligned} da &= -u^{-1}P(\sigma)dQ\langle V\rangle dB - \frac{1}{2}u^{-1}Ric\langle V\rangle d\lambda \\ &= u^{-1}dQ\langle V\rangle Q(\sigma)dB - \frac{1}{2}u^{-1}Ric\langle V\rangle d\lambda, \end{aligned}$$

with $a(0) = \dot\alpha(0)\in\tau_o M$, where Ric is the Ricci tensor defined by

$$Ric\langle v_m\rangle \equiv \sum_{i=1}^d R\langle v_m, e_i\rangle e_i$$

where $\{e_i\}_{i=1}^d$ is an orthonormal basis for $\tau_m M$.

Proof. First suppose that $\xi(s)\in\mathbb{R}^N$ is any continuous semimartingale such that $\xi(s)\in\tau_{\sigma(s)}M$ for all s and let $w\in End(\mathbb{R}^N)$ be the unique solution to the stochastic differential equation

$$\delta w - w\Gamma\langle\delta\sigma\rangle = 0 \quad \text{with} \quad w(0) = I.$$

A simple computation shows that $\delta(wu) = 0$. Since $wu = I$ at $s = 0$ it follows that $wu = I$ for all s and hence $w = u^{-1}$. Therefore

$$\begin{aligned} d(u^{-1}\xi) &= u^{-1}\{\Gamma\langle\delta\sigma\rangle\xi + \delta\xi\} \\ &= u^{-1}\{dQ\langle\delta\sigma\rangle\xi + \delta\xi\}, \end{aligned}$$

wherein we have used the definition of Γ in (3.34) and the assumption that $Q(\sigma(s))\xi(s) = 0$. To simplify notation, write $p(s) \equiv P(\sigma(s))$ and $q(s) = Q(\sigma(s))$. Since $\delta q = dQ\langle\delta\sigma\rangle$ and $q\xi = 0$, the last displayed equation may be written as

$$d(u^{-1}\xi) = u^{-1}\{\delta q \cdot \xi + q\delta\xi + p\delta\xi\} = u^{-1}\{\delta(q\xi) + p\delta\xi\} = u^{-1}p\delta\xi. \tag{4.28}$$

Taking $\xi = v$ shows that

$$da = u^{-1}p\delta v = u^{-1}pdv + \frac{1}{2}(d(u^{-1}p))dv.$$

For any $c \in \mathbb{R}^N$, we may apply (4.28) to $\xi = pc$ to find that $d(u^{-1}pc) = u^{-1}p\delta pc$, i.e. $d(u^{-1}p) = u^{-1}p\delta p$. Therefore we have shown that

$$da = u^{-1}pdv + \frac{1}{2}u^{-1}pdpdv. \tag{4.29}$$

Recall that $\Sigma(s,t)$ solves

$$\delta\Sigma = P(\Sigma)\delta B \quad \text{with} \quad \Sigma(0,t) = \alpha(t), \tag{4.30}$$

where $\delta\Sigma(s,t) \equiv \Sigma(\delta s,t)$ is the Stratonovich differential of Σ in the s parameter. Hence, differentiating (4.30) at with respect to t at $t = 0$ show that v satisfies $\delta v = \dot{p}\delta B$, where

$$\dot{p}(s) \equiv \frac{d}{dt}|_0 P(\Sigma(s,t)) = dP\langle v(s)_{\sigma(s)}\rangle.$$

Hence $dv = \dot{p}dB + \frac{1}{2}d\dot{p}dB$ which in combination with (4.29) shows that

$$\begin{aligned} da &= u^{-1}p\dot{p}dB + \frac{1}{2}u^{-1}\{pd\dot{p}dB + pdp\dot{p}dB\}. \\ &= -u^{-1}P(\sigma)dQ\langle V\rangle dB + \frac{1}{2}u^{-1}\{S\}. \end{aligned} \tag{4.31}$$

Differentiating the identity $P(\Sigma) = P(\Sigma)^2$ with respect to t at $t = 0$ implies $\dot{p} = p\dot{p} + \dot{p}p$ and hence

$$\delta\dot{p} = \delta p\dot{p} + \dot{p}\delta p + p\delta\dot{p} + \delta\dot{p}p.$$

Solving for $p\delta\dot{p}$ gives:

$$p\delta\dot{p} = \delta\dot{p}q - \delta p\dot{p} - \dot{p}\delta p.$$

Therefore, letting $S \equiv \{pd\dot{p} + pdp\dot{p}\}dB$, we have

$$\begin{aligned} S &= \{d\dot{p}q - dp\dot{p} - \dot{p}dp + pdp\dot{p}\}dB \\ &= \{d\dot{p}q - qdp\dot{p} - \dot{p}dp\}dB. \end{aligned}$$

By Lemma 4.24 and the identity

$$qdp\dot{p}dB = dp\dot{p}qdB = P\langle pdB\rangle\dot{p}qdB,$$

it follows that $qdp\dot{p}dB = 0$ and hence $S = \{d\dot{p}q - \dot{p}dp\}dB$. To deal with the term $d\dot{p}$, let $\theta(m,\xi) \equiv dP\langle(P(m)\xi)_m\rangle$ for all $\xi \in \mathbb{R}^N$ and $m \in M$. Then $\dot{p} = \theta(\sigma, v)$, so that

$$d\dot{p}qdB = \theta'(\sigma,v)\langle pdB\rangle qdB + \theta(\sigma,dv)qdB,$$

where $\theta'(m,\xi)\langle w\rangle \equiv w_m(\theta(\cdot,\xi))$ for all $w_m \in TM$ and $\xi \in \mathbb{R}^N$. Again by Lemma 4.24, it follows that $\theta'(\sigma,v)\langle pdB\rangle qdB = 0$, so that

$$d\dot{p}qdB = dP\langle (pdv)_\sigma\rangle qdB = dP\langle (p\dot{p}dB)_\sigma\rangle qdB = dP\langle (\dot{p}qdB)_\sigma\rangle qdB.$$

