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Heat Equation Derivative Formulas for Vector Bundles

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We use martingale methods to give Bismut type derivative formulas for differentials and co-differentials of heat semigroups on forms, and more generally for sections of vector bundles. The formulas are mainly in terms of Weitzenböck curvature terms; in most cases derivatives of the curvature are not involved. In particular, our results improve B. K. Driver's formula in (1997, *J. Math. Pures Appl. (9)* **76**, 703–737) for logarithmic derivatives of the heat kernel measure on a Riemannian manifold. Our formulas also include the formulas of K. D. Elworthy and X.-M. Li (1998, *C. R. Acad. Sci. Paris Sér. I Math.* **327**, 87–92). © 2001 Academic Press

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1. INTRODUCTION

Let M be an n-dimensional oriented Riemannian manifold (not necessarily complete) without boundary and E a smooth Hermitian vector bundle over M. Denote by $\Gamma(E)$ the smooth sections of E. Further assume that L is a second order elliptic differential operator on $\Gamma(E)$ whose principle symbol is the dual of the Riemannian metric on M tensored with the identity section of $\mathrm{Hom}(E)$. In this paper we derive stochastic calculus formulas for $De^{tL}\alpha$ and $e^{tL}D\alpha$ where $\alpha \in \Gamma(E)$ and D is an appropriately chosen first order differential operator on $\Gamma(E)$.

As an example of the kind of formula found in this paper, let us consider one representative special case. Namely suppose that M is a compact spin manifold, $E = S$ is a spinor bundle over M, D is the Dirac operator on $\Gamma(S)$ and $L = -D^2$. Let scal denote the scalar curve of M. Then

$$(e^{-TD^2/2}D\alpha)(x) = (De^{-TD^2/2}\alpha)(x)$$
$$= \frac{1}{T}\mathbb{E}[e^{-(1/8)\int_0^T \mathrm{scal}(X_t(x))\,dt}\gamma_{B_T}//_T^{-1}\alpha(X_T(x))],$$

where $X_t(x)$ is a Brownian motion on M starting at $x \in M$, $//_t$ is stochastic parallel translation along $X_\bullet(x)$ in S relative to the spin connection, B_t is a T_xM-valued Brownian motion associated to $X_t(x)$ and γ_{B_T} is the Clifford multiplication of B_T on S_x. This result is described in more detail in Section 5.2 below.

It is also possible to get a formula for $D^2e^{-TD^2/2}\alpha$ by iterating a minor generalization of the previous formula. For example if $0 < T_1 < T$ then

$$(D^2e^{-TD^2/2}\alpha)(x)$$
$$= \frac{1}{T_1(T-T_1)}\mathbb{E}[e^{-(1/8)\int_0^T \mathrm{scal}(X_t(x))\,dt}\gamma_{B_{T_1}}\gamma_{B_T - B_{T_1}}//_T^{-1}\alpha(X_T(x))];$$

see Theorem 7.4 below.

There are a number of other related approaches to derivative formulas in the literature; see for example Norris [44], Elworthy and Li [24, 26], Stroock and Turetsky [54, 55] and Hsu [34, 35].

Let us end the introduction with a short outline of the paper. Section 2 introduces the basic stochastic and differential geometric notation along with the standing Assumption 1 used throughout the paper. Geometric examples satisfying Assumption 1 are also presented here. More details on the notation and the examples of Section 2 may be found in Appendix A.

Section 3 introduces a number of local martingales associated to the geometric data of Section 2. Theorem 3.7 and Corollary 3.9 of this section are fundamental to the rest of the paper. In Section 4 we describe some general heat equation derivative formulas under the assumption that the local martingales introduced in Section 3 are in fact martingales, see in particular Eqs. (4.22) and (4.23).

Section 5 illustrates our results for compact M. Dirac operators are covered in Section 5.2 and the differential d and co-differential d^* are treated in Section 5.3. Heat equation derivative formulas involving covariant derivatives are covered in Section 5.4, and Section 5.5 in the context of 1-forms. In Section 6 we show how to relax the compactness assumption on M. Some technical heat semi-group properties used in the section are gathered in Appendix B.

Section 7 is devoted to higher derivative formulas. The ideas are illustrated in Theorems 7.4 and 7.7. Theorem 7.4 gives a formula for the square of the Dirac operator on spinor valued solutions to the heat equation while Theorem 7.7 gives a formula for the Hessian of a solution to the scalar heat equation.

2. GENERAL STOCHASTIC AND GEOMETRIC NOTATION

2.1. *Brownian Motion on M*

Let M be an n-dimensional oriented Riemannian manifold (not necessarily complete) without boundary, $(\cdot, \cdot)$ be the Riemannian metric on M, and ∇^{TM} be the Levi–Civita covariant derivative on TM. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space satisfying the usual hypothesis, and for each $x \in M$ let $\{X_t(x): t < \zeta(x)\}$ be an M-valued Brownian motion on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geqslant 0}, \mathbb{P})$, starting from x, with possibly finite lifetime $\zeta(x)$. Recall that $X_t = X_t(x)$ is said to be an M-valued Brownian motion provided that X is a diffusion process such that

$$M_t^f := f(X_t) - f(X_0) - \tfrac{1}{2} \int_0^t \Delta f(X_s)\, ds \qquad \text{on} \quad \{t < \zeta(x)\}$$

is a real local martingale for every $f \in C^\infty(M)$. Here Δf denotes the Riemannian Laplacian of f.

2.2. *Covariant Derivatives and Parallel Translation*

Let $E \to M$ and $\tilde{E} \to M$ be two vector bundles over the Riemannian manifold M. Further assume that E and $\tilde{E}$ are equipped with covariant derivatives, ∇^E and $\nabla^{\tilde{E}}$ respectively. In the case that E and $\tilde{E}$ are Riemannian vector bundles with fiber metrics $(\cdot, \cdot)_E$ and $(\cdot, \cdot)_{\tilde{E}}$ respectively, we will **always** assume that ∇^E and $\nabla^{\tilde{E}}$ are metric compatible covariant derivatives. Given a smooth curve $\sigma\colon [0, \infty[\to M$, let

$$//_t^{TM}(\sigma)\colon T_{\sigma(0)}M \to T_{\sigma(t)}M, \qquad //_t^E(\sigma)\colon E_{\sigma(0)} \to E_{\sigma(t)} \qquad \text{and}$$
$$//_t^{\tilde{E}}(\sigma)\colon \tilde{E}_{\sigma(0)} \to \tilde{E}_{\sigma(t)}$$

denote parallel translation along σ up to time t relative to ∇^{TM}, ∇^E, and $\nabla^{\tilde{E}}$ respectively. The corresponding *stochastic parallel translations* along the Brownian motion $X_{\bullet}(x)$ will simply be denoted by $//_t^{TM}\colon T_xM \to T_{X_t(x)}M$, $//_t^E\colon E_x \to E_{X_t(x)}$ and $//_t^{\tilde{E}}\colon \tilde{E}_x \to \tilde{E}_{X_t(x)}$ respectively.

Notation 2.1. In what follows, let $\langle \cdot, \cdot \rangle$ denote the pairing between a vector space V and its dual space V^*. For a linear operator $C\colon V \to W$, let $C^{\mathrm{tr}}\colon W^* \to V^*$ denote the transpose of C. If V and W are inner product spaces, $C^*\colon W \to V$ will denote the adjoint of C relative to the inner products on V and W. The inner product on V will be denoted by $(\cdot, \cdot)$ or $(\cdot, \cdot)_V$.

The covariant derivatives ∇^{TM}, ∇^E, and $\nabla^{\tilde{E}}$ induce covariant derivatives on any vector bundle $\mathscr{E}$ over M which is constructed by taking tensor products of the bundles TM, E, $\tilde{E}$ and their dual bundles. In order to simplify notation we will simply write ∇ for the induced covariant derivative on any such vector bundle $\mathscr{E}$ and similarly we will write $//_t\colon \mathscr{E}_x \to \mathscr{E}_{X_t(x)}$ for stochastic parallel translation and R for the curvature tensor relative to this covariant derivative.

EXAMPLE 2.2. Suppose that $\mathscr{E} = TM \otimes E^* \otimes \tilde{E}$ and that $v \otimes \alpha \otimes \xi \in \mathscr{E}_x$, then

$$//_t(v \otimes \alpha \otimes \xi) = (//_t^{TM} v) \otimes ((//_t^{E\,\mathrm{tr}})^{-1}\, \alpha) \otimes (//_t^{\tilde{E}} \xi) \tag{2.1}$$

and for $a, b \in T_xM$,

$$R(a, b)(v \otimes \alpha \otimes \xi) = R^{TM}(a, b)\, v \otimes \alpha \otimes \xi$$
$$+ v \otimes (-\alpha \circ R^E(a, b)) \otimes \xi + v \otimes \alpha \otimes R^{\tilde{E}}(a, b)\, \xi,$$

where $(//_t^{E\,\mathrm{tr}})^{-1}\, \alpha := \alpha \circ (//_t^E)^{-1}$ and R^{TM}, R^E, and $R^{\tilde{E}}$ are the curvature tensors for ∇^{TM}, ∇^E and $\nabla^{\tilde{E}}$ respectively. Our convention of denoting parallel translation and the curvature tensor as $//_t$ and R respectively on all bundles associated to TM, E, and $\tilde{E}$ leads to the strange looking identities, $//_t^{\mathrm{tr}} = //_t^{-1}$ and $R^{\mathrm{tr}} = -R$. For example, on $\mathscr{E} = TM \otimes E^* \otimes \tilde{E}$,

$$\begin{aligned} //_t^{\mathrm{tr}} &= (//_t^{TM})^{\mathrm{tr}} \otimes ((//_t^{E\,\mathrm{tr}})^{-1})^{\mathrm{tr}} \otimes (//_t^{\tilde{E}})^{\mathrm{tr}} \\ &= (//_t^{T^*M})^{-1} \otimes (//_t^{E})^{-1} \otimes (//_t^{\tilde{E}^*})^{-1} \end{aligned}$$

which is $//_t^{-1}$ on $\mathscr{E}^* = T^*M \otimes E \otimes \tilde{E}^*$. Similarly, $R^{\mathrm{tr}} = -R$ is a consequence of the fact that, in general, $R^{E^*} = -(R^E)^{\mathrm{tr}}$.

Definition 2.3. The orthogonal frame bundle of M will be denoted by $O(M)$. Given a point $x \in M$, the principle bundle $O(M)$ may be realized as $\bigcup_{m \in M} O_m(M)$, where

$$O_m(M) := \{u\colon T_xM \to T_mM \mid u \text{ is an isometry}\}.$$

(The base point x will be suppressed from the notation since different x's lead to equivalent principle bundles.) Let $\pi\colon O(M) \to M$ be the fiber projection map defined by $\pi(u) = m$ if $u \in O_m(M)$ and let ϑ be the T_xM-valued one-form on $O(M)$ defined by $\vartheta(\xi) = u^{-1}\pi_*\xi$ for all $\xi \in T_uO(M)$.

Definition 2.4 (Brownian Motion on T_xM). Associated to the Brownian motion $X_t(x)$ on M is a T_xM-valued local martingale B_t defined on $[0, \zeta(x)[$ by the Fisk Stratonovich stochastic integral,

$$B_t := \int_0^t //_s^{-1}\,\delta X_s(x) := \int_0^t \vartheta(\delta //_s^{TM}),$$

see [22, 28].

2.3. *Laplacians and First Order Differential Operators*

For a vector bundle E over M, let $\Gamma(E)$ denote the smooth sections of E. In case of a Riemannian vector bundle E over M, let $L^2(E)$ denote the square-integrable sections relative to the Riemannian volume measure on M and L^2-$\Gamma(E)$ the square-integrable smooth sections of E.

Definition 2.5 (Horizontal Laplacians). The horizontal Laplacians $\square\colon \Gamma(E) \to \Gamma(E)$ and $\tilde{\square}\colon \Gamma(\tilde{E}) \to \Gamma(\tilde{E})$ are the second order differential operators given by

$$\square a := \sum_{i=1}^n \nabla^2_{e_i \otimes e_i} a \qquad \text{and} \qquad \tilde{\square}\tilde{a} := \sum_{i=1}^n \nabla^2_{e_i \otimes e_i} \tilde{a},$$

where $\{e_i\}_{i=1}^n$ is any local orthonormal frame of TM and for $a \in \Gamma(E)$

$$\nabla^2_{e_i \otimes e_i} a := (\nabla^E_{e_i})^2\, a - \nabla^E_{\nabla^{TM}_{e_i} e_i} a \tag{2.2}$$

with $\nabla^2_{e_i\otimes e_i}\tilde{a}$ being defined analogously. In other words, $\Box$ is given as the following composition,

$$\Box\colon \Gamma(E) \xrightarrow{\nabla^E} \Gamma(T^*M\otimes E) \xrightarrow{\nabla^{T^*M\otimes E}} \Gamma(T^*M\otimes T^*M\otimes E) \xrightarrow{\mathrm{tr}} \Gamma(E).$$

Definition 2.6 A *multiplication map* from E to $\tilde{E}$ is a smooth section m of the vector bundle $\mathrm{Hom}(T^*M\otimes E, \tilde{E}) \cong TM\otimes E^*\otimes \tilde{E}$. To each multiplication map m we define a first order differential operator D_m: $\Gamma(E)\to\Gamma(\tilde{E})$ by

$$D_m a = m\,\nabla a. \tag{2.3}$$

Notation 2.7. For $v\in T_xM$, let $m_v\in\mathrm{Hom}(E_x, \tilde{E}_x)$ be given by

$$m_v\xi := m((v,\cdot)\otimes\xi)\in\tilde{E}_x$$

for all $\xi\in E_x$. In the following we will typically describe m by describing m_v for $v\in TM$.

Definition 2.8 (Compatibility with ∇). A multiplication map m is said to be *compatible* with ∇ provided $\nabla m = 0$, i.e.

$$\nabla^{\tilde{E}}_v(m_X a) = m_{(\nabla^{TM}_v X)}a + m_X(\nabla^E_v a)$$

for all $X\in\Gamma(TM)$, $a\in\Gamma(E)$, and $v\in TM$.