Hence

$$\begin{aligned} S &= dP\langle (\dot{p}qdB)_\sigma\rangle qdB - \dot{p}dpdB \\ &= dP\langle (dP\langle v_\sigma\rangle Q(\sigma)dB\rangle Q(\sigma)dB - dP\langle v_\sigma\rangle dP\langle P(\sigma)dB\rangle dB \\ &= \rho\langle v_\sigma\rangle d\lambda, \end{aligned}$$

where

$$\begin{aligned} \rho\langle v_m\rangle &\equiv \sum_{i=1}^N \{dP\langle (dP\langle v_m\rangle Q(m)e_i\rangle Q(m)e_i - dP\langle v_m\rangle dP\langle P(m)e_i\rangle e_i\} \\ &\equiv \sum_{i=1}^N (dP\langle (dP\langle v_m\rangle Q(m)e_i\rangle Q(m)e_i - dP\langle P(m)e_i\rangle dP\langle v_m\rangle e_i) \\ &\qquad - \sum_{i=1}^N [dP\langle v_m\rangle, dP\langle P(m)e_i\rangle]e_i. \end{aligned}$$

For given $m \in M$, choose the basis $\{e_i\}$ such that $\{e_i\}_{i=1}^d$ is an orthonormal basis for $\tau_m M$ and write $n_j \equiv e_{i+j}$ for $j = 1, 2, \ldots, N-d$, so that $\{n_j\}_{j=1}^{N-d}$ is an orthonormal basis for $\tau_m M^\perp$. Noting that

$$[dP\langle v_m\rangle, dP\langle P(m)e_i\rangle] = R\langle v_m, P(m)e_i\rangle,$$

we find that

$$\rho\langle v_m\rangle = \sum_{j=1}^{N-d} dP\langle (dP\langle v_m\rangle n_j\rangle n_j - \sum_{i=1}^d dP\langle e_i\rangle dP\langle v_m\rangle e_i - \mathrm{Ric}\langle v_m\rangle.$$

Assembling the last four equation with (4.31), the Theorem follows if we can show

$$\begin{aligned} 0 &= \sum_{j=1}^{N-d} dP\langle (dP\langle v_m\rangle n_j\rangle n_j - \sum_{i=1}^d dP\langle e_i\rangle dP\langle v_m\rangle e_i \\ &= \sum_{j=1}^{N-d} dQ\langle (dQ\langle v_m\rangle n_j\rangle n_j - \sum_{i=1}^d dQ\langle e_i\rangle dQ\langle v_m\rangle e_i, \end{aligned}$$

or equivalently that

$$\sum_{j=1}^{N-d} dQ\langle dQ\langle v\rangle n_j\rangle n_j = \sum_{i=1}^d dQ\langle e_i\rangle dQ\langle v\rangle e_i \quad \forall v \in \tau_m M. \tag{4.32}$$

Because both sides of (4.32) are in $\tau_m M$, to prove (4.32) it suffices to show

$$\sum_{j=1}^{N-d} (dQ\langle dQ\langle v\rangle n_j\rangle n_j, w) = \sum_{i=1}^d (dQ\langle e_i\rangle dQ\langle v\rangle e_i, w) \tag{4.33}$$

for all v and w in $\tau_m M$. Using the fact that $dQ\langle v\rangle$ is symmetric and the identity (see Proposition 3.28):

$$dQ\langle v_m\rangle w = dQ\langle w_m\rangle v \quad \forall v_m, w_m \in T_m M, \tag{4.34}$$

Eq. (4.33) is equivalent to:

$$\sum_{j=1}^{N-d}(n_j, dQ\langle v\rangle dQ\langle w\rangle n_j) = \sum_{i=1}^{d}(dQ\langle v\rangle e_i, dQ\langle w\rangle e_i). \tag{4.35}$$

Thus (4.32) is valid iff

$$\mathrm{tr}[Q(m)dQ\langle v\rangle dQ\langle w\rangle] = \mathrm{tr}[P(m)dQ\langle w\rangle dQ\langle v\rangle] \tag{4.36}$$

But

$$\begin{aligned}\mathrm{tr}[P(m)dQ\langle w\rangle dQ\langle v\rangle] &= -\mathrm{tr}[dP\langle w\rangle Q(m)dQ\langle v\rangle]\\ &= \mathrm{tr}[dQ\langle w\rangle Q(m)dQ\langle v\rangle]\\ &= \mathrm{tr}[Q(m)dQ\langle v\rangle dQ\langle w\rangle].\end{aligned}$$

■

Lemma 4.26. *Let B be any $\mathbb{R}^N$-valued semi-martingale, σ is the solution to $\delta\sigma = P(\sigma)\delta B$ with $\sigma(0) = o$, and $b \equiv \int u^{-1}\delta\sigma = \int u^{-1}P(\sigma)\delta B$. Then*

$$b = \int u^{-1}P(\sigma)dB. \tag{4.37}$$

Moreover if B is a standard Brownian motion then (b,β) is a standard Brownian motion on $\mathbb{R}^N$, where

$$\beta \equiv \int u^{-1}Q(\sigma)dB. \tag{4.38}$$

In particular, the "normal" Brownian motion β is independent of b and hence σ and u.

Proof. Again let $p = P(\sigma)$, then

$$\begin{aligned}d(u^{-1}P(\sigma))dB &= u^{-1}\{\Gamma\langle\delta\sigma\rangle pdB + dP\langle\delta\sigma\rangle dB\}\\ &= u^{-1}\{dQ\langle pdB\rangle pdB - dQ\langle pdB\rangle dB\}\\ &= u^{-1}\{dQ\langle pdB\rangle pdB - dQ\langle pdB\rangle pdB\} = 0,\end{aligned}$$

where we have again used $pdB \otimes qdB = 0$. This proves (4.37).

Now suppose that B is a Brownian motion. Since $(b,\beta) = \int u^{-1}dB$ and u is an orthogonal process, it easily follow's using Lévy's criteria that (b,β) is a standard Brownian motion. Since (σ,u) satisfies the coupled pair of stochastic differential equations

$$d\sigma = u\delta b \quad \text{with} \quad \sigma(0) = o$$

and

$$du + \Gamma\langle u\delta b\rangle u = 0 \quad \text{with} \quad u(0) = id \in End(\mathbb{R}^N),$$

it follows that (σ,u) is a functional of b and hence σ and u are independent of β. ■

5. Calculus on $W(M)$

In this section, we will introduce a geometry on $W(M)$. This induces a gradient D and a divergence operator D^* for $W(M)$. We will investigate the necessary integration by parts formulas to conclude that D^* is densely defined. Then we will examine S. Fang's beautiful theorem on the existence of a mass or spectral gap for the Ornstein Uhlenbeck operator $\mathcal{L} = D^*D$. It has been shown in Driver and Röckner [48] that this operator generates a diffusion on $W(M)$. This last result also holds for pinned paths on M and free loops on $\mathbb{R}^N$, see [16] for the $\mathbb{R}^N$ case.

5.1. Tangent spaces and Riemannian metrics on $W(M)$.

Let σ be a Brownian motion on M starting at o. We will associate the processes u and b to σ in the usual way so that u is parallel translation along σ and b is a T_oM-valued Brownian motion. In this section, we assume that the filtration $\{\mathcal{F}_s\}$ on Ω is the one generated by the Brownian motion b (or equivalently σ).

Definition 5.1. The **continuous tangent space** to $W(M)$ at $\sigma \in W(M)$ is the set $CT_\sigma W(M)$ of continuous vector-fields along σ which are zero at $s = 0$:

(5.1)
$$CT_\sigma W(M) = \{X \in C([0,1], TM) | X(s) \in T_{\sigma(s)}M \ \forall \ s \in [0,1] \text{ and } X(0) = 0\}.$$

To motivate the above definition, consider a differentiable curve in $W(M)$ going through σ at $t = 0$: $(t \to f(t,\cdot)) : (-1,1,) \to W(M)$. The derivative $X(s) \equiv \frac{d}{dt}|_0 f(t,s)$ of such a curve should by definition be a tangent vector $W(M)$ at σ. This is indeed the case.

We now wish to define a Riemannian metric on $W(M)$. We know from the case that $M = \mathbb{R}^d$, that the continuous tangent space is too large for most purposes, see for example the Cameron-Martin theorem. We will have to introduce the Riemannian structure on a sub-bundle which we call the Cameron-Martin tangent space. In the sequel, set

$$H \equiv \{h : [0,1] \to T_oM : h(0) = 0, \quad \text{and} \quad (h,h) \equiv \int_0^1 |h'(s)|^2_{T_oM} ds < \infty\}.$$

H is just the usual Cameron-Martin space with $\mathbb{R}^d$ replaced by the isometric inner-product space (T_oM).

Definition 5.2. A Cameron-Martin process h is a T_oM-valued process such that $s \to h(s)$ is in H a.s.. Contrary to our earlier assumptions, we do not assume that h is adapted unless explicitly stated.

Definition 5.3. A TM-valued process X is said to be a Cameron-Martin vector-field if $h(s) \equiv u^{-1}(s)X(s)$ is a Cameron-Martin process and

$$\langle\langle X, X\rangle\rangle \equiv E[(h,h)_H] < \infty. \tag{5.2}$$

A Cameron-Martin vector field X is said to be adapted if $h \equiv u^{-1}X$ is adapted. The set of Cameron-Martin vector-fields will be denoted by $\mathcal{X}$ and those which are adapted will be denoted by $\mathcal{X}_a$.

Remark 5.4. Notice that $\mathcal{X}$ is a Hilbert space with the inner product determined by $\langle\langle\cdot,\cdot\rangle\rangle$ in (5.2). Furthermore, $\mathcal{X}_a$ is a Hilbert-subspace of $\mathcal{X}$.

Notation 5.5. Given Cameron-Martin process h, let $X^h \equiv uh$. In this way we may identify Cameron-Martin processes with Cameron-Martin vector fields.

We define a "metric" (G) on $\mathcal{X}$ by

$$G\langle X^h, X^h\rangle = (h,h). \tag{5.3}$$

With this notation we may write

$$\langle\langle X, X\rangle\rangle = EG\langle X, X\rangle.$$

Remark 5.6. Notice, if σ is a smooth curve then the expression in (5.3) could be written as

$$G\langle X, X\rangle = \int_0^1 g\langle \frac{\nabla}{ds}X(s), \frac{\nabla}{ds}X(s)\rangle ds,$$

where $\frac{\nabla}{ds}$ denotes the covariant derivative along the curve σ which is induced from the covariant derivative ∇. This is a typical metric used by differential geometers on path and loop spaces.

The function G is to be interpreted as a Riemannian metric on $W(M)$.

5.2. Divergence and Integration by Parts.

Definition 5.7. A function $f : W(M) \to \mathbb{R}$ is called a smooth cylinder if there exists a partition $\{0 = s_0 < s_1 < s_2 \cdots < s_n = 1\}$ of $[0,1]$ and $F \in C^\infty(M^{n+1})$ such that $f(\sigma) = F(\sigma(s_0), \sigma(s_1), \ldots, \sigma(s_n))$.