Since $a\in T^*_xM\otimes E_x \cong \mathrm{Hom}(T_xM, E)$ may be written as $a=\sum_{i=1}^n (e_i,\cdot)\otimes a(e_i)$ where $\{e_i\}_{i=1}^n$ is an orthonormal frame for T_xM, it follows that

$$ma = \sum_{i=1}^n m_{e_i}a(e_i). \tag{2.4}$$

In particular, the operator D_m defined in Eq. (2.3) may be expressed as

$$(D_m a)_x = \sum_{i=1}^n m_{e_i}\nabla_{e_i}a, \tag{2.5}$$

where $a\in\Gamma(E)$ and $\{e_i\}_{i=1}^n$ is an orthonormal frame for T_xM.

Definition 2.9. To each multiplication map m, there is a *dual* multiplication map m^{tr} which is the smooth section of the bundle $\mathrm{Hom}(T^*M\otimes \tilde{E}^*, E^*)$ determined by $m^{\mathrm{tr}}_v = (m_v)^{\mathrm{tr}}$ for all $v\in TM$. If E and $\tilde{E}$ are Riemannian vector bundles, the *adjoint* multiplication map m^* from $\tilde{E}$ to E is determined by $m^*_v := (m_v)^*$ for all $v\in TM$.

Remark 2.10. It is easily checked that if m is compatible with ∇ then so is its dual m^{tr} and its adjoint m^*. Moreover, if m is ∇-compatible, then $//_t^{-1} m = m$, $//_t^{-1} m^{\mathrm{tr}} = m^{\mathrm{tr}}$ and $//_t^{-1} m^* = m^*$. More precisely

$$m_v = (//_t^{\tilde{E}})^{-1}\, m_{//_t^{TM} v}\, //_t^{E},$$
$$m_v^{\mathrm{tr}} = (//_t^{E})^{\mathrm{tr}}\, m^{\mathrm{tr}}_{//_t^{TM} v}\, (//_t^{\tilde{E}\,\mathrm{tr}})^{-1},$$

and

$$m_v^* = (//_t^{E})^{-1}\, m^*_{//_t^{TM} v}\, //_t^{\tilde{E}}$$

for all $v \in T_x M$. Also recall that when ∇^E and $\nabla^{\tilde{E}}$ are metric compatible covariant derivatives, then $(//_t^E)^{-1} = (//_t^E)^*$ and $(//_t^{\tilde{E}})^{-1} = (//_t^{\tilde{E}})^*$.

In geometrically natural situations one is often presented with the following formalism.

Assumption 1. We suppose that m is a multiplication operator and L and $\tilde{L}$ are given second order differential operators on $\Gamma(E)$ and $\Gamma(\tilde{E})$ respectively which satisfy the following conditions.

1. The operators $D := D_m$, L, and $\tilde{L}$ obey the commutation relation:

$$DL = \tilde{L} D - \rho,$$

for some $\rho \in \Gamma(\mathrm{Hom}(E, \tilde{E}))$.

2. The operators $\mathscr{R} := \square - L\colon\ \Gamma(E) \to \Gamma(E)$ and $\tilde{\mathscr{R}} := \tilde{\square} - \tilde{L}$: $\Gamma(\tilde{E}) \to \Gamma(\tilde{E})$ are zeroth order operators, i.e. $\mathscr{R}$ and $\tilde{\mathscr{R}}$ are sections in $\Gamma(\mathrm{End}(E))$ and $\Gamma(\mathrm{End}(\tilde{E}))$ respectively.

2.4. *Examples*

Let us now gives some examples where Assumption 1 is satisfied. More detailed comments about these examples and the general setup may be found in Section A.3 of Appendix A.

EXAMPLE 2.11 (Exterior Bundle Examples). Let $\Lambda T^*M := \bigoplus_{k=0}^n \Lambda^k T^*M$ denote the exterior bundle over M, $\Omega^k(M)$ denote the sections of $\Lambda^k T^*M$ and $\Omega(M) := \bigoplus_{k=0}^n \Omega^k(M)$ be the space of differential forms over M. Let d denote the exterior differential on $\Omega(M)$, d^* be the co-differential and

$$\Delta = -(d + d^*)^2 = -(d^*d + dd^*)$$

be the de Rham–Hodge Laplacian on $\Omega(M)$. We now give three related examples satisfying Assumption 1 with $\rho = 0$.

1. Let $E := \Lambda^k T^*M$, $\tilde{E} := \Lambda^{k+1} T^*M$ and let $m_v = C_v$ be the exterior product or creation operator which is defined by $C_v \alpha := (v, \cdot) \wedge \alpha$ for $v \in T_x M$, $\alpha \in \Lambda^k T_x^* M$ and $x \in M$. Then

$$D_C = d_k = d \mid \Omega^k(M), \qquad L := \Delta_k := \Delta \mid \Omega^k(M),$$
$$\tilde{L} := \Delta_{k+1} = \Delta \mid \Omega^{k+1}(M)$$

satisfy Assumption 1 with $\rho = 0$.

2. Let $E := \Lambda^k T^*M$, $\tilde{E} := \Lambda^{k-1} T^*M$ and $m_v = A_v$ be the interior product or annihilation operator which is defined by $A_v \alpha := \alpha(v, \cdot, ..., \cdot)$ for all $v \in T_x M$, $\alpha \in \Lambda^k T_x^* M$ and $x \in M$. Then

$$D_A = -d_k^* = -d^* \mid \Omega^k(M), \qquad L = \Delta_k = \Delta \mid \Omega^k(M),$$
$$\tilde{L} = \Delta_{k-1} = \Delta \mid \Omega^{k-1}(M)$$

satisfy Assumption 1 with $\rho = 0$.

3. Let $E = \tilde{E} := \Lambda T^*M$, $L = \tilde{L} := \Delta$ and let $m = \gamma$ be the "Clifford" multiplication defined by $\gamma = C - A$. Then

$$D_\gamma = d + d^*, \qquad L = \tilde{L} = \Delta$$

satisfy the Assumption 1 with $\rho = 0$.

Item 3 of the last example generalizes to differential forms with values in a vector bundle. This is described in the next example.

EXAMPLE 2.12 (Vector-valued Forms). Let $S \to M$ be a Euclidean vector bundle over M (with fiber inner product denoted by $(\cdot, \cdot)_S$) equipped with a metric compatible covariant derivative ∇^S. Let $E = \tilde{E} = \Lambda T^*M \otimes S$ and let m be the Clifford multiplication γ determined by $\gamma = C - A$, where as above $C_v a := (v, \cdot) \wedge a$ and $A_v a = a(v, \cdot, ..., \cdot)$ for all $v \in T_x M$, $a \in \Lambda^k T_x^* M \otimes S$ and $x \in M$ (see Section A.1 of Appendix A for our conventions). Then $D_\gamma = d_\nabla + d_\nabla^*$ where d_∇ and d_∇^* are the covariant differential and co-differential on $\mathscr{A}(E) := \Gamma(\Lambda T^*M \otimes S)$. Then $L = \tilde{L} := -D_\gamma^2 = -(d_\nabla + d_\nabla^*)^2$ and γ satisfy Assumption 1 with $\rho = 0$.

EXAMPLE 2.13 (Dirac Operator on a Spin Manifold). Assume now that M is a spin manifold and $S \to M$ a spinor bundle over M. Let ∇^S denote the spin connection on S and let $\tilde{S} = S$. Further let $m_v \alpha = \gamma_v \alpha$ denote the Clifford action of $v \in T_x M$ on $\alpha \in S_x$. Then D_γ is the Dirac operator on $\Gamma(S)$ and $L = \tilde{L} := -D_\gamma^2$ satisfy Assumption 1 with $\rho = 0$ and $\mathscr{R} = \tilde{\mathscr{R}} = \frac{1}{4}$ scal, where scal is the scalar curvature of M. For details on this and the next example, see (for instance) Theorem 3.52 on p. 126 of [4].

EXAMPLE 2.14 (Twisted Dirac Operators). The spinor bundle S is tensored with an auxiliary Riemannian/Hermitian vector bundle F over M to give a Dirac operator of the form

$$D: \Gamma(S \otimes F) \xrightarrow{\nabla^{S \otimes F}} \Gamma(T^*M \otimes S \otimes F) \xrightarrow{\gamma \otimes 1} \Gamma(S \otimes F), \tag{2.6}$$

where as in the previous example γ denotes Clifford multiplication. Then again $L = \tilde{L} = -D^2$ satisfy Assumption 1 with $\rho = 0$.

PROPOSITION 2.15 (∇ on a General Vector Bundle). *Let $E \to M$ be a vector bundle with covariant derivative ∇^E, R^E be the curvature tensor of ∇^E,*

$$\tilde{E} := T^*M \otimes E \cong \mathrm{Hom}(TM, E)$$

and m: $T^*M \otimes E \to \tilde{E} = T^*M \otimes E$ be the identity map considered as a multiplication map from E to $\tilde{E}$. (Notice that $D_m = \nabla^E$.) Given $\mathcal{R} \in \Gamma(\mathrm{End}(E))$ let

$$\tilde{\mathcal{R}} = \mathrm{Ric}^{\mathrm{tr}} \otimes 1_E - 2R^E \cdot + 1_{T^*M} \otimes \mathcal{R} \in \Gamma(\mathrm{End}(\tilde{E})) \tag{2.7}$$

and

$$\rho = \nabla \cdot R^E + \nabla^{\mathrm{End}(E)} \mathcal{R} \in \Gamma(\mathrm{Hom}(E, \tilde{E})), \tag{2.8}$$

where for $\eta \in \tilde{E}_x = T_x^*M \otimes E_x \cong \mathrm{Hom}(T_xM, E_x)$, $v \in T_xM$, $a \in E_x$, and $\{e_i\}$ again an orthogonal frame for T_xM,

$$(R^E \cdot \eta)(v) = \sum_{i=1}^n R^E(v, e_i)\, \eta(e_i), \tag{2.9}$$

$$(\nabla \cdot R^E a)(v) \equiv \sum_{i=1}^n (\nabla_{e_i} R^E)(e_i, v)\, a, \tag{2.10}$$

$$(\nabla^{\mathrm{End}(E)} \mathcal{R} a)(v) = (\nabla_v^{\mathrm{End}(E)} \mathcal{R})\, a, \tag{2.11}$$

and Ric *is the Ricci tensor of M,*

$$\mathrm{Ric}\, v \equiv \sum_{i=1}^n R^{TM}(v, e_i)\, e_i. \tag{2.12}$$

Then $L := \square - \mathcal{R}$, $\tilde{L} = \tilde{\square} - \tilde{\mathcal{R}}$, ρ, and $m = \mathrm{id}$ satisfy Assumption 1.

Proof. By Eq. (A.18) of Appendix A,

$$\begin{aligned}\nabla_v^E La &= \nabla_v^E(\square - \mathcal{R})\, a = \nabla_v^E \square a - (\nabla_v^{\mathrm{End}(E)} \mathcal{R})\, a - \mathcal{R}\, \nabla_v^E a \\ &= (\tilde{\square}\, \nabla^E a)(v) - \nabla_{\mathrm{Ric}\, v} a + 2 \sum_{i=1}^n R^E(v, e_i)\, \nabla_{e_i} a \\ &\quad - (\nabla \cdot R^E a)(v) - (\nabla_v^{\mathrm{End}(E)} \mathcal{R})\, a - \mathcal{R}\, \nabla_v^E a \\ &= (\tilde{L}\, \nabla^E a)(v) - (\rho a)(v). \qquad \blacksquare\end{aligned}$$

2.5. *The Adjoint of D_m*

We will end this section by computing the formal adjoint of D_m when m is compatible with ∇.

LEMMA 2.16. *Suppose that E and $\tilde{E}$ are vector bundles with fiber metrics $(\cdot, \cdot)_E$ and $(\cdot, \cdot)_{\tilde{E}}$ and that ∇^E and $\nabla^{\tilde{E}}$ are metric compatible covariant derivatives on E and $\tilde{E}$ respectively. Also assume that m is a ∇-compatible multiplication map and let m^* be the adjoint multiplication map. Then the operator $D_{m^*}\colon \Gamma(\tilde{E}) \to \Gamma(E)$ is the formal adjoint of $-D_m$. More precisely,*

$$\int_M (D_m S, T)_{\tilde{E}}\, d\,\mathrm{vol} = -\int_M (S, D_{m^*} T)_E\, d\,\mathrm{vol}$$

for all smooth sections $S \in \Gamma(E)$ and $T \in \Gamma(\tilde{E})$ such that $S \otimes T$ has compact support.

Proof. For S and T fixed as above, let X be the compactly supported vector field on M determined by

$$(m_v S, T)_{\tilde{E}} = (X, v)_{TM}$$

for all $v \in TM$. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame on M, then

$$\begin{aligned}e_i(X, e_i)_{TM} &= e_i(m_{e_i} S, T)_{\tilde{E}} = (\nabla_{e_i}(m_{e_i} S), T)_{\tilde{E}} + (m_{e_i} S, \nabla_{e_i} T)_{\tilde{E}} \\ &= (m_{\nabla_{e_i} e_i} S, T)_{\tilde{E}} + (m_{e_i} \nabla_{e_i} S, T)_{\tilde{E}} + (S, m_{e_i}^* \nabla_{e_i} T)_E \\ &= (X, \nabla_{e_i} e_i)_{TM} + (m_{e_i} \nabla_{e_i} S, T)_{\tilde{E}} + (S, m_{e_i}^* \nabla_{e_i} T)_E.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{div}(X) &= \sum_{i=1}^n (\nabla_{e_i} X, e_i)_{TM} = \sum_{i=1}^n [e_i(X, e_i)_{TM} - (X, \nabla_{e_i} e_i)_{TM}] \\ &= \sum_{i=1}^n [(m_{e_i} \nabla_{e_i} S, T)_{\tilde{E}} + (S, m_{e_i}^* \nabla_{e_i} T)_E] \\ &= (D_m S, T)_{\tilde{E}} + (S, D_{m^*} T)_E.\end{aligned}$$

The lemma now follows by integrating this last expression over M and using Stokes' theorem to conclude that

$$\int_M \operatorname{div}(X)\, d\,\mathrm{vol} = \int_M d(A_X \mathrm{vol}) = 0. \quad \blacksquare$$

Remark 2.17. An analogous proof shows that if m is a ∇-compatible multiplication map, then

$$\int_M \langle D_m S, T\rangle \, d\,\mathrm{vol} = -\int_M \langle S, D_{m^{\mathrm{tr}}} T\rangle \, d\,\mathrm{vol}$$

for all smooth sections $S \in \Gamma(E)$ and $T \in \Gamma(\tilde{E}^*)$ such that $S \otimes T$ has compact support.