Given a Cameron-Martin vector field X on $W(M)$, let Xf denote the random variable

$$Xf \equiv \sum_{i=0}^n (\mathrm{grad}_i f(\sigma(s)), X(s_i)), \tag{5.4}$$

where $\mathrm{grad}_i f$ denotes the gradient of f relative to the i'th variable. We also define the gradient operator D on smooth cylinder functions on $W(M)$ by requiring Df to be the unique Cameron-Martin process such that $G\langle Df, X\rangle = Xf$ for all $X \in \mathcal{X}$. The explicit formula for D is

$$Df(s) = u(s)\sum_{i=0}^n s \wedge s_i u(s_i)^{-1}\mathrm{grad}_i f(\sigma(s)).$$

In the next Theorem, it will be shown that X is in the domain of D^* when X is an adapted Cameron-Martin vector field. From this fact it will easily follow that D^* is densely defined.

Theorem 5.8. *Let X be an adapted Cameron-Martin vector field on $W(M)$, and $h \equiv u^{-1}X$. Then $X \in \mathcal{D}(D^*)$ and*

$$D^*X = \int_0^1 h' \cdot db + \frac{1}{2}\int_0^1 Ric_u\langle h\rangle \cdot db \equiv \int_0^1 B(h)\cdot db, \tag{5.5}$$

where B is the random linear operator mapping H to $L^2(ds, T_oM)$ given by

$$B(h) \equiv h' + \frac{1}{2}Ric_u\langle h\rangle, \tag{5.6}$$

and $Ric_u\langle h\rangle \equiv u^{-1}Ric\langle uh\rangle$. (Recall that $v\cdot w$ denotes the standard dot product of $v, w \in \mathbb{R}^N$.)

Remark 5.9. Notice that for each $\omega \in \Omega$ (recall Ω is the probability space) $B_\omega(h) \equiv h' + \frac{1}{2}Ric_{u(\omega)}\langle h\rangle$ is a bounded linear operator from H to $L^2(ds, T_oM)$ and the bound can be chosen independent of ω. The bound only depends on the Ricci tensor.

Proof. I will only sketch the proof here, the interested reader may find complete details in [46]. We start by proving the theorem under the additional assumption that $h \equiv u^{-1}X$ satisfies

$$\sup_{s\in[0,1]} |h'_s| \le C,$$

where C is a non-random constant. Using Theorem 3.42 as motivation, the "pull back" X by the development map $(b \to \sigma)$ should be the "vector-field" Y on $W(T_oM)$ given by:

$$Y = h + \int (\int \Omega_u\langle h, \delta b\rangle)\delta b.$$

Writing this in Itô form:

$$Y = \int C db + \int r d\lambda,$$

where $C \equiv \int \Omega_u\langle h, \delta b\rangle$ and

$$r = h' + \frac{1}{2} Ric_u\langle h\rangle.$$

Key Point: The process C is skew-adjoint because of the skew-symmetry properties of the curvature tensor, see Eq. 3.22.

Following Bismut, (also see Fang and Malliavin), for each $t \in \mathbb{R}$ let $B(t,\cdot)$ be the process given by:

$$B(t,\cdot) = \int e^{tC} db + t\int r d\lambda. \tag{5.7}$$

Notice that $B(t,\cdot)$ is not the flow of the vector-field Y but does have the property that $\frac{d}{dt}|_0 B(t,\cdot) = Y$. It is also easy to concluded by Girsanov's theorem that $B(t,\cdot)$ (for fixed t) is a Brownian motion relative to $Z_t \cdot \mu$, where

$$Z_t = \exp - \left\{ \int_0^1 t(r, e^{tC}db) + \frac{1}{2}t^2 \int_0^1 (r,r)ds \right\}. \tag{5.8}$$

For $t \in \mathbb{R}$, let $\Sigma(t,\cdot) \equiv \phi(B(t,\cdot))$ as in Theorem 4.20. After choosing a good version of Σ it is possible to show using a stochastic analogue of Theorem 3.43 that $\dot\Sigma(0,\cdot) = X$, so the $Xf = \frac{d}{dt}|_0 f(\Sigma(t,\cdot))$. Now if f is a smooth cylinder function on $W(M)$, then

$$E(f(\Sigma(t,\cdot)Z_t) = Ef(\sigma)$$

for all t. Differentiating this last expression relative to t at $t = 0$ gives:

$$E(Xf(\sigma)) - E(f\int_0^1 (r, db)) = 0.$$

This last equation may be written alternatively as

$$\langle\langle Df, X\rangle\rangle = EG(Df, X) = (f, \int_0^1 B(h)\cdot db))_{L^2}.$$

Hence it follows that $X \in \mathcal{D}(D^*)$ and

$$D^*X = \int_0^1 B(h)\cdot db.$$

This proves the theorem in the special case that h' is uniformly bounded.

Let X be a general adapted Cameron-Martin vector-field and $h \equiv u^{-1}X$. For each $n \in \mathbb{N}$, let $h_n(s,\sigma) \equiv \int_0^s h'(\tau,\sigma)\cdot 1_{|h'(\tau,\sigma)|\le n} d\tau$. (Notice that h_n is still adapted.) Set

$X^n \equiv uh_n$, then by the special case above we know that $X^n \in \mathcal{D}(D^*)$ and $D^*X^n = \int_0^1 B(h_n)\cdot db$. It is easy to check that $\langle\langle X-X^n, X-X^n\rangle\rangle = E(h-h_n, h-h_n)_H \to 0$ as $n\to\infty$. Furthermore,

$$\begin{aligned} E[D^*(X^m - X^n), D^*(X^m - X^n)] &= E\int_0^1 |B(h_m - h_n)|^2 ds \\ &\le CE(h_m - h_n, h_m - h_n)_H, \end{aligned}$$

from which it follows that D^*X^m is convergent. Because D^* is a closed operator, it follows that $X \in \mathcal{D}(D^*)$ and

$$D^*X = \lim_{n\to\infty} D^*X^n = \lim_{n\to\infty}\int_0^1 B(h_n)\cdot db = \int_0^1 B(h)\cdot db,$$

since

$$E\int_0^1 |B(h-h_n)|^2 ds \le CE(h-h_n, h-h_n)_H \to 0 \text{ as } n\to\infty.$$

■

Corollary 5.10. *The operator D^* is densely defined. In particular D is closable. (Let $\bar{D}$ denote the closure of D.)*

Proof. Let $h \in H$, $X^h \equiv uh$, and f and g be a smooth cylinder functions. Then by the product rule:

$$\begin{aligned} \langle\langle Df, gX^h\rangle\rangle + E[f(Dg, X^h)] &= \langle\langle gDf + fDg, X^h\rangle\rangle \\ &= \langle\langle D(fg), X^h\rangle\rangle = E(fgD^*X^h), \end{aligned}$$

from which we learn that $gX^h \in \mathcal{D}(D^*)$ (the domain of D^*) and

$$D^*(gX^h) = gD^*X^h - (Dg, X^h).$$

Since $\{gX^h | h \in H \text{ and } g \text{ is a cylinder function}\}$ is a dense subset of $\mathcal{X}$, D^* is densely defined. ■

Theorem 5.11 may be extended to allow for vector-fields on the paths of M which are not based. This is important for Hsu's proof of Logarithmic Sobolev inequalities for the Ornstein-Uhlenbeck operator $\mathcal{L} = D^*\bar{D}$.

Theorem 5.11. *Let h be an adapted T_oM-valued process such that $h(0)$ is independent of ω and $h - h(0)$ is a Cameron-Martin process. Let E_x denote the path space expectation for a Brownian motion starting at $x \in M$. Let $f : C([0,1] \to M) \to \mathbb{R}$, be a cylinder function as in 5.7. As before let $X \equiv X^h \equiv uh$ and $X^h f$ be defined as in (5.4). Then*

$$E_o[X^h f] = E_o[fD^*X^h] + \langle d(E_{(\cdot)}f), h(0)_o\rangle, \tag{5.9}$$

where

$$D^*X^h \equiv \int_0^1 h'\cdot db + \frac{1}{2}\int_0^1 Ric_u\langle h\rangle\cdot db \equiv \int_0^1 B(h)\cdot db,$$

as in (5.5) and $B(h)$ is defined in (5.6).

Proof. Start by choosing a smooth curve α in M such that $\dot{\alpha}(0) = h(0)_o$. Let C, r, $B(t,\cdot)$, and Z_t be defined by the same formulas as in the proof of the previous theorem. Let $u_0(t)$ denote parallel translation along α, that is

$$du_0(t)/dt + \Gamma\langle\dot{\alpha}(t)\rangle u_0(t) = 0 \quad \text{with} \quad u_0(0) = id.$$

For $t \in \mathbb{R}$, define $\Sigma(t,\cdot)$ by

$$\Sigma(t,\delta s) = u(t,\delta s)B(t,\delta s) \quad \text{with} \quad \Sigma(t,0) = \alpha(t)$$

and

$$u(t,\delta s) + \Gamma\langle u(t,s)B(t,\delta s)\rangle u(t,s) = 0 \quad \text{with} \quad u(t,0) = u_o(t).$$

Appealing to a stochastic version of Theorem 3.44 (after choosing a good version of Σ) it is possible to show that $\dot\Sigma(0,\cdot) = X$, so the $Xf = \frac{d}{dt}|_0 f(\Sigma(t,\cdot))$. As in the above proof $B(t,\cdot)$ is a Brownian motion relative to the expectation E_t defined by $E_t(F) \equiv E(Z_t F)$. From this it is easy to see that $\Sigma(t,\cdot)$ is a Brownian motion on M starting at $\alpha(t)$ relative to the expectation E_t. Therefore, if f is a smooth cylinder function on $W(M)$, then

$$E(f(\Sigma(t,\cdot)Z_t) = E_{\alpha(t)}f$$

for all t. Differentiating this last expression relative to t at $t = 0$ gives:

$$E(Xf(\sigma)) - E(f\int_0^1 r\cdot db) = \langle dE_{(\cdot)}f, h(0)_o\rangle.$$

The rest of the proof is identical to the previous proof. ■

5.3. **Hsu' s Derivative Formula.** As a corollary Theorem 5.11 we get Elton Hsu's derivative formula which plays a key role in his proof of a Logarithmic Sobolev inequality on $W(M)$, see [82]. Hsu's original proof was by a coupling argument. The idea is similar, the only question is how one describes the perturbed process $\Sigma(t,\cdot)$ of the last proof.

Corollary 5.12 (Hsu's Derivative Formula)**.** *Let $v_o \in T_oM$. Define h to be the adapted T_oM-valued process solving the differential equation:*

$$h' + \frac{1}{2}Ric_u\langle h\rangle = 0 \quad \textit{with} \quad h(0) = v_o. \tag{5.10}$$

Then

$$\langle d(E_{(\cdot)}f), v_o\rangle = E_o[X^h f]. \tag{5.11}$$

Proof. Apply the previous theorem to X^h with h defined by (5.10). Notice that h has been constructed so that $B(h) \equiv 0$, i.e. $D^*X^h = 0$. ■

The following theorem was first proved by Hsu [82] with an independent proof given shortly thereafter by Aida and Elworthy [8]. Hsu's proof relies on a modification of the additivity property for Logarithmic Sobolev inequalities adapted to the case where there is a Markov dependence. A key point in Hsu's proof is Corollary 5.12. On the other hand Aida and Elworthy show, using the projection construction of Brownian motion, the logarithmic Sobolev inequality on $W(M)$ is a consequence of Gross' [69] original logarithmic Sobolev inequality on the classical Wiener space $W(\mathbb{R}^N)$. As mentioned earlier, Theorem 4.25 is a key step in Aida's and Elworthy's proof.