3. LOCAL MARTINGALES

In this section, suppose that m is a multiplication map and $L = \square - \mathcal{R}$ and $\tilde{L} = \tilde{\square} - \tilde{\mathcal{R}}$ are second order differential operators on $\Gamma(E)$ and $\Gamma(\tilde{E})$ satisfying Assumption 1, i.e. $\rho := \tilde{L} D_m - D_m L \in \Gamma(\mathrm{Hom}(E, \tilde{E}))$. Let $Q_t \in \mathrm{End}(E_x)$ and $\tilde{Q}_t \in \mathrm{End}(\tilde{E}_x)$ denote the solutions to the ordinary differential equations,

$$\frac{d}{dt} Q_t = -\frac{1}{2} Q_t \mathcal{R}_{//_t} \qquad \text{with} \quad Q_0 = \mathrm{id}_{E_x} \tag{3.1}$$

and

$$\frac{d}{dt} \tilde{Q}_t = -\frac{1}{2} \tilde{Q}_t \tilde{\mathcal{R}}_{//_t} \qquad \text{with} \quad \tilde{Q}_0 = \mathrm{id}_{\tilde{E}_x}, \tag{3.2}$$

where $\mathcal{R}_{//_t} := (//_t^E)^{-1} \mathcal{R} \, //_t^E$ and $\tilde{\mathcal{R}}_{//_t} := (//_t^{\tilde{E}})^{-1} \tilde{\mathcal{R}} \, //_t^{\tilde{E}}$ are linear operators on E_x and $\tilde{E}_x$ respectively.

Notation 3.1. A time dependent section $\{a_t\}_{0 \leqslant t < T}$ of E is said to be smooth if $(t, x) \mapsto a_t(x)$ is infinitely differentiable for $(t, x) \in \,]0, T[\, \times M$ with derivatives extending continuously to $[0, T[\, \times M$.

Proposition 3.2. *Let m, L and $\tilde{L}$ be as in Assumption* 1 *and Q and $\tilde{Q}$ be defined by Eqs.* (3.1) *and* (3.2) *respectively. Suppose further that*

$\{a_t\}_{0\leqslant t<T}$ is a smooth time dependent section of E which satisfies the backwards heat equation,

$$\frac{\partial}{\partial t}a_t+\frac{1}{2}La_t=0. \tag{3.3}$$

For t in the stochastic interval $[0,\zeta(x)\wedge T[$, let

$$N_t:=Q_t//_t^{-1}a_t(X_t(x)) \tag{3.4}$$

and

$$\tilde N_t:=\tilde Q_t//_t^{-1}\,Da_t(X_t(x)). \tag{3.5}$$

Then the stochastic differentials of N_t and $\tilde N_t$ are given by

$$dN_t=Q_t//_t^{-1}\,\nabla_{//_t^{TM}\,dB_t}a_t(X_t(x)) \tag{3.6}$$

and

$$d\tilde N_t=\tilde Q_t//_t^{-1}\,\nabla_{//_t\,dB_t}Da_t(X_t(x))+\tfrac{1}{2}\tilde Q_t//_t^{-1}(\rho a_t)(X_t(x))\,dt, \tag{3.7}$$

where $\rho:=\tilde L D_m-D_m L\in\Gamma(\mathrm{Hom}(E,\tilde E))$ as in Assumption 1.

Proof. The proof is an application of Itô's lemma and the commutation relations in Assumption 1. In more detail we have,

$$\begin{aligned}dN_t&=Q_t//_t^{-1}\,\nabla_{//_t\,dB_t}a_t(X_t(x))\\&\quad+\frac{1}{2}\begin{pmatrix}-Q_t\mathscr{R}_{//_t}//_t^{-1}a_t(X_t(x))+Q_t//_t^{-1}\Box a_t(X_t(x))\\-Q_t//_t^{-1}La_t(X_t(x))\end{pmatrix}dt\\&=Q_t//_t^{-1}\,\nabla_{//_t\,dB_t}a_t(X_t(x)),\end{aligned}$$

where in the last equality we used the identity,

$$-\mathscr{R}_{//_t}//_t^{-1}+//_t^{-1}\Box=//_t^{-1}(\Box-\mathscr{R})=//_t^{-1}L.$$

Similarly,

$$\begin{aligned}d\tilde N_t&=\tilde Q_t//_t^{-1}\,\nabla_{//_t\,dB_t}Da_t(X_t(x))\\&\quad+\frac{1}{2}\begin{pmatrix}-\tilde Q_t\tilde{\mathscr{R}}_{//_t}//_t^{-1}\,Da_t(X_t(x))+\tilde Q_t//_t^{-1}\tilde\Box\,Da_t(X_t(x))\\-\tilde Q_t//_t^{-1}\,DLa_t(X_t(x))\end{pmatrix}dt\\&=\tilde Q_t//_t^{-1}\,\nabla_{//_t\,dB_t}Da_t(X_t(x))+\tfrac{1}{2}\tilde Q_t//_t^{-1}(\rho a)(X_t(x))\,dt\end{aligned}$$

because

$$-\tilde{\mathscr{R}}_{//_t}//_t^{-1}\,D+//_t^{-1}\tilde\Box\,D-//_t^{-1}\,DL=//_t^{-1}(\tilde LD-DL)=//_t^{-1}\rho.\quad\blacksquare$$

3.1. *Dual Pairs*

DEFINITION 3.3. A pair of adapted continuous processes $\ell_t \in E_x^*$ and $\tilde{\ell}_t \in \tilde{E}_x^*$ is called a *dual pair* if $Z_t = \langle \tilde{N}_t, \tilde{\ell}_t \rangle - \langle N_t, \ell_t \rangle$ is a local martingale.

THEOREM 3.4. *Suppose that $\ell_t \in E_x^*$ and $\tilde{\ell}_t \in \tilde{E}_x^*$ are continuous semimartingales such that*

$$d\ell_t = \alpha_t \, dB_t + \beta_t \, dt$$

and

$$d\tilde{\ell}_t = \tilde{\alpha}_t \, dB_t + \tilde{\beta}_t \, dt,$$

where $\alpha_t \in \operatorname{Hom}(T_x M, E_x^*)$, $\tilde{\alpha}_t \in \operatorname{Hom}(T_x M, \tilde{E}_x^*)$, $\beta_t \in E_x^*$ and $\tilde{\beta}_t \in \tilde{E}_x^*$ are predictable processes. Then ℓ and $\tilde{\ell}$ is a dual pair provided

$$\tilde{\alpha} \equiv 0, \tag{3.8}$$

$$\tfrac{1}{2}\rho^{\mathrm{tr}} /\!/_t \tilde{Q}_t^{\mathrm{tr}} \tilde{\ell}_t = /\!/_t Q_t^{\mathrm{tr}} \beta_t, \qquad \text{and} \tag{3.9}$$

$$m^{\mathrm{tr}}_{/\!/_t v} /\!/_t \tilde{Q}_t^{\mathrm{tr}} \tilde{\beta}_t = /\!/_t Q_t^{\mathrm{tr}} \alpha_t v \qquad \text{for each} \quad v \in T_x M. \tag{3.10}$$

Proof. Let $Z_t = \langle \tilde{N}_t, \tilde{\ell}_t \rangle - \langle N_t, \ell_t \rangle$ and write $dX \simeq dY$ if $X - Y$ is a local martingale. Computing dZ_t using Proposition 3.2 gives

$$\begin{aligned}
dZ_t &\simeq \langle d\tilde{N}_t, \tilde{\ell} \rangle + \langle \tilde{N}_t, d\tilde{\ell}_t \rangle + \langle d\tilde{N}_t, d\tilde{\ell}_t \rangle - \langle N_t, d\ell_t \rangle - \langle dN_t, d\ell_t \rangle \\
&\simeq \langle \tfrac{1}{2} \tilde{Q}_t /\!/_t^{-1} (\rho a_t)(X_t(x)), \tilde{\ell}_t \rangle \, dt \\
&\quad + \langle \tilde{N}_t, \tilde{\beta}_t \rangle \, dt + \langle \tilde{Q}_t /\!/_t^{-1} \nabla_{/\!/_t dB_t} Da_t(X_t(x)), \tilde{\alpha}_t \, dB_t \rangle \\
&\quad - \langle N_t, \beta_t \rangle \, dt - \langle Q_t /\!/_t^{-1} \nabla_{/\!/_t dB_t} a_t(X_t, (x)), \alpha_t \, dB_t \rangle \\
&= \tfrac{1}{2} \langle \tilde{Q}_t /\!/_t^{-1} (\rho a_t)(X_t(x)), \tilde{\ell}_t \rangle \, dt \\
&\quad + \sum_{i=1}^{n} \langle \tilde{Q}_t /\!/_t^{-1} m_{/\!/_t e_i} \nabla_{/\!/_t e_i} a_t(X_t(x)), \tilde{\beta}_t \rangle \, dt \\
&\quad + \sum_{i=1}^{n} \langle \tilde{Q}_t /\!/_t^{-1} \nabla_{/\!/_t e_i} Da_t(X_t(x)), \tilde{\alpha}_t e_i \rangle \, dt \\
&\quad - \langle Q_t /\!/_t^{-1} a_t(X_t(x)), \beta_t \rangle \, dt \\
&\quad - \sum_{i=1}^{n} \langle Q_t /\!/_t^{-1} \nabla_{/\!/_t e_i} a_t(X_t(x)), \alpha_t e_i \rangle \, dt.
\end{aligned}$$

Keeping in mind that $//_t = (//_t^{-1})^{\mathrm{tr}}$ (i.e. $//_t^{E^*} = ((//_t^{E})^{-1})^{\mathrm{tr}}$) we find, by comparing terms involving a, $\nabla_{//_t e_i} a$, and $\nabla_{//_t e_i} Da$, that $dZ_t \simeq 0$ provided that Eqs. (3.8), (3.9) and (3.10) are satisfied. ∎

The upshot of the previous theorem is that to make a dual pair we should choose $\tilde{\alpha} = 0$ (i.e. $d\tilde{\ell}_t = \tilde{\beta}_t\, dt$),

$$\beta_t = \tfrac{1}{2}(Q_t^{\mathrm{tr}})^{-1} //_t^{-1} \rho^{\mathrm{tr}} //_t \tilde{Q}_t^{\mathrm{tr}} \tilde{\ell}_t \qquad \text{and}$$
$$\alpha_t v = (Q_t^{\mathrm{tr}})^1 //_t^{-1} m_{//_t v}^{\mathrm{tr}} //_t \tilde{Q}_t^{\mathrm{tr}} \tilde{\beta}_t \qquad \text{for each} \quad v \in T_x M.$$

Because the processes Q_t^{tr} and $\tilde{Q}_t^{\mathrm{tr}}$ will arise often in the sequel, it is convenient to introduce the following notation.

Notation 3.5. Let $\mathcal{Q}_t := Q_t^{\mathrm{tr}} \in \mathrm{End}(E_x^*)$ and $\tilde{\mathcal{Q}}_t := \tilde{Q}_t^{\mathrm{tr}} \in \mathrm{End}(\tilde{E}_x^*)$. Taking the transposes of Eqs. (3.1) and (3.2) shows that $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ solve the following ordinary differential equations:

$$\frac{d}{dt}\mathcal{Q}_t = -\frac{1}{2}\mathcal{R}_{//_t}^{\mathrm{tr}} \mathcal{Q}_t \qquad \text{with} \quad \mathcal{Q}_0 = \mathrm{id}_{E_x^*} \tag{3.11}$$

and

$$\frac{d}{dt}\tilde{\mathcal{Q}}_t = -\frac{1}{2}\tilde{\mathcal{R}}_{//_t}^{\mathrm{tr}} \tilde{\mathcal{Q}}_t \qquad \text{with} \quad \tilde{\mathcal{Q}}_0 = \mathrm{id}_{\tilde{E}_x^*}. \tag{3.12}$$

DEFINITION 3.6 (Finite Energy Process). Let V be a finite dimensional vector space. A V-valued process $\{\ell_s\}_{s \in [0, T[}$ is said to be a *finite energy process* provided ℓ is adapted and (on a set of measure one) $s \to \ell_s$ is absolutely continuous and $\int_0^T |d\ell_s/ds|_V^2\, ds < \infty$, where $|\cdot|_V$ denotes any one of the equivalent norms on V. If in addition there is a $p \in [1, \infty)$ such that

$$\mathbb{E}\left[\left(\int_0^T |d\ell_s/ds|_V^2\, ds\right)^{p/2}\right] < \infty,$$

then we say that $\{\ell_s\}_{s \in [0, T[}$ is an *L^p-finite energy process.*

We have the following immediate consequence of Theorem 3.4. In this theorem and in the rest of the paper we will write ℓ rather than $\tilde{\ell}$ for a finite energy process with values in $\tilde{E}_x^*$.