Theorem 5.13 (Logarithmic Sobolev Inequality)**.** *Let M be a compact Riemannian manifold, then there is a constant C depending on M such that*

$$E(f^2 \log f^2) \le CE(Df, Df) + Ef^2 \log Ef^2,$$

for all smooth cylinder function f on $W(M)$.

For a proof of this theorem the reader is referred to [82, 8]. These paper should be quite accessible after reading these notes.

5.4. **Fang's Spectral Gap Theorem and Proof.** It is well known that logarithmic Sobolev inequalities imply "spectral gap" inequalities. Hence a spectral gap inequality on $W(M)$ is a Corollary of 5.13. In fact, this inequality was already known by the work of Fang [58]. In this section, I will present Fang's [58] spectral gap theorem and his elegant proof.

Theorem 5.14. *Let $\bar{D}$ be the closure of D and $\mathcal{L}$ be the selfadjoint operator on $L^2(W(M))$ defined by $\mathcal{L} = D^*\bar{D}$. (Note, if $M = \mathbb{R}^n$ then $\mathcal{L}$ would be an infinite dimensional Ornstein Uhlenbeck operator.) Then the null-space of $\mathcal{L}$ consists of the constant functions on $W(M)$ and $\mathcal{L}$ has a spectral gap, i.e. there is a constant $c > 0$ such that $(\mathcal{L}f, f)_{L^2} \geq c(f, f)_{L^2}$ for all $f \in \mathcal{D}(\mathcal{L})$ which are perpendicular to the constant functions.*

The proof of this theorem will be given at the end of this subsection. We first will need to represent F in terms of DF.

Lemma 5.15. *For each $F \in L^2(\mu)$, there is a unique adapted Cameron-Martin vector field X on $W(M)$ such that*

$$F = E(F) + D^*X.$$

Proof. By the Martingale representation theorem (see Corollary 6.2 in the appendix below), there is a predictable T_oM–valued process (a) (which is not in general continuous) such that

$$E\int_0^1 |a_s|^2 ds < \infty,$$

and

$$F = E(F) + \int_0^1 a_s \cdot db(s). \tag{5.12}$$

Define $h \equiv B^{-1}(a)$, i.e. let h be the solution to the differential equation:

$$h_s' + A_s h_s = a_s \quad \text{with} \quad h_0 = 0, \tag{5.13}$$

where for any $\xi \in T_oM$,

$$A_s\xi \equiv \frac{1}{2} Ric_{u_s}\langle\xi\rangle.$$

Claim: B_ω^{-1} is a bounded linear map from $L^2(ds, T_oM) \to H$ for each $\omega \in \Omega$, and furthermore the norm of B_ω^{-1} is bounded independent of $\omega \in \Omega$.

To prove the claim, let M_s be the $End(T_oM)$–valued solution to the differential equation

$$M_s' + A_s M_s = 0 \quad \text{with} \quad M_0 = I, \tag{5.14}$$

then the solution to (5.13) can be written as:

$$h_s = \int_0^s M_s M_\tau^{-1} a_\tau d\tau. \tag{5.15}$$

Since, $\rho_s \equiv M_s M_\tau^{-1}$ solves the differential equation

$$\rho_s' + A_s \rho_s = 0 \quad \text{with} \quad \rho_\tau = I$$

it is easy to show from the boundedness of A and an application of Gronwall's inequality that $|M_s M_\tau^{-1}| = |\rho_s| \le C$, where C is a non-random constant independent of s and τ. Therefore,

$$\begin{aligned}(h,h)_H &= \int_0^1 |a_s - A_s h_s|^2 ds \\ &\le 2\int_0^1 |a_s|^2 ds + 2\int_0^1 |A_s h_s|^2 ds \\ &\le 2(1+C^2K^2)\int_0^1 |a_s|^2 ds,\end{aligned}$$

where K is a bound on the process A_s. This proves the claim.

Because of the claim, $h \equiv B^{-1}(a)$ satisfies $E(h,h)_H < \infty$. It is also easy to see that h is adapted (see (5.15)). Hence, $X \equiv uh$ is an adapted Cameron-Martin vector field and

$$D^*X = \int_0^1 B(h)\cdot db = \int_0^1 a \cdot db.$$

The existence part of the theorem now follows from this equation and equation (5.12).

The uniqueness assertion follows from the energy identity:

$$E(D^*X)^2 = E\int_0^1 |B(h)(s)|^2 ds \ge CE(h,h)_H.$$

Indeed if $D^*X = 0$, then $h = 0$ and hence $X = uh = 0$. ■

The next goal is to find an expression for the vector-field X in the above Lemma in terms of the function F itself. This will be the content of the next theorem.

Notation 5.16. Let

$$L^2_a(P : L^2(ds, T_oM)) = \{v \in L^2(P : L^2(ds, T_oM)) | v \text{ is adapted}\}.$$

Define the bounded linear operator $\bar{B}$ from $\mathcal{X}_a$ to $L^2_a(P : L^2(ds, T_oM))$ by $\bar{B}(X) = B(u^{-1}X)$. Also let $\mathcal{Q} : \mathcal{X} \to \mathcal{X}$ denote the orthogonal projection of $\mathcal{X}$ onto $\mathcal{X}_a$.

Remark 5.17. Notice that $D^*X = \int_0^1 \bar{B}(X)\cdot db$ for all $X \in \mathcal{X}_a$. We have seen that $\bar{B}$ has a bounded inverse, in fact $\bar{B}^{-1}(a) = uB^{-1}(a)$.

Theorem 5.18. *As above let $\bar{D}$ denote the closure of D. Also let $T : \mathcal{X} \to \mathcal{X}_a$ be the bounded linear operator defined by*

$$T(X) = (\bar{B}^*\bar{B})^{-1}\, \mathcal{Q}X$$

for all $X \in \mathcal{X}$. Then for all $F \in \mathcal{D}(\bar{D})$,

$$F = EF + D^*T\bar{D}F. \tag{5.16}$$

It is worth pointing out that $\bar{B}^*$ is not uB^* but is instead given by $\mathcal{Q}uB^*$. This is because uB^* does not take adapted processes to adapted processes. This is the reason it is necessary to introduce the orthogonal projection.