THEOREM 3.7. *Let a, N, and $\tilde{N}$ be as in Proposition 3.2, $\ell_t \in \tilde{E}_x^*$ be a finite energy process and define the E_x^*-valued process,*

$$U_t^\ell := \int_0^t \mathcal{Q}_s^{-1} //_s^{-1} m_{//_s dB_s}^{\mathrm{tr}} //_s \tilde{\mathcal{Q}}_s \ell'_s + \tfrac{1}{2}\int_0^t \mathcal{Q}_s^{-1} \rho_{//_s}^{\mathrm{tr}} \tilde{\mathcal{Q}}_s \ell_s\, ds, \tag{3.13}$$

where $\rho_{//_s} := //_s^{-1} \rho(X_s(x)) //_s$ *and*

$$\rho_{//_s}^{\mathrm{tr}} = //_s^{\mathrm{tr}} \rho^{\mathrm{tr}}(X_s(x))(//_s^{-1})^{\mathrm{tr}} = //_s^{-1} \rho^{\mathrm{tr}}(X_s(x)) //_s. \tag{3.14}$$

(*See the remarks at the end of Example* 2.2 *explaining the identity* $//_s^{\mathrm{tr}} = //_s^{-1}$.) *Then*

$$Z_t^\ell := \langle \tilde{N}_t, \ell_t \rangle - \langle N_t, U_t^\ell \rangle \tag{3.15}$$

is a local martingale on $[0, \zeta(x) \wedge T[$ *and*

$$\begin{aligned} dZ_t^\ell &= \langle \tilde{Q}_t //_t^{-1} \nabla_{//_t dB_t} Da_t(X_t(x)), \ell_t \rangle \\ &\quad - \langle Q_t //_t^{-1} \nabla_{//_t dB_t} a_t(X_t(x)), U_t^\ell \rangle \\ &\quad - \langle N_t, \mathcal{Q}_t^{-1} //_t^{-1} m_{//_t dB_t}^{\mathrm{tr}} //_t \tilde{\mathcal{Q}}_t \ell'_t \rangle. \end{aligned} \tag{3.16}$$

Proof. Because of Theorem 3.4 and the comments after its proof, we need only prove Eq. (3.16). Since we already know that Z_t^ℓ is a local martingale, we need only keep track of those terms in dZ_t^ℓ which depend linearly on dB_t. Hence Eq. (3.16) follows from the identity

$$\begin{aligned} dZ_t^\ell &= \langle d\tilde{N}_t, \ell_t \rangle + \langle \tilde{N}_t, d\ell_t \rangle + \langle d\tilde{N}_t, d\ell_t \rangle \\ &\quad - \langle dN_t, U_t^\ell \rangle - \langle N_t, dU_t^\ell \rangle - \langle dN_t, dU_t^\ell \rangle \end{aligned}$$

and Eqs. (3.6), (3.7) and (3.13). ∎

By Remark 2.10, if the multiplication operator m is compatible with ∇, then

$$m_v^{\mathrm{tr}} = //_s^{\mathrm{tr}} m_{//_s v}^{\mathrm{tr}} (//_s^{-1})^{\mathrm{tr}} = //_s^{-1} m_{//_s v}^{\mathrm{tr}} //_s$$

for all $v \in T_x M$. Hence, the formula for U_t^ℓ in Eq. (3.13) simplifies to

$$U_t^\ell = \int_0^t \mathcal{Q}_s^{-1} m_{dB_s}^{\mathrm{tr}} \tilde{\mathcal{Q}}_s \ell'_s + \tfrac{1}{2} \int_0^t \mathcal{Q}_s^{-1} \rho_{//_s}^{\mathrm{tr}} \tilde{\mathcal{Q}}_s \ell_s \, ds. \tag{3.17}$$

Remark 3.8. Suppose that $\ell_t \in \tilde{E}_x$ is a finite energy process. Under the further assumption that E and $\tilde{E}$ are Riemannian vector bundles and m is compatible with ∇, the results of Theorem 3.7 remains true with U_t^ℓ and Z_t^ℓ defined by

$$U_t^\ell := \int_0^t (Q_s^{-1})^* \, m_{dB_s}^* \tilde{Q}_s^* \ell'_s + \tfrac{1}{2} \int_0^t (Q_s^{-1})^* \, \rho_{//_s}^* \tilde{Q}_s^* \ell_s \, ds \tag{3.18}$$

and

$$Z_t^{\ell} := (\tilde{N}_t, \ell_t)_{\tilde{E}} - (N_t, U_t^{\ell})_E, \tag{3.19}$$

where $\rho^*_{//_s} := //_s^{-1} \rho^*(X_s(x)) //_s$.

We will finish this section with another equivalent version of Theorem 3.7.

Corollary 3.9. *Suppose that m is compatible with* ∇ *(for simplicity),* k_t *is an* $\tilde{E}_x^*$*-valued finite energy process and a and N are as in Proposition* 3.2. *Also define*:

$$\hat{N}_t := //_t^{-1} Da_t(X_t(x)), \tag{3.20}$$

$$\hat{U}_t^k := \int_0^t \mathcal{Q}_s^{-1} m^{\mathrm{tr}}_{dB_s}(k'_s + \tfrac{1}{2}\tilde{\mathcal{R}}^{\mathrm{tr}}_{//_s} k_s) + \tfrac{1}{2}\int_0^t \mathcal{Q}_s^{-1} \rho^{\mathrm{tr}}_{//_s} k_s \, ds \tag{3.21}$$

and

$$\hat{Z}_t^k := \langle \hat{N}_t, k_t \rangle - \langle N_t, \hat{U}_t^k \rangle. \tag{3.22}$$

Then $\hat{Z}_t^k$ *is a local martingale on* $[0, \zeta(x) \wedge T[$ *whose Itô differential is given by*

$$\begin{aligned} d\hat{Z}_t^k &= \langle //_t^{-1} \nabla_{//_t dB_t} Da_t(X_t(x)), k_t \rangle \\ &\quad - \langle Q_t //_t^{-1} \nabla_{//_t dB_t} a_t(X_t(x)), \hat{U}_t^k \rangle \\ &\quad - \langle Q_t^{-1} N_t, m^{\mathrm{tr}}_{dB_t}(k'_t + \tfrac{1}{2}\tilde{\mathcal{R}}^{\mathrm{tr}}_{//_t} k_t) \rangle. \end{aligned} \tag{3.23}$$

Proof. Comparing Eqs. (3.5) and (3.20) shows that $\tilde{N}_t = \tilde{Q}_t \hat{N}_t$. Define the finite energy process ℓ_s by $\ell_s := (\tilde{\mathcal{Q}}_s)^{-1} k_s$ so that $k_s = \tilde{\mathcal{Q}}_s \ell_s$. Because

$$k'_s = \tilde{\mathcal{Q}}_s \ell'_s - \tfrac{1}{2}\tilde{\mathcal{R}}^{\mathrm{tr}}_{//_s} \tilde{\mathcal{Q}}_s \ell_s = \tilde{\mathcal{Q}}_s \ell'_s - \tfrac{1}{2}\tilde{\mathcal{R}}^{\mathrm{tr}}_{//_s} k_s, \tag{3.24}$$

it follows from Eq. (3.17) that $\hat{U}_t^k := U_t^{\ell}$. This identity and

$$\langle \tilde{N}_t, \ell_t \rangle = \langle \tilde{Q}_t \hat{N}_t, (\tilde{\mathcal{Q}}_t)^{-1} k_t \rangle = \langle \hat{N}_t, k_t \rangle$$

shows that $\hat{Z}_t^k = Z_t^{\ell}$ as well. By Theorem 3.7 $\hat{Z}_t^k = Z_t^{\ell}$ is a local martingale on $[0, \zeta(x) \wedge T[$ and by Eq. (3.16) and Eq. (3.24) the differential of $\hat{Z}_t^k$ is given by

$$\begin{aligned} d\hat{Z}_t^k &= \langle \tilde{Q}_t //_t^{-1} \nabla_{//_t dB_t} Da_t(X_t(x)), (\tilde{\mathcal{Q}}_t)^{-1} k_t \rangle \\ &\quad - \langle Q_t //_t^{-1} \nabla_{//_t dB_t} a_t(X_t(x)), \hat{U}_t^k \rangle \\ &\quad - \langle N_t, \mathcal{Q}_t^{-1} m^{\mathrm{tr}}_{dB_t}(k'_t + \tfrac{1}{2}\tilde{\mathcal{R}}^{\mathrm{tr}}_{//_t} k_t) \rangle \end{aligned}$$

which is the same as Eq. (3.23). ∎

4. THE FUNDAMENTAL DERIVATIVE FORMULAS

In this section, we will give a number of derivative formulas under the assumption that the local martingales in Proposition 3.2 and Theorem 3.7 are in fact martingales. In the later sections we will verify this hypothesis in a number of cases. To simplify notation, we will from now on assume that the multiplication map m is compatible with the covariant derivative ∇. Thus the assumptions which are in force in the remainder of the paper are:

Assumption 2 (Standing Assumptions). Let E and $\tilde{E}$ be vector bundles endowed with covariant derivatives. Assume that L, $\tilde{L}$, and m are given satisfying Assumption 1 and that m is compatible with the covariant derivatives.

All of our examples in Section 3 satisfy this assumption. The following theorem contains the basic derivative formulas in this paper.

THEOREM 4.1 (Basic Derivative Formula I). *Let a be a solution to Eq. (3.3) and Q and $\tilde{Q}$ be given by (3.1) and (3.2). Also let τ be a stopping time bounded by $T<\infty$ such that $\tau<\zeta(x)$ and let $\ell_t \in \tilde{E}_x^*$ be a finite energy process on the stochastic interval $[0,\tau]$. Assume that τ and ℓ have been chosen such that*

$$\mathbb{E}\,|\langle \tilde{Q}_\tau //_\tau^{-1} Da_\tau(X_\tau(x)), \ell_\tau\rangle| < \infty, \quad \mathbb{E}\,|\langle Q_\tau //_\tau^{-1} a_\tau(X_\tau(x)), U_\tau^\ell\rangle| < \infty,$$

where U^ℓ is defined in Eq. (3.17). Further assume $(Z^\ell)_t^\tau := Z_{t\wedge\tau}^\ell$ is a martingale where Z_t^ℓ is the local martingale in Eq. (3.15). Then

$$\mathbb{E}[\langle Da_0(x), \ell_0\rangle] = \mathbb{E}[\langle \tilde{Q}_\tau //_\tau^{-1} Da_\tau(X_\tau(x)), \ell_\tau\rangle] - \mathbb{E}[\langle Q_\tau //_\tau^{-1} a_\tau(X_\tau(x)), U_\tau^\ell\rangle]. \tag{4.1}$$

Therefore,

1. *if $\ell_\tau = 0$ and $\ell_0 = \xi \in \tilde{E}_x^*$ then*

$$\langle Da_0(x), \xi\rangle = -\mathbb{E}[\langle Q_\tau //_\tau^{-1} a_\tau(X_\tau(x)), U_\tau^\ell\rangle], \tag{4.2}$$

2. *or if $\ell_0 = 0$ and $\ell_\tau = \xi \in \tilde{E}_x^*$ (where ξ may be random here) then*

$$\mathbb{E}[\langle \tilde{Q}_\tau //_\tau^{-1} Da_\tau(X_\tau(x)), \xi\rangle] = \mathbb{E}[\langle Q_\tau //_\tau^{-1} a_\tau(X_\tau(x)), U_\tau^\ell\rangle]. \tag{4.3}$$

Proof. Since we have assumed that the expression $(Z^\ell)^\tau$ in Eq. (3.15) of Theorem 3.7 is a martingale, it follows that $\mathbb{E}Z_0^\ell = \mathbb{E}(Z^\ell)_T^\tau = \mathbb{E}Z_\tau^\ell$, i.e.

$$\mathbb{E}\langle \tilde{N}_\tau, \ell_\tau\rangle - \mathbb{E}\langle \tilde{N}_0, \ell_0\rangle = \mathbb{E}\langle N_\tau, U_\tau^\ell\rangle$$

which along with the definitions of N, $\tilde{N}$ in Proposition 3.2 proves Eq. (4.1). ∎

Remark 4.2. Suppose that $\ell_t \in \tilde{E}_x$ is a finite energy process. Under the further assumption that E and $\tilde{E}$ are Riemannian vector bundles and m is compatible with ∇, Theorem 4.1 remains valid with all pairings $\langle \cdot, \cdot \rangle$ replaced by the appropriate inner products $(\cdot, \cdot)_E$ or $(\cdot, \cdot)_{\tilde{E}}$ and all transposes replaced by adjoints and U^ℓ defined as in Eq. (3.18) above.

THEOREM 4.3 (Basic Derivative Formula II). *Let a be a solution to Eq.* (3.3) *and Q be given by* (3.1). *Also let τ be a stopping time bounded by $T < \infty$ such that $\tau < \zeta(x)$ and let $k_t \in \tilde{E}_x^*$ be a finite energy process on the stochastic interval $[0, \tau]$. Assume that τ and k have been chosen such that*

$$\mathbb{E}\,|\langle //_\tau^{-1}\, Da_\tau(X_\tau(x)), k_\tau \rangle| < \infty, \qquad \mathbb{E}\,|\langle Q_\tau //_\tau^{-1} a_\tau(X_\tau(x)), \hat{U}_\tau^k \rangle| < \infty,$$

where $\hat{U}^k$ is defined in Eq. (3.21). *Further assume $(\hat{Z}^k)_t^\tau := \hat{Z}^k_{t \wedge \tau}$ is a martingale where $\hat{Z}_t^k$ is the local martingale in Eq.* (3.22). *Then*

$$\mathbb{E}[\langle Da_0(x), k_0 \rangle] = \mathbb{E}[\langle //_\tau^{-1}\, Da_\tau(X_\tau(x)), k_\tau \rangle] - \mathbb{E}[\langle Q_\tau //_\tau^{-1} a_\tau(X_\tau(x)), \hat{U}_\tau^k \rangle]. \tag{4.4}$$

Therefore,

1. *if $k_\tau = 0$ and $k_0 = \xi \in \tilde{E}_x^*$ then*

$$\langle Da_0(x), \xi \rangle = -\mathbb{E}[\langle Q_\tau //_\tau^{-1} a_\tau(X_\tau(x)), \hat{U}_\tau^k \rangle], \tag{4.5}$$

2. *or if $k_0 = 0$ and $k_\tau = \xi \in \tilde{E}_x^*$ (where ξ may be random here) then*

$$\mathbb{E}[\langle //_\tau^{-1}\, Da_\tau(X_\tau(x)), \xi \rangle] = \mathbb{E}[\langle Q_\tau //_\tau^{-1} a_\tau(X_\tau(x)), \hat{U}_\tau^k \rangle]. \tag{4.6}$$

Proof. The proof is the same as Theorem 4.1 except that we use Corollary 3.9 in place of Theorem 3.7. ∎

The formulas appearing in Theorem 3.7 take on a simpler form when m is compatible with ∇ and ℓ_t is of the form $\ell_t = \bar{\ell}_t \xi$ where $\bar{\ell}_t$ is an $\mathbb{R}$-valued finite energy process and $\xi \in \tilde{E}^*$. To write out these formula, it is convenient to introduce the process $V_t^{\bar{\ell}} \in \mathrm{Hom}(E_x, \tilde{E}_x)$ by

$$V_t^{\bar{\ell}} = \int_0^t \bar{\ell}'_s \tilde{Q}_s m_{dB_s} Q_s^{-1} + \tfrac{1}{2} \int_0^t \bar{\ell}_s \tilde{Q}_s \rho_{//_s} Q_s^{-1}\, ds, \tag{4.7}$$

where $\rho_{//_s} := //_s^{-1} \rho(X_s(x)) //_s$ and the $\tilde{E}_x$-valued process

$$\bar{Z}_t^{\bar{\ell}} := \bar{\ell}_t \tilde{N}_t - V_t^{\bar{\ell}} N_t. \tag{4.8}$$

Corollary 4.4. *Let a be a solution to Eq.* (3.3) *and Q and $\tilde{Q}$ be given by* (3.1) *and* (3.2). *Also let τ be a stopping time bounded by $T<\infty$ such that $\tau<\zeta(x)$ and let $\bar{\ell}_t$ be an $\mathbb{R}$-valued finite energy process on the stochastic interval $[0,\tau]$. Assume that τ and $\bar{\ell}$ have been chosen such that*

$$\mathbb{E}\,|\bar{\ell}_\tau \tilde{Q}_\tau //_\tau^{-1} Da_\tau(X_\tau(x))|<\infty, \qquad \mathbb{E}\,|V_\tau^{\bar{\ell}} Q_\tau //_\tau^{-1} a_\tau(X_\tau(x))|<\infty,$$

where $V^{\bar{\ell}}$ is defined in Eq. (4.7). *Further assume $(\bar{Z}^{\bar{\ell}})_t^\tau := \bar{Z}_{t\wedge\tau}^{\bar{\ell}}$ is a martingale where $\bar{Z}_t^{\bar{\ell}}$ is the local martingale in Eq.* (4.8). *Then*

$$Da_0(x)=\mathbb{E}[\bar{\ell}_\tau \tilde{Q}_\tau //_\tau^{-1} Da_\tau(X_\tau(x))]-\mathbb{E}[V_\tau^{\bar{\ell}} Q_\tau //_\tau^{-1} a_\tau(X_\tau(x))]. \tag{4.9}$$

Moreover,

1. *if $\bar{\ell}_\tau=0$ and $\bar{\ell}_0=1$ then*

$$Da_0(x)=-\mathbb{E}[V_\tau^{\bar{\ell}} Q_\tau //_\tau^{-1} a_\tau(X_\tau(x))], \tag{4.10}$$

2. *or if $\bar{\ell}_0=0$ and $\bar{\ell}_\tau=1$ then*

$$\mathbb{E}[\tilde{Q}_\tau //_\tau^{-1} Da_\tau(X_\tau(x))]=\mathbb{E}[V_\tau^{\bar{\ell}} Q_\tau //_\tau^{-1} a_\tau(X_\tau(x))]. \tag{4.11}$$

In the remainder of Section 4, we will illustrate the type of formulas that can be derived from Theorem 4.1. For now we will just assume that the hypothesis of Theorem 4.1 or Corollary 4.4 are satisfied. The subsequent sections will address the issue of verifying these hypotheses in a number of different contexts.

In order to apply Theorem 4.1, it is necessary to have a solution a to Eq. (3.3). There are basically two choices which will be used in the sequel: (i) $a_t=e^{(T-t)L/2}\alpha$ where $\alpha\in\Gamma(E)$ and (ii) $a_t=\alpha$ where $L\alpha=0$. The second case is formally a special case of the first.

4.1. *Feynman–Kac Formulas*

For this subsection, we will assume that the lifetime $\xi(x)$ of the Brownian motion is infinite. We begin by recalling the Feynman–Kac representation for e^{tL} and $e^{t\tilde{L}}$.

Proposition 4.5 (Feynman–Kac). *Suppose that α and $\tilde{\alpha}$ are smooth sections of E and $\tilde{E}$ such that there exist smooth solutions to the partial differential equations*

$$\frac{\partial}{\partial t}u(t)=\frac{1}{2}Lu(t) \qquad \textit{with} \quad u(0)=\alpha \tag{4.12}$$

and

$$\frac{\partial}{\partial t}\tilde{u}(t)=\frac{1}{2}\tilde{L}\tilde{u}(t) \qquad \text{with} \quad \tilde{u}(0)=\tilde{\alpha}. \tag{4.13}$$

Further assume that there is a solution to Eq. (4.13) *when* $\tilde{\alpha}=D\alpha$. *We will write* $e^{tL/2}\alpha$ *for* $u(t)$, *respectively* $e^{t\tilde{L}/2}\tilde{\alpha}$ *for* $\tilde{u}(t)$, *and* $e^{t\tilde{L}/2}\,D\alpha$ *for the solution to Eq.* (4.13) *when* $\tilde{\alpha}=D\alpha$. *Let* $a_t=u(T-t)=e^{(T-t)\,L/2}\alpha$, *respectively,* $\tilde{a}_t=\tilde{u}(T-t)=e^{(T-t)\,\tilde{L}/2}\tilde{\alpha}$, *and let* N_t *and* $\tilde{N}_t$ *be defined by Eqs.* (3.4) *and* (3.5). *If the local martingales* N_t *and*

$$t\to\tilde{Q}_t//_t^{-1}\tilde{a}_t(X_t(x))$$

are in fact martingales for $0\leqslant t\leqslant T$, *then*

$$(e^{TL/2}\alpha)(x)=\mathbb{E}[Q_T//_T^{-1}\alpha(X_T(x))], \tag{4.14}$$

$$(e^{T\tilde{L}/2}\tilde{\alpha})(x)=\mathbb{E}[\tilde{Q}_T//_T^{-1}\tilde{\alpha}(X_T(x))]. \tag{4.15}$$

Under the further assumptions,

$$\text{(i)} \quad \int_0^T \mathbb{E}\,|\tilde{Q}_t\rho_{//_t}Q_t^{-1}N_T|\,dt<\infty \tag{4.16}$$

(*with* $\rho_{//_t}$ *as in Eq.* (3.14)) *and* (ii) *the local martingale*

$$t\to\int_0^t \tilde{Q}_r//_r^{-1}\,\nabla_{//_r dB_r}Da_r(X_r(x)) \tag{4.17}$$

is a martingale on $[0, T]$, *then*

$$\begin{aligned}(e^{T\tilde{L}/2}\,D\alpha)(x)&=(De^{TL/2}\alpha)(x)\\&\quad+\tfrac{1}{2}\mathbb{E}\left[\left(\int_0^T \tilde{Q}_t\rho_{//_t}Q_t^{-1}\,dt\right)Q_T//_T^{-1}\alpha(X_T(x))\right].\end{aligned} \tag{4.18}$$

Remarks 4.6. (i) Equation (4.18) is a stochastic version of Duhamel's principle.

(ii) The hypotheses of the previous proposition are easily verified when M is compact, see Proposition 5.1 below.

Proof (of Proposition 4.5). Since N_t is assumed to be a martingale,

$$(e^{TL/2}\alpha)(x)=\mathbb{E}[N_0]=\mathbb{E}[N_T]=\mathbb{E}[Q_T//_T^{-1}\alpha(X_T(x))]$$

which proves Eq. (4.14). The proof of Eq. (4.15) is similar and so we omit it.

Taking expectations of Eq. (3.7) of Proposition 3.2 and using Eq. (4.15) with $\tilde{\alpha} = D\alpha$, we find that

$$\begin{aligned}
(e^{T\tilde{L}/2} D\alpha)(x) &= \mathbb{E}[\tilde{Q}_T //_T^{-1} D\alpha(X_T(x))] = \mathbb{E}[\tilde{N}_T] \\
&= \mathbb{E}[\tilde{N}_0] + \tfrac{1}{2}\mathbb{E}\int_0^T \tilde{Q}_t //_t^{-1}(\rho a_t)(X_t(x))\,dt \\
&= (De^{TL/2}\alpha)(x) \\
&\quad + \tfrac{1}{2}\mathbb{E}\int_0^T \tilde{Q}_t //_t^{-1}\rho(X_t(x)) //_t Q_t^{-1} Q_t //_t^{-1} a_t(X_t(x))\,dt \\
&= (De^{TL/2}\alpha)(x) + \tfrac{1}{2}\mathbb{E}\int_0^T \tilde{Q}_t \rho_{//_t} Q_t^{-1} N_t\,dt. \qquad (4.19)
\end{aligned}$$

Note that Eq. (4.16), along with the martingale property of N_t, implies as well that

$$\int_0^T \mathbb{E}\,|\tilde{Q}_t \rho_{//_t} Q_t^{-1} N_t|\,dt < \infty. \qquad (4.20)$$

To simplify notation, let $\Gamma_t := \tilde{Q}_t \rho_{//_t} Q_t^{-1}$. From the bounds in Eqs. (4.16) and (4.20) and Fubini's Theorem, it follows that

$$\mathbb{E}\,|\Gamma_t N_T| < \infty \qquad \text{and} \qquad \mathbb{E}\,|\Gamma_t N_t| < \infty \qquad (4.21)$$

for almost every $t \in [0, T]$. At any t where Eq. (4.21) holds, one shows using the martingale property of N_t that $\mathbb{E}[\Gamma_t N_t] = \mathbb{E}[\Gamma_t N_T]$. (This is done by first truncating Γ_t and then passing to the limit.) Since $\mathbb{E}[\Gamma_t N_t] = \mathbb{E}[\Gamma_t N_T]$ for almost every t, we find that

$$\mathbb{E}\int_0^T \tilde{Q}_t \rho_{//_t} Q_t^{-1} N_t\,dt = \int_0^T \mathbb{E}[\Gamma_t N_t]\,dt = \int_0^T \mathbb{E}[\Gamma_t N_T]\,dt,$$

which along with Eq. (4.19) implies Eq. (4.18) since

$$N_T = Q_T //_T^{-1} a_T(X_T(x)) = Q_T //_T^{-1} \alpha(X_T(x)). \quad \blacksquare$$

4.2. *Semigroup Derivative Formulas*

Let $a_t = e^{(T-t)\,L/2}\alpha$ where $\alpha \in \Gamma(E)$ and $e^{tL/2}$ is the semigroup generated by a suitable extension of L. Taking $\tau = T$ in Theorem 4.1, we find the following derivative formula from Eq. (4.2),

$$\langle De^{TL/2}\alpha(x), \xi \rangle = -\mathbb{E}[\langle Q_T //_T^{-1}\alpha(X_T(x)), U_T^{\ell} \rangle] \qquad \text{for any} \quad \xi \in \tilde{E}_x^* \qquad (4.22)$$

where ℓ is an appropriate finite energy process with values in $\tilde{E}_x^*$ such that $\ell_T = 0$ and $\ell_0 = \xi$. Similarly, using Proposition 4.5 and Eq. (4.3),

$$\begin{aligned}\langle e^{T\tilde{L}/2} D\alpha(x), \xi\rangle &= \mathbb{E}[\langle \tilde{Q}_T //_T^{-1} D\alpha(X_T(x)), \xi\rangle] \\ &= \mathbb{E}[\langle Q_T //_T^{-1} \alpha(X_T(x)), U_T^\ell\rangle] \\ &\qquad \text{for any} \quad \xi \in \tilde{E}_x^*, \end{aligned} \tag{4.23}$$

where now $\ell_T = \xi$ and $\ell_0 = 0$.

Remark 4.7. It is interesting to notice that by choosing ℓ_T to be random, we may also use Eq. (4.3) of Theorem 4.1 to get a formula for $e^{T\hat{L}/2} D\alpha(x)$ for any operator $\hat{L}$ on $\Gamma(\tilde{E})$ of the form $\hat{L} = \tilde{\square} - \hat{\mathcal{R}}$. To do this, let

$$\frac{d}{dt}\hat{Q}_t = -\frac{1}{2}\hat{Q}_t \hat{\mathcal{R}}_{//_t} \qquad \text{with} \quad \hat{Q}_0 = \mathrm{id}_{\tilde{E}_x}, \tag{4.24}$$

where $\hat{\mathcal{R}}_{//_t} := (//_t^{\tilde{E}})^{-1} \hat{\mathcal{R}} //_t^{\tilde{E}}$, and choose ℓ such that $\ell_0 = 0$ and $\ell_T = (\hat{Q}_T \tilde{Q}_T^{-1})^{\mathrm{tr}} \xi$ with $\xi \in \tilde{E}_x^*$ non-random. Then by the Feynman–Kac formula (Proposition 4.5)

$$\begin{aligned}\langle e^{T\hat{L}/2} D\alpha(x), \xi\rangle &= \mathbb{E}[\langle \hat{Q}_T //_T^{-1} Da_T(X_T(x)), \xi\rangle] \\ &= \mathbb{E}[\langle \tilde{Q}_T //_T^{-1} Da_T(X_T(x)), (\hat{Q}_T \tilde{Q}_T^{-1})^{\mathrm{tr}} \xi\rangle] \\ &= \mathbb{E}[\langle \tilde{Q}_T //_T^{-1} Da_T(X_T(x)), \ell_T\rangle]. \end{aligned} \tag{4.25}$$

This equation along with Eq. (4.3) and $a_T = \alpha$ implies that

$$\langle e^{T\hat{L}/2} D\alpha(x), \xi\rangle = \mathbb{E}[\langle Q_T //_T^{-1} \alpha(X_T(x)), U_T^\ell\rangle], \tag{4.26}$$

where ℓ is an $\tilde{E}_x^*$-valued process such that $\ell_0 = 0$ and $\ell_T = (\hat{Q}_T \tilde{Q}_T^{-1})^{\mathrm{tr}} \xi$.

Remark 4.8. Equations (4.22) and (4.23) provide stochastic formulas for $De^{TL/2}\alpha$ and $e^{T\tilde{L}/2} D\alpha$, not containing derivatives of the section α. These and related formulas rely on the fact that one of the local martingales given by Eq. (3.15), or Eq. (3.19), or Eq. (3.22), is a martingale for certain choices of a finite energy process $\{\ell_t\}$, respectively $\{k_t\}$. Nevertheless there is an essential difference between formula (4.22) and (4.23). To get the formula for $De^{TL/2}\alpha$, we need to know that the local martingale

$$Z_t^\ell = \langle \tilde{N}_t, \ell_t\rangle - \langle N_t, U_t^\ell\rangle,$$

as given by Eq. (3.15), is a martingale for a finite energy process ℓ such that $\ell_0 = \xi$ and $\ell_T = 0$. As we shall see (Section 6), this can always be achieved (independently of whether M is compact or complete), for instance, by taking $\ell_s = 0$ already for $s \geqslant \tau \wedge T$ where τ is the first exit time of $X(x)$ from some relatively compact neighborhood of x. On the other hand, to get the formula for $e^{T\tilde{L}/2}\, D\alpha$, the martingale property for Z^ℓ is exploited where now $\ell_0 = 0$ and $\ell_T = \xi$, which no longer is a local problem. It implies the second equality in (4.23), which combined with the Feynman–Kac representation for $e^{T\tilde{L}/2}\, D\alpha$ gives both equalities in Eq. (4.23). As will be seen in Section 5, the martingale property in this case is easily checked if M is compact (for most applications, completeness of M is sufficient), but it is not automatically satisfied in general. For instance, sticking to the case $\rho = 0$ for simplicity, and under the assumption that Z^ℓ is a martingale also for $\ell_s \equiv \xi$, we get the formula

$$\langle (De^{T\tilde{L}/2}\alpha)(x), \xi \rangle = \mathbb{E}[\langle \tilde{Q}_T //_T^{-1}\, D\alpha(X_T(x)), \xi \rangle]$$

which combined with Eq. (4.23) shows that $De^{T\tilde{L}2}\alpha = e^{T\tilde{L}/2}\, D\alpha$, see also Eq. (4.18). The validity of such commutation rules is known to be an intriguing question for (non-complete) Riemannian manifolds.