Proof. Let $Y \in \mathcal{X}_a$ be given, $X \in \mathcal{X}_a$ such that $F = EF + D^*X$. Then

$$\begin{aligned}\langle\langle Y,\ \mathcal{Q}\bar{D}F\rangle\rangle &= \langle\langle Y, \bar{D}F\rangle\rangle = E(D^*Y \cdot F) \\ &= E(D^*Y \cdot D^*X) = E(\bar{B}(Y), \bar{B}(X))_{L^2(ds)} \\ &= \langle\langle Y, \bar{B}^*\bar{B}(X)\rangle\rangle,\end{aligned}$$

where in going from the first to the second line we have used $E(D^*Y) = 0$. From the above displayed equation it follows that $\mathcal{Q}\bar{D}F = \bar{B}^*\bar{B}(X)$ and hence $X = (\bar{B}^*\bar{B})^{-1}\,\mathcal{Q}\bar{D}F = T(\bar{D}F)$. ∎

Proof. *Proof of Theorem 5.14.* Let $F \in \mathcal{D}(\bar{D})$, then by the above theorem

$$E(F - EF)^2 = E(D^*T\bar{D}F)^2 = E|\bar{B}(T\bar{D}F)|^2_{L^2(ds,T_oM)} \le C\langle\langle \bar{D}F, \bar{D}F\rangle\rangle.$$

In particular if $F \in \mathcal{D}(\mathcal{L})$, then $\langle\langle \bar{D}F, \bar{D}F\rangle\rangle = E[\mathcal{L}F \cdot F]$, and hence

$$(\mathcal{L}F, F)_{L^2} \ge C^{-1}(F - EF, F - EF)_{L^2}.$$

Therefore, if $F \in \mathrm{nul}(\mathcal{L})$, it follows that $F = EF$, i.e. F is a constant. Moreover if $F \perp 1$ (i.e. $EF = 0$) then

$$(\mathcal{L}F, F)_{L^2} \ge C^{-1}(F, F)_{L^2},$$

proving Theorem 5.14 with $c = C^{-1}$. ∎

6. Appendix: Martingale Representation Theorem

We continue the notation of Sections 4 and 5. In particular σ is a Brownian motion on M starting at $o \in M$ and $b = \Psi(\sigma)$ is the Brownian motion on $\mathbb{R}^n$ associated to σ described before Theorem 4.20.

Lemma 6.1. *Let F be the smooth cylinder function on $W(M)$,*

$$F(\sigma) = f(\sigma(s_1), \ldots, \sigma(s_n)),$$

where $0 < s_1 < s_2 \cdots < s_n \le 1$. Then

$$F = E(F) + \int_0^1 a_s \cdot db(s), \tag{6.1}$$

where a_s is a bounded, piecewise-continuous (in s), and predictable process. Furthermore, the jumps points of a are contained in the set $\{s_1, \ldots, s_n\}$ and $a_s \equiv 0$ is $s \ge s_n$.

Proof. The proof will be by induction on n. First assume that $n = 1$, so that $F(\sigma) = f(\sigma(\tau))$ for some $0 < \tau \le 1$. Let $H(s, m) \equiv (e^{(\tau - s)\Delta/2}f)(m)$ for $0 \le s \le \tau$ and $m \in M$. Then it is easy to compute:

$$dH(s, \sigma(s)) = \mathrm{grad}H(s, \sigma(s)) \cdot u_s db(s).$$

Hence upon integrating this last equation from 0 to τ gives:

$$F(\sigma) = (e^{\tau\Delta/2}f)(o) + \int_0^\tau u_s^{-1}\mathrm{grad}H(s, \sigma(s)) \cdot db(s) = E(F) + \int_0^1 a_s \cdot db(s),$$

where $a_s = 1_{s \le \tau} u_s^{-1}\mathrm{grad}H(s, \sigma(s))$. This proves the $n = 1$ case. To finish the proof it suffices to show that we may reduce the assertion of the lemma at the level n to the assertion at the level $n - 1$.

Let $F(\sigma) = f(\sigma(s_1), \ldots, \sigma(s_n))$, where $0 < s_1 < s_2 \cdots < s_n \le 1$. Let

$$(\Delta_n f)(x_1, x_2, \ldots, x_n) = (\Delta g)(x_n)$$

where $g(x) \equiv f(x_1, x_2, \ldots, x_{n-1}, x)$. Similarly, let grad_n denote the gradient acting on the n'th variable of a function $f \in C^\infty(M^n)$. Set

$$H(s, \sigma) \equiv (e^{(s_n - s)\Delta_n/2}f)(\sigma(s_1), \ldots, \sigma(s_{n-1}), \sigma(s))$$

for $s_{n-1} \le s \le s_n$. Again it is easy to show that

$$dH(s,\sigma) = (\mathrm{grad}_n e^{(s_n - s)\Delta_n/2} f)(\sigma(s_1), \ldots, \sigma(s_{n-1}), \sigma(s)) \cdot u_s db(s)$$

for $s_{n-1} \le s \le s_n$. Integrating this last expression from s_{n-1} to s_n yields:

$$\begin{aligned} F(\sigma) &= (e^{(s_n - s_{n-1})\Delta_n/2} f)(\sigma(s_1), \ldots, \sigma(s_{n-1}), \sigma(s_{n-1})) \\ &\quad + \int_{s_{n-1}}^{s_n} (\mathrm{grad}_n e^{(s_n - s)\Delta_n/2} f)(\sigma(s_1), \ldots, \sigma(s_{n-1}), \sigma(s)) \cdot u_s db(s) \\ &= (e^{(s_n - s_{n-1})\Delta_n/2} f)(\sigma(s_1), \ldots, \sigma(s_{n-1}), \sigma(s_{n-1})) + \int_{s_{n-1}}^{s_n} \alpha_s \cdot db(s), \end{aligned}$$

where $\alpha_s \equiv u_s^{-1}(\mathrm{grad}_n e^{(s_n - s)\Delta_n/2} f)(\sigma(s_1), \ldots, \sigma(s_{n-1}), \sigma(s))$ for $s \in (s_{n-1}, s_n)$. By induction we know that the smooth cylinder function

$$(e^{(s_n - s_{n-1})\Delta_n/2} f)(\sigma(s_1), \ldots, \sigma(s_{n-1}), \sigma(s_{n-1}))$$

may be written as a constant plus $\int_0^1 u^{-1} a_s \cdot db(s)$, where a_s is bounded and piecewise continuous and $a_s \equiv 0$ if $s \ge s_{n-1}$. Hence it follows by replacing a_s by $a_s + 1_{(s_{n-1}, s_n)}(s)\alpha_s$ that

$$F(\sigma) = C + \int_0^{s_n} a_s \cdot db(s)$$

for some constant C. By taking expectations of both sides of this equation, it follows that $C = EF(\sigma)$. ∎

Corollary 6.2. *Let $F \in L^2(\mu)$, then there is a predictable process (a) such that $E\int_0^1 |a_s|^2 ds < \infty$, and $F = E(F) + \int_0^1 a_s \cdot db$.*

Proof. Choose a sequence of smooth cylinder functions $\{F_n\}$ such that $F_n \to F$ as $n \to \infty$. By replacing F by $F - EF$ and F_n by $F_n - EF_n$, we may assume that $EF = 0$ and $EF_n = 0$. Let a^n be predictable processes such that $F_n = \int_0^1 a^n \cdot db$. Notice that

$$E \int_0^1 |a_s^n - a_s^m|^2 ds = E(F_n - F_m)^2 \to 0 \text{ as } m, n \to \infty.$$

Hence, if $a \equiv L^2(ds \times d\mu) - \lim_{n\to\infty} a^n$, then

$$F_n = \int_0^1 a^n \cdot db \to \int_0^1 a \cdot db \text{ as } n \to \infty.$$

This show that $F = \int_0^1 a \cdot db$. ∎

Corollary 6.3. *Let F be a smooth cylinder function, then there is a predictable, piecewise continuously differentiable Cameron-Martin vector field X such that $F = E(F) + D^*X$.*

Proof. Just follow the proof of Lemma 5.15 using Lemma 6.1 in place of Corollary 6.2. ∎

7. Comments on References

A rather large number of references are given below. This list is long but by no means complete. Some of the references have been cited in the text above where as most have not. In this section I will make a few miscellaneous remarks about some of the articles listed below. It is left to the reader to glean from the titles the contents of any articles in the References not explicitly mentioned in the text.

7.1. **Articles by Topic.**

(1) **Manifolds and Geometry:** See [1, 17, 22, 34, 37, 64, 68, 77, 86, 87, 88, 89, 90, 114, 122]. The classic texts among these are those by Kobayashi and Nomizu. I also highly recommend [64] and [37]. The books by Klingenberg give an idea of why differential geometers are interested in loop spaces.
(2) **Lie Groups:** There are a vast number of books on Lie groups. Here are two which I have found very useful, [18, 125].
(3) **Stochastic Calculus on Manifolds:** See [21, 23, 24, 49, 50, 51, 55, 83, 95, 105, 111, 116, 117, 118, 126]. The books by Elworthy [51], Emery [55], and Ikeda and Watanabe [83] are highly recommended. Also see the articles by Elworthy [52], Meyer [105], and Norris [111].
(4) **Integration by Parts Formulas:** Many people have now proved some version of integration by parts for path and loop spaces in one context or another, see for example [24, 25, 26, 27, 28, 39, 40, 43, 56, 57, 61, 63, 94, 104, 112, 119, 120, 121]. We have followed Bismut in these notes who proved integration by parts formulas for cylinder functions depending on one time. However, as is pointed out by Leandre and Malliavin and Fang, Bismut's technique works with out any essential change for arbitrary cylinder functions. In [39, 40], the flow associated to a general class of vector fields on paths and loop spaces of a manifold were constructed. Moreover, it was shown that these flows left Wiener measure quasi-invariant. From these facts one can also derive integration by parts formulas.
(5) **Spectral Gap and Logarithmic Sobolev Inequalities:** See [8, 58, 69, 71, 82]. The paper by S. Fang was the first to show that the operator $\mathcal{L}$ defined in Section 5 has a spectral gap. The paper [69] by Gross was the pioneering work on logarithmic Sobolev inequalities. It is shown there that logarithmic Sobolev inequalities hold for Gaussian measure spaces and in particular path and loop spaces on Euclidean spaces. The first proof of a logarithmic Sobolev inequality for paths on a general Riemannian manifold was given by E. Hsu in [82]. Shortly after Aida and Elworthy gave a "non-intrinsic" proof of the same result. The issue of the spectral gap and Logarithmic Sobolev inequalities for general loop spaces is still an open problem. In [71], Gross has prove a Logarithmic Sobolev inequality **with an added potential term** for a special geometry on loop groups. Here Gross uses pinned Wiener measure as the reference measure. In Driver and Lohrenz [47], it is shown that a Logarithmic Sobolev inequality **without** a potential term does hold on the Loop group provided one replace pinned Wiener measure by a "heat kernel" measure. The question as to when or if the potential is needed in Gross's setting for logarithmic Sobolev inequalities is still an open question. It is worth pointing out that the potential term is definitely needed if the group is not simply connected, see [71] for an explanation.

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