4.3. *Harmonic Section Derivative Formula*

Suppose that $a \in \Gamma(E)$ is a L-harmonic section (i.e. $La = 0$) defined locally in a neighborhood $\mathcal{V}$ of x. Let τ be the first exit time of $X_t(x)$ from some relatively compact neighborhood of x which is contained in $\mathcal{V}$. If $\{\ell_t\}_{0 \leqslant t < \tau}$ is a bounded L^1-finite energy process (i.e. $(\int_0^\tau |\ell'(s)|^2\, ds)^{1/2} \in L^1$) such that $\ell_0 = \xi \in \tilde{E}_x^*$ and $\ell_\tau = 0$, then

$$\langle Da(x), \xi \rangle = -\mathbb{E}[\langle Q_\tau //_\tau^{-1} a(X_\tau(x)), U_\tau^\ell \rangle] \tag{4.27}$$

where U^ℓ is the process defined in Eq. (3.17).

5. APPLICATIONS FOR COMPACT M

In order to avoid technical complications we will first demonstrate some applications of the previous results under the assumption that M is a compact manifold without boundary. Applications in the case that M is not compact will be given in the next section. One key consequence of the compactness of M is that $\zeta(x) = \infty$ a.s. for all $x \in M$. Again recall that Assumption 2 (at the beginning of Section 4) is in force throughout the remainder of this paper.

By standard elliptic P.D.E. theory and the Minakshisundaram–Pleijel method for constructing heat kernels we have the following facts. The heat equation

$$\partial_t a_t = L a_t \qquad \text{with} \quad a_0 = \alpha \in \Gamma(E) \tag{5.1}$$

has a unique solution which we write as $P_t \alpha$. The linear operator P_t: $\Gamma(E) \to \Gamma(E)$ extends continuously to a one-parameter semigroup on $L^2(E)$, see for example Chapter 2 of [4]. The L^2 generator of this semigroup is the closure $\bar{L}$ of L. To verify this last assertion, let $\hat{L}$ be the L^2 generator of P_t and $L^\dagger$ denote the formal adjoint of L. Then using the result in [4], $L^\dagger$ generates (in the same way L generated P) the adjoint semigroup P_t^*. Therefore, for $a \in \mathcal{D}(\hat{L})$ and $b \in \Gamma(E)$, we have that

$$(\hat{L}a, b) = \frac{d}{dt}\bigg|_{0+} (P_t a, b) = \frac{d}{dt}\bigg|_{0+} (a, P_t^* b) = (a, L^\dagger b).$$

This shows that $L \subset \bar{L} \subset \hat{L} \subset (L^\dagger)^*$. However, by basic elliptic regularity theory, $\mathcal{D}((L^\dagger)^*) = \mathcal{D}(\bar{L}) = H^2$—the Sobolev space with two derivatives in L^2. Therefore, $\bar{L} = \hat{L} = (L^\dagger)^*$. From now on we will write e^{tL} for P_t.

Proposition 5.1. *Suppose that α and $\tilde{\alpha}$ are L^2 sections of E and $\tilde{E}$ respectively, then the Feynman–Kac formula in Eqs.* (4.14) *and* (4.15) *of Proposition* 4.5 *are valid. If we further assume that α is an H^1 section of E* (*i.e. α is an L^2 section of E with one weak derivative in L^2*), *then Eq.* (4.18) *is valid as well.*

Proof. Using a continuity argument, it suffices to prove Eqs. (4.14), (4.15) and (4.18) under the assumption that α and $\tilde{\alpha}$ are smooth sections of E and $\tilde{E}$ respectively. Since M is compact, Q_t, Q_t^{-1} and $\tilde{Q}_t$ are bounded by e^{Kt}, where K is a non-random constant depending on the bounds on $\mathcal{R}$ and $\tilde{\mathcal{R}}$. Using these facts and the assumed smoothness of α and $\tilde{\alpha}$, it is easy to see that the assumptions in Proposition 4.5 are satisfied. ∎

Remark 5.2. A simple consequence of Eq. (4.18) is that $De^{tL/2}\alpha = e^{t\tilde{L}/2} D\alpha$ when $\rho = 0$. Of course this may be proved directly as well. Indeed, for $\alpha \in \Gamma(E)$, $a_t = De^{tL/2}\alpha$ and $b_t = e^{t\tilde{L}/2} D\alpha$ are both solutions to the heat equation $\frac{d}{dt} a_t = \frac{1}{2} \tilde{L} a_t$ with initial condition $a_t|_{t=0} = D\alpha$. Uniqueness of solutions to the $\tilde{L}$ heat equation gives $a_t = b_t$, i.e.

$$De^{tL/2}\alpha = e^{t\tilde{L}/2} D\alpha \qquad \text{for all} \quad \alpha \in \Gamma(E).$$

By continuity, the previous equation extends to all H^1 sections α of E.

For a more general account on the elliptic theory of the heat equation (5.1), including elliptic boundary problems in the case of compact manifolds with boundary, the reader is referred to Agranovich [1], Grubb [33], as well as Seeley [47, 48, 49].

5.1. *Corollaries of Theorem* 4.1

COROLLARY 5.3. *Suppose that* α *is a bounded measurable section of* E *and* $\xi \in \tilde{E}_x^*$. *Let* $\{\ell_t\}_{0 \leqslant t \leqslant T}$ *be an* L^1*-finite energy process with values in* $\tilde{E}_x^*$ (*see Definition* 3.6) *such that* $\ell_T = 0$. *Then, with* $\xi := \ell_0 \in \tilde{E}_x^*$,

$$\langle De^{TL/2}\alpha(x), \xi\rangle = -\mathbb{E}[\langle Q_T //_T^{-1} \alpha(X_T(x)), U_T^\ell\rangle], \tag{5.2}$$

where U^ℓ *is given by Eq.* (3.17). *More generally, letting* $p \in (1, \infty)$ *and* $q = p/(p-1)$ *be the conjugate exponent of* p, *if* $\{\ell_t\}_{0 \leqslant t \leqslant T}$ *is an* L^q*-finite energy process and* α *is an* L^p *section of* E, *then Eq.* (5.2) *is still valid.*

Proof. We will apply Theorem 4.1 with $a_t(x) := (e^{(T-t)L/2}\alpha)(x)$. By the Burkholder–Davis–Gundy inequality, there exists a constant C, depending only on $q \in [1, \infty)$, K and T, such that

$$\mathbb{E}\left|\int_0^T \mathcal{Q}_s^{-1} m_{dB_s}^{\mathrm{tr}} \tilde{\mathcal{Q}}_s \ell'_s\right|^q \leqslant C\mathbb{E}\left[\left(\int_0^T |\ell'_t|^2\, dt\right)^{q/2}\right]$$

and also

$$\begin{aligned}\mathbb{E}\left|\int_0^T \mathcal{Q}_s^{-1} \rho_{//_s}^{\mathrm{tr}} \tilde{\mathcal{Q}}_s \ell_s\, ds\right|^q &\leqslant C\mathbb{E}\left[\left(\int_0^T |\ell_s|^2\, ds\right)^{q/2}\right] \\ &\leqslant C\left(\ell_0^q + \mathbb{E}\left[\left(\int_0^T |\ell'_s|^2\, ds\right)^{q/2}\right]\right).\end{aligned}$$

These equations together show that U_T^ℓ is L^q-integrable.

First suppose that α is a bounded measurable section of E. As in the proof of Proposition 5.1, the local martingale N of Proposition 3.2 and Z^ℓ of Theorem 3.7 are already martingales. In case $|De^{tL/2}\alpha|$ is not bounded on $(0, T] \times M$ it may be necessary to first modify ℓ such that $\ell(T-\varepsilon) = 0$ for some small $\varepsilon > 0$ and then to take the limit as $\varepsilon \to 0$, see [58] for details.

Finally, if $p \in (1, \infty)$ and $\{\ell_t\}_{0 \leqslant t \leqslant T}$ is an L^q-finite energy process, then both sides of Eq. (5.2) depend continuously on $\alpha \in L^p(E)$, hence it suffices to prove Eq. (5.2) when α is smooth. But for smooth α it is easy to verify the hypothesis of Theorem 4.1. ∎

COROLLARY 5.4. *Suppose that* $\alpha \in \Gamma(E)$ *is an* L*-harmonic section (i.e.* $L\alpha = 0$*) and* $\xi \in \tilde{E}_x^*$. *Let* $\{\ell_t\}_{0 \leqslant t \leqslant T}$ *be an* L^1*-finite energy process with values in* $\tilde{E}_x^*$ *such that* $\ell_T = 0$ *and* $\ell_0 = \xi \in \tilde{E}_x^*$. *Then*

$$\langle D\alpha(x), \xi \rangle = -\mathbb{E}[\langle Q_T //_T^{-1} \alpha(X_T(x)), U_T^\ell \rangle], \tag{5.3}$$

where again U^ℓ *is given by Eq.* (3.17).

Proof. This is a consequence of Theorem 4.1 with $a_t := \alpha$ or directly from the previous Corollary upon noting that $e^{TL/2}\alpha = \alpha$ since $L\alpha = 0$. ∎

COROLLARY 5.5. *Suppose that* α *is a* C^1 *section of* E *and* $\xi \in \tilde{E}_x^*$. *Let* $\{\ell_t\}_{0 \leqslant t \leqslant T}$ *be an* L^1*-finite energy process with values in* $\tilde{E}_x^*$ *such that* $\ell_0 = 0$ *and* $\ell_T = \xi \in \tilde{E}_x^*$. *Under Assumption* 2,

$$\langle e^{T\tilde{L}/2} D\alpha(x), \xi \rangle = \mathbb{E}[\langle Q_T //_T^{-1} \alpha(X_T(x)), U_T^\ell \rangle], \tag{5.4}$$

where U^ℓ *is given by Eq.* (3.17). *If instead we choose* ℓ *as above except with* $\ell_0 = \xi \in \tilde{E}_x^*$ *and* $\ell_T = 0$, *then*

$$\langle e^{T\tilde{L}/2} D\alpha(x), \xi \rangle = \mathbb{E}[\langle Q_T //_T^{-1} \alpha(X_T(x)), W_T^\ell \rangle], \tag{5.5}$$

where

$$W_T^\ell := \frac{1}{2}\left(\int_0^T \tilde{Q}_t \rho_{//_t} Q_t^{-1}\, dt\right)^{\mathrm{tr}} \xi - U_T^\ell.$$

Proof. Equation (5.4) follows from Eq. (4.3) of Theorem 4.1 with $a_t := e^{(T-t)L/2}\alpha$. Equation (5.5) is a consequence of Eq. (4.2) of Theorem 4.1 and Eq. (4.18) which imply,

$$\begin{aligned}
&-\mathbb{E}[\langle Q_T //_T^{-1} a_T(X_T(x)), U_T^\ell \rangle] \\
&\quad = \langle De^{TL/2}\alpha(x), \xi \rangle \\
&\quad = \langle (e^{T\tilde{L}/2} D\alpha)(x), \xi \rangle \\
&\qquad - \frac{1}{2}\mathbb{E}\left[\left\langle \left(\int_0^T \tilde{Q}_t \rho_{//_t} Q_t^{-1}\, dt\right) Q_T //_T^{-1} \alpha(X_T(x)), \xi \right\rangle\right].
\end{aligned}$$ ∎

Remark 5.6. Suppose that E and $\tilde{E}$ are Hermitian vector bundles with metric compatible covariant derivatives. Under these conditions, the results of Corollaries 5.3, 5.4, and 5.5 may be rewritten by replacing the dual space $\tilde{E}_x^*$ by $\tilde{E}_x$ and the dual parings $\langle \cdot, \cdot \rangle$ by the appropriate Hermitian metrics and then using Eq. (3.18) to define the process U_t^ℓ.

5.2. *Formulas for Dirac operators*

Let us specialize Corollaries 5.3 and 5.5 to the case where $E=\tilde{E}=S$ is a spin bundle over M. Recall that S is Riemannian vector bundle with metric compatible spin connection ∇^S, see Example 2.13. In this case $\mathcal{R}=\tilde{\mathcal{R}}=\frac{1}{4}\mathrm{scal}$, $\rho=0$,

$$Q_T=\tilde{Q}_T=e^{-(1/8)\int_0^T \mathrm{scal}(X_t(x))\,dt}\,\mathrm{id}_{S_x}$$

and

$$U_T^{\ell}=-\int_0^T \gamma_{dB_s}\ell'_s,$$

wherein we have used the fact that $\gamma^*=-\gamma$. Corollary 5.3 (see Remark 5.6) becomes

$$(De^{-TD^2/2}\alpha(x),\xi)_{S_x}$$
$$=\mathbb{E}\left[e^{-(1/8)\int_0^T \mathrm{scal}(X_t(x))\,dt}\left(//_T^{-1}\alpha(X_T(x)),\int_0^T \gamma_{dB_s}\ell'_s\right)_{S_x}\right],$$

where ℓ is an L^1-finite energy process with values in S_x such that $\ell_T=0$ and $\ell_0=\xi\in S_x$. Taking $\ell_t=h_t\xi$ in the previous equation (where $\xi\in S_x$ and h_t is an L^1-finite energy process with values in $\mathbb{R}$) gives

$$(De^{-TD^2/2}\alpha)(x)$$
$$=-\mathbb{E}\left[e^{-(1/8)\int_0^T \mathrm{scal}(X_t(x))\,dt}\left(\int_0^T h'_s\gamma_{dB_s}\right)//_T^{-1}\alpha(X_T(x))\right] \tag{5.6}$$

wherein we have used the fact that $\gamma_v^*=-\gamma_v$. Choosing $h_t=1-t/T$ in this equation implies

$$(De^{-TD^2/2}\alpha)(x)=\frac{1}{T}\mathbb{E}[e^{-(1/8)\int_0^T \mathrm{scal}(X_t(x))\,dt}\gamma_{B_T}//_T^{-1}\alpha(X_T(x))]. \tag{5.7}$$

Similarly, using Eq. (5.4) of Corollary 5.5 (with $\xi\in S_x$),

$$(e^{-TD^2/2}D\alpha(x),\xi)_{S_x}$$
$$=-\mathbb{E}\left[e^{-(1/8)\int_0^T \mathrm{scal}(X_t(x))\,dt}\left(//_T^{-1}\alpha(X_T(x)),\int_0^T \gamma_{dB_s}\ell'_s\right)_{S_x}\right],$$

where ℓ is an L^1-finite energy process with values in S_x such that $\ell_T = \xi$ and $\ell_0 = 0$. Taking $\ell_t = t\xi/T$ in the previous equation gives the same formula for $e^{-TD^2/2} D\alpha(x)$ as in the right side of Eq. (5.7). This of course should be the case since $[D, L] = 0$.

5.3. *Application to d and d**

Let $E = \tilde{E} = \Lambda T^*M$, $L = \tilde{L} = \Delta$ and $P_t := e^{t\bar{\Delta}/2}$ where $\bar{\Delta}$ is the self-adjoint extension of Δ. An application of Corollary 5.3 with m being either the interior product A or the exterior product C (see Example 2.11 and Definition A.1 in Appendix A for the notation) gives the following theorem.

THEOREM 5.7. *Let M be a compact manifold, a be a bounded measurable section of ΛT^*M and $v \in \Lambda T_x M$ for some $x \in M$. Then*

$$\langle (dP_T a)_x, v \rangle = -\mathbb{E}\left[\left\langle //_T^{-1} a(X_T(x)), \mathcal{Q}_T \int_0^T \mathcal{Q}_t^{-1}(A_{dB_t} \mathcal{Q}_t \ell'_t)\right\rangle\right] \tag{5.8}$$

$$\langle (d^* P_T a)_x, v \rangle = -\mathbb{E}\left[\left\langle //_T^{-1} a(X_T(x)), \mathcal{Q}_T \int_0^T \mathcal{Q}_t^{-1}(dB_t \wedge \mathcal{Q}_t \ell'_t)\right\rangle\right], \tag{5.9}$$

where ℓ is any L^1-finite energy process with values in $\Lambda T_x M$ such that $\ell_0 = v$ and $\ell_T = 0$, $\mathcal{Q}_t$ is the solution to the differential equation

$$\frac{d}{dt}\mathcal{Q}_t = -\frac{1}{2}\mathcal{R}^{\mathrm{tr}}_{//_t}\mathcal{Q}_t \qquad \text{with} \quad \mathcal{Q}_0 = \mathrm{id}_{\Lambda T_x M}, \tag{5.10}$$

$\mathcal{R}^{\mathrm{tr}}_{//_t} = //_t^{-1}\mathcal{R}^{\mathrm{tr}}//_t$ *and $\mathcal{R}^{\mathrm{tr}}$ is the Weitzenböck curvature term described in Eq.* (A.16) *of Lemma* A.9, *Appendix* A.

Similarly an application of Corollary 5.5 with m being either the interior product A or the exterior product C implies the following theorem.

THEOREM 5.8. *Let M be a compact manifold, a be a C^1 section (H^1 would do) of ΛT^*M and $v \in \Lambda T_x M$ for some $x \in M$. Then*

$$\langle (P_T da)_x, v \rangle = \mathbb{E}\left[\left\langle //_T^{-1} a(X_T(x)), \mathcal{Q}_T \int_0^T \mathcal{Q}_t^{-1}(A_{dB_t} \mathcal{Q}_t \ell'_t)\right\rangle\right] \tag{5.11}$$

$$\langle (P_T d^* a)_x, v \rangle = \mathbb{E}\left[\left\langle //_T^{-1} a(X_T(x)), \mathcal{Q}_T \int_0^T \mathcal{Q}_t^{-1}(dB_t \wedge \mathcal{Q}_t \ell'_t)\right\rangle\right], \tag{5.12}$$

where ℓ is any L^1-finite energy process with values in $\Lambda T_x M$ such that $\ell_0 = 0$ and $\ell_T = v$ and $\mathcal{Q}_t$ solves Eq. (5.10) *as above.*

This theorem also follows from Theorem 5.7 because $dP_T a = P_T da$ and $d^*P_T a = P_T d^*a$ for C^1 sections a of ΛT^*M, see Remark B.9.

In the special case where $\ell' \equiv$ constant, the formulas (5.9) and (5.10) have been derived by Elworthy and Li [26] from a non-intrinsic formula by the techniques of filtering out redundant noise, as developed in Elworthy and Yor [27]; see [23] for a general account on this.

Specializing Eq. (5.8) to zero forms gives the following Bismut type formula, see [24, 56, 58].

COROLLARY 5.9. *Let $f\colon M \to \mathbb{R}$ be bounded and measurable, $x \in M$ and $v \in T_xM$. Then, for any L^1-finite energy process ℓ with values in T_xM such that $\ell_0 = v$, and $\ell_T = 0$,*

$$\langle (dP_T f)_x, v\rangle = -\mathbb{E}\left[f(X_T(x)) \int_0^T (\mathcal{Q}_t \ell'_t, dB_t)\right],$$

where $\mathcal{Q}_t$ is the $\operatorname{Aut}(T_xM)$*-valued process satisfying the differential equation:*

$$\frac{d}{dt}\mathcal{Q}_t = -\frac{1}{2}\operatorname{Ric}_{//_t}\mathcal{Q}_t \qquad \text{with} \quad \mathcal{Q}_0 = \operatorname{id}_{T_xM}. \tag{5.13}$$

Proof. Letting $a = f \in \Lambda^0(T^*M)$ in Eq. (5.8) and using the fact that $//_t$ and Q_t act as the identity on 0-forms we find that

$$\begin{aligned}\langle (dP_T f)_x, v\rangle &= -\mathbb{E}\left[f(X_T(x)) \int_0^T A_{dB_t}\mathcal{Q}_t \ell'_t\right] \\ &= -\mathbb{E}\left[f(X_T(x)) \int_0^T (\mathcal{Q}_t \ell'_t, dB_t)\right],\end{aligned}$$

where $\mathcal{Q}_t$ is the restriction of the solution of Eq. (5.10) to T_xM. By Eq. (A.17) of Appendix A $\mathcal{R}^{\mathrm{tr}} \mid TM = \operatorname{Ric}$, and thus $\mathcal{Q}_t$ restricted to T_xM solves Eq. (5.13). ∎

The following theorem is a special case of Eq. (5.9) of Theorem 5.7 and improves a result in [19] (see Corollary 5.18 below) by giving a formula for $\mathbb{E}[\nabla \cdot Y(X_T(x))]$ which does not contain derivatives of the curvature tensor.

THEOREM 5.10. *Let M be a compact Riemannian manifold, $\mathcal{Q}_t$ denote the solution to Eq. (5.13), Y be a smooth vector field on M, $\nabla \cdot Y$ denote the* *divergence of Y, $T>0$ and ℓ be a real-valued L^1-finite energy process. If $\ell_0=0$ and $\ell_T=1$, then*

$$\mathbb{E}[\nabla\cdot Y(X_T(x))] = -\mathbb{E}\left[\left(//_T^{-1}\,Y(X_T(x)),\ \mathcal{Q}_T\int_0^T \ell'_t\mathcal{Q}_t^{-1}\,dB_t\right)\right]. \tag{5.14}$$

Proof. Let $a=(Y,\cdot)\in\Omega^1(M)$. By Eq. (A.9) of Appendix A,

$$d^*a = -\sum_i A_{e_i}\nabla_{e_i}a = -\sum_i A_{e_i}(\nabla_{e_i}Y,\cdot) = -\nabla\cdot Y.$$

Applying Eq. (5.12) with $v=1\in\Lambda^0(T_xM)$ implies

$$\begin{aligned}\mathbb{E}[\nabla\cdot Y(X_T(x))] &= -\langle (P_T d^*a)_x, 1\rangle\\ &= -\mathbb{E}\left[\left\langle //_T^{-1}a(X_T(x)),\ \mathcal{Q}_T\int_0^T \mathcal{Q}_t^{-1}\ell'_t\,dB_t\right\rangle\right],\end{aligned}$$

where we have used the fact that $\mathcal{R}^{\mathrm{tr}}=0$ on $\Lambda^0(TM)$ and $\mathcal{R}^{\mathrm{tr}}=\mathrm{Ric}$ on $\Lambda^1(TM)$ and hence that $\mathcal{Q}_t\,|\,\Lambda^0(T_xM)=\mathrm{id}$ and $\mathcal{Q}_t\,|\,\Lambda^1(T_xM)$ solves Eq. (5.13). This proves Eq. (5.14) because $//_T^{-1}a(X_T(x))=(//_T^{-1}\,Y(X_T(x)),\cdot)$. ∎

Note that compactness of M is not essential here: the formulas in Theorem 5.8 only require the martingale property of (3.15), or (3.19), for some ℓ such that $\ell_0=v$ and $\ell_T=0$. As indicated in Remark 4.8, this can always be achieved and gives a formula for $(dP_Ta)(x)$, respectively. $(d^*P_Ta)(x)$, as long as P_Ta is well-defined, i.e. $Q_T^{-1}//_T^{-1}a(X_T(x))\in L^1$. In particular, a need not be differentiable. Also the finite lifetime of the Brownian motion only effects the stochastic representation of P_Ta, see Section B.1 of Appendix B but not the given argument. (See Section 6 for precise statements in this direction.) In the situation of Theorem 5.10 this shows that the right-hand side of Eq. (5.14) is just $-d^*P_Ta$ where $a=(Y,\cdot)$. To verify however that

$$d^*P_Ta = -\mathbb{E}[(\nabla\cdot Y)(X_T(x))] \qquad (\equiv P_Td^*a) \tag{5.15}$$

requires assumptions (in particular, differentiability of a): it precisely reflects the property that the local martingale (3.5) is actually a martingale, from where (5.15) follows by taking expectations.

5.3.1. *An integrated logarithmic gradient estimate for the heat kernel on M.* As an application of formula (5.14) we get the following integrated estimate for the gradient of the logarithmic derivative of the heat kernel on M. Of course this result may be derived by partial differential equation techniques as well.

THEOREM 5.11. *Let M be a compact Riemannian manifold without boundary. For every $q \in [1, \infty[$ there is a constant $C_q < \infty$ such that for all $t \leqslant 1$,*

$$\left(\int_M |\nabla_z \log p_t(x, z)|^q \, p_t(x, z) \operatorname{vol}(dz) \right)^{1/q} \leqslant C_q e^{2Kt} t^{-1/2}, \tag{5.16}$$

where K is a bound on the Ricci curvature (Ric) *of M.*

Remark 5.12. This estimate with $q = 1$ may be used to show that pinned Brownian motion on a compact manifold is a semimartingale.

Proof. Without loss of generality, we may assume that $q \geqslant 2$. Since

$$\begin{aligned} \mathbb{E}[\nabla \cdot Y(X_T(x))] &= \int_M p_T(x, z) \, \nabla \cdot Y(z) \operatorname{vol}(dz) \\ &= -\int_M (Y(z), \nabla_z \log p_T(x, z))_{TM} \, p_T(x, z) \operatorname{vol}(dz), \end{aligned}$$

Equation (5.14) is equivalent to

$$\begin{aligned} &\int_M (Y(z), \nabla_z \log p_T(x, z))_{TM} \, p_T(x, z) \operatorname{vol}(dz) \\ &\quad = \mathbb{E}\left[\left(//_T^{-1} Y(X_T(x)), \mathcal{Q}_T \int_0^T \ell'_t \mathcal{Q}_t^{-1} \, dB_t \right) \right], \end{aligned}$$

where ℓ is a real-valued L^1-finite energy process such that $\ell_0 = 0$ and $\ell_T = 1$.
If K is a bound on Ric and $1/p + 1/q = 1$, Hölder's and the Burkholder–Davis–Gundy inequalities imply

$$\begin{aligned} &\left| \int_M (Y(z), \nabla_z \log p_T(x, z))_{TM} \, p_T(x, z) \operatorname{vol}(dz) \right| \\ &\quad \leqslant e^{KT} \mathbb{E}\left[|Y(X_T(x))| \left| \int_0^T \ell'_t \mathcal{Q}_t^{-1} \, dB_t \right| \right] \\ &\quad \leqslant e^{KT} (\mathbb{E} \, |Y(X_T(x))|^p)^{1/p} \left(\mathbb{E} \left| \int_0^T \ell'_t \mathcal{Q}_t^{-1} \, dB_t \right|^q \right)^{1/q} \\ &\quad \leqslant C_q e^{KT} (\mathbb{E} \, |Y(X_T(x))|^p)^{1/p} \left(\mathbb{E} \left| \int_0^T \operatorname{tr}[(\mathcal{Q}_t^{-1})^* \, \mathcal{Q}_t^{-1}] \, \ell'^2_t \, dt \right|^{q/2} \right)^{1/q} \\ &\quad \leqslant C_q e^{2KT} (\mathbb{E} \, |Y(X_T(x))|^p)^{1/p} \left(\mathbb{E} \left| \int_0^T \ell'^2_t \, dt \right|^{q/2} \right)^{1/q} \end{aligned}$$

for some constant $C_q < \infty$. Choosing $\ell_t := t/T$ in this inequality shows,

$$\begin{aligned}\left|\int_M (Y(z), \nabla_z \log p_T(x,z))_{TM}\, p_T(x,z)\,\mathrm{vol}(dz)\right| &\leqslant T^{-1/2} C_q e^{2KT} (\mathbb{E}\,|Y(X_T(x))|^p)^{1/p} \\ &= T^{-1/2} C_q e^{2KT} \left(\int_M |Y(z)|^p\, p_T(x,z)\,\mathrm{vol}(dz)\right)^{1/p}.\end{aligned}$$

Now choose $Y(z) := |\nabla_z \log p_T(x,z)|^{q-2}\,\nabla_z \log p_T(x,z)$ to get

$$\begin{aligned}&\int_M |\nabla_z \log p_T(x,z)|^q\, p_T(x,z)\,\mathrm{vol}(dz) \\ &\quad\leqslant T^{-1/2} C_q e^{2KT} \left(\int_M |\nabla_z \log p_T(x,z)|^{p(q-1)}\, p_T(x,z)\,\mathrm{vol}(dz)\right)^{1/p} \\ &\quad= T^{-1/2} C_q e^{2KT} \left(\int_M |\nabla_z \log p_T(x,z)|^{q}\, p_T(x,z)\,\mathrm{vol}(dz)\right)^{1-1/q}.\end{aligned}$$

Solving this equation for $(\int_M |\nabla_z \log p_T(x,z)|^q\, p_T(x,z)\,\mathrm{vol}(dz))^{1/q}$ proves Eq. (5.16). ∎

5.4. *Formulas for* $\nabla e^{TL/2}$

In this subsection, we will write out the results in Corollaries 5.3, 5.5 and Theorem 4.3 when $D=\nabla$. Let $E \to M$ be a Riemannian vector bundle with metric compatible covariant derivative ∇^E, $\mathcal{R} \in \Gamma(\mathrm{End}(E))$, $\tilde{E} := T^*M \otimes E$, $m = \mathrm{id}$ (the identity multiplication map), $D_m = \nabla^E$, and $L = \square - \mathcal{R}$. We also define $\tilde{\mathcal{R}} \in \Gamma(\mathrm{End}(\tilde{E}))$ and $\rho \in \Gamma(\mathrm{Hom}(E, \tilde{E}))$ by

$$\tilde{\mathcal{R}} = \mathrm{Ric}^{\mathrm{tr}} \otimes \mathrm{id}_E - 2R^E\cdot + \mathrm{id}_{T^*M} \otimes \mathcal{R},$$

and

$$\rho = \nabla \cdot R^E + (\nabla^{\mathrm{End}(E)} \mathcal{R}).$$

By Proposition 2.15, $L := \square - \mathcal{R}$, $\tilde{L} = \tilde{\square} - \tilde{\mathcal{R}}$, ρ, and m satisfy Assumption 2. We have the following immediate consequences of Corollaries 5.3 and 5.5.

THEOREM 5.13. *Suppose that α is a bounded measurable section of E and $\xi \in \tilde{E}_x^* = T_x M \otimes E_x^*$ and $\{\ell_t\}_{0 \leqslant t \leqslant T}$ a L^1-finite energy process with values in $\tilde{E}_x^*$ (see Definition* 3.6*) such that $\ell_T = 0$ and $\ell_0 = \xi \in \tilde{E}_x^*$. Then*

$$\langle \nabla e^{TL/2} \alpha(x), \xi \rangle = -\mathbb{E}[\langle Q_T //_T^{-1} \alpha(X_T(x)), U_T^\ell \rangle], \tag{5.17}$$

where U^{ℓ} is given by Eq. (3.17). If we further assume that α is an H^1 section of E and $\{\ell_t\}_{0 \leqslant t \leqslant T}$ is an L^1-finite energy process with values in $\tilde{E}^*_x$ such that $\ell_0 = 0$, then

$$\langle e^{T\tilde{L}/2} \nabla\alpha(x), \xi \rangle = \mathbb{E}[\langle Q_T //_T^{-1} \alpha(X_T(x)), U^{\ell}_T \rangle]. \tag{5.18}$$

Remark 5.14. We may extend Eq. (5.18) using Remark 4.7 as follows. Let $\hat{\mathcal{R}}$ be an arbitrary section of $\mathrm{End}(\tilde{E})$, $\hat{L} = \tilde{\square} - \hat{\mathcal{R}}$ and

$$\frac{d}{dt} \hat{Q}_t = -\frac{1}{2} \hat{Q}_t \hat{\mathcal{R}}_{//_t} \qquad \text{with} \quad \hat{Q}_0 = \mathrm{id}_{\tilde{E}_x}, \tag{5.19}$$

where as before $\hat{\mathcal{R}}_{//_t} := (//_t^{\tilde{E}})^{-1} \hat{\mathcal{R}} //_t^{\tilde{E}}$. Then

$$\langle e^{T\hat{L}/2} \nabla\alpha(x), \xi \rangle = \mathbb{E}[\langle Q_T //_T^{-1} \alpha(X_T(x)), U^{\ell}_T \rangle],$$

where ℓ_t is any L^1-finite energy process such that $\ell_0 = 0$ and $\ell_T = (\hat{Q}_T \tilde{Q}_T^{-1})^{\mathrm{tr}} \xi$.

By using Theorem 4.3, we may get another (more explicit) formula for $\nabla e^{TL/2}$. This theorem will be given after the following preparatory Lemma.

LEMMA 5.15. *The transpose of the multiplication map $m = \mathrm{id}$ is the "annihilation" operator A, where A_v: $\tilde{E}^* = TM \otimes E^* \to E^*$ is determined by*

$$A_v(w \otimes \alpha) = (v, w)\, \alpha \qquad \textit{for all} \quad v, w \in T_x M, \quad \alpha \in E^*_x, \quad x \in M.$$

Also

$$\tilde{\mathcal{R}}^{\mathrm{tr}} = \mathrm{Ric} \otimes \mathrm{id}_{E^*} + 2 \sum_{i=1}^{n} e_i \otimes R^{E^*}(\cdot, e_i) + \mathrm{id}_{TM} \otimes \mathcal{R}^{\mathrm{tr}}, \tag{5.20}$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame and $e_i \otimes R^{E^*}(\cdot, e_i) \in \Gamma(\mathrm{End}(\tilde{E}^*))$ is determined by

$$e_i \otimes R^{E^*}(\cdot, e_i)(v \otimes \alpha) = e_i \otimes R^{E^*}(v, e_i)\, \alpha = -e_i \otimes \alpha \circ R^E(v, e_i)$$

for $v \in T_x M$ and $\alpha \in E^*_x$. Moreover,

$$\rho^{\mathrm{tr}} = (\nabla \cdot R^E)^{\mathrm{tr}} + (\nabla^{\mathrm{End}(E)} \mathcal{R})^{\mathrm{tr}} \in \Gamma(\mathrm{Hom}(TM \otimes E^*, E^*)), \tag{5.21}$$

where

$$(\nabla \cdot R^E)^{\mathrm{tr}}\, (v \otimes \alpha) = \alpha \circ (\nabla \cdot R^E)(v) = \sum_{i=1}^{n} \alpha \circ ((\nabla_{e_i} R^E)(e_i, v)) \tag{5.22}$$

and

$$(\nabla^{\mathrm{End}(E)} \mathcal{R})^{\mathrm{tr}}\, (v \otimes \alpha) = \alpha \circ (\nabla_v^{\mathrm{End}(E)} \mathcal{R}). \tag{5.23}$$

Proof. Let $v, w \in T_xM$, $\xi \in E_x$ and $\alpha \in E_x^*$. Since $m_v\colon E_x \to \tilde{E}_x = T_x^*M \otimes E_x$ is given by $m_v\xi = (v, \cdot) \otimes \xi$, it follows that

$$\langle m_v\xi, w \otimes \alpha\rangle = (v, w)\langle \xi, \alpha\rangle = \langle \xi, (v, w)\,\alpha\rangle = \langle \xi, A_v(w \otimes \alpha)\rangle$$

which shows that $m_v^{\mathrm{tr}} = A_v$. Now let $\beta \in \mathrm{Hom}(T_xM, E_x) \cong T_x^*M \otimes E_x$. Then Eq. (5.20) follows by taking transposes of Eq. (2.7) along with the computation,

$$\begin{aligned}\langle (R^E\cdot)^{\mathrm{tr}}\,(v \otimes \alpha), \beta\rangle &= \langle v \otimes \alpha, R^E \cdot \beta\rangle \\ &= \sum_{i=1}^n \langle \alpha, R^E(v, e_i)\,\beta(e_i)\rangle \\ &= \sum_{i=1}^n \langle e_i \otimes R^E(v, e_i)^{\mathrm{tr}}\,\alpha, \beta\rangle \\ &= -\sum_{i=1}^n \langle e_i \otimes R^{E^*}(v, e_i)\,\alpha, \beta\rangle.\end{aligned}$$

Eqs. (5.21), (5.22) and (5.23) are proved similarly:

$$\begin{aligned}\langle (\nabla \cdot R^E)\,\xi, v \otimes \alpha\rangle &= \sum_{i=1}^n \langle (\nabla_{e_i} R^E)(e_i, v)\,\xi, \alpha\rangle \\ &= \sum_{i=1}^n \langle \xi, \alpha \circ ((\nabla_{e_i} R^E)(e_i, v))\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle (\nabla^{\mathrm{End}(E)}\mathcal{R})\,\xi, v \otimes \alpha\rangle &= \langle (\nabla_v^{\mathrm{End}(E)}\mathcal{R})\,\xi, \alpha\rangle \\ &= \langle \xi, (\nabla^{\mathrm{End}(E)}\mathcal{R})^{\mathrm{tr}}\,(v \otimes \alpha)\rangle \\ &= \langle \xi, \alpha \circ (\nabla_v^{\mathrm{End}(E)}\mathcal{R})\rangle. \quad \blacksquare\end{aligned}$$

THEOREM 5.16. *Let $\mathcal{R} \in \Gamma(\mathrm{End}(E))$, Q be as in Eq. (3.1), $L = \square - \mathcal{R}$, and $m = \mathrm{id}$. Suppose that α is a bounded measurable section of E, ℓ_s is a T_xM-valued L^1-finite energy process and*

$$V_t^\ell := \int_0^t [(\ell_s' + \tfrac{1}{2}\,\mathrm{Ric}_{//_s}\ell_s, dB_s) - R_{//_s}^E(\ell_s, \delta B_s) + \tfrac{1}{2}(\ell_s, \overleftarrow{dB}_s)\,\mathcal{R}_{//_s}]\,Q_s^{-1},$$

where δB_s, and $\overleftarrow{dB}_s$ denotes the Fisk-Stratonovich and backwards Itô differential respectively. (More precisely, if X is another semimartingale, then

$X_s\,\delta B_s = X_s\,dB_s + \frac{1}{2}\,dX_s\,dB_s$ *and* $X_s\,d\overleftarrow{B}_s = X_s\,dB_s + dX_s\,dB_s$.) *If* $\ell_0 = v \in T_xM$ *and* $\ell_T = 0$ *then*

$$\nabla_v e^{TL/2}\alpha(x) = -\mathbb{E}[\,V_T^{\ell} Q_T //_T^{-1}\alpha(X_T(x))\,], \tag{5.24}$$

and assuming in addition that $a \in \Gamma(E)$, *if* $\ell_0 = 0$ *and* $\ell_T = v \in T_xM$ *then*

$$(e^{T\tilde{\Box}/2}\,\nabla\alpha)_v = \mathbb{E}[\,V_T^{\ell} Q_T //_T^{-1}\alpha(X_T(x))\,]. \tag{5.25}$$

Proof. Let $a_t := e^{(T-t)\,L/2}\alpha$, $\xi \in E_x^*$, ℓ_s be an L^1-finite energy process with values in T_xM and $k_s = \ell_s \otimes \xi$. In order to apply Theorem 4.3 we need to work out $\hat{U}_t^k$ defined in Eq. (3.21). Using Lemma 5.15,

$$\begin{aligned}
m_{dB_s}^{\mathrm{tr}}(k_s' + \tfrac{1}{2}\tilde{\mathscr{R}}_{//_s}^{\mathrm{tr}} k_s) &= A_{dB_s}(\ell_s' \otimes \xi + \tfrac{1}{2}\tilde{\mathscr{R}}_{//_s}^{\mathrm{tr}}(\ell_s \otimes \xi)) \\
&= (\ell_s', dB_s)\,\xi + \tfrac{1}{2}(\mathrm{Ric}_{//_s}\,\ell_s, dB_s)\,\xi \\
&\quad + \sum_{i=1}^{n} (e_i, dB_s)\,R_{//_s}^{E^*}(\ell_s, e_i)\,\xi + \tfrac{1}{2}(\ell_s, dB_s)\,\mathscr{R}_{//_s}^{\mathrm{tr}}\xi \\
&= (\ell_s' + \tfrac{1}{2}\,\mathrm{Ric}_{//_s}\,\ell_s, dB_s)\,\xi + R_{//_s}^{E^*}(\ell_s, dB_s)\,\xi \\
&\quad + \tfrac{1}{2}(\ell_s, dB_s)\,\mathscr{R}_{//_s}^{\mathrm{tr}}\xi.
\end{aligned}$$

Similarly,

$$\rho_{//_s}^{\mathrm{tr}} k_s = (\nabla\cdot R^E)_{//_s}^{\mathrm{tr}}\,(\ell_s)\,\xi + (\nabla^{\mathrm{End}(E)}\mathscr{R})_{//_s}^{\mathrm{tr}}\,(\ell_s)\,\xi$$

and hence from Eq. (3.21),

$$\begin{aligned}
\hat{U}_t^k &= \int_0^t \mathscr{Q}_s^{-1}[\,(\ell_s' + \tfrac{1}{2}\,\mathrm{Ric}_{//_s}\,\ell_s, dB_s)\,\xi + R_{//_s}^{E^*}(\ell_s, dB_s)\,\xi \\
&\quad + \tfrac{1}{2}(\ell_s, dB_s)\,\mathscr{R}_{//_s}^{\mathrm{tr}}\xi\,] \\
&\quad + \tfrac{1}{2}\int_0^t \mathscr{Q}_s^{-1}((\nabla\cdot R^E)_{//_s}^{\mathrm{tr}}\,(\ell_s)\,\xi + (\nabla^{\mathrm{End}(E)}\mathscr{R})_{//_s}^{\mathrm{tr}}\,(\ell_s)\,\xi)\,ds \\
&= \int_0^t \mathscr{Q}_s^{-1}[\,(\ell_s' + \tfrac{1}{2}\,\mathrm{Ric}_{//_s}\,\ell_s, dB_s)\,\xi + R_{//_s}^{E^*}(\ell_s, \delta B_s)\,\xi \\
&\quad + \tfrac{1}{2}(\ell_s, d\overleftarrow{B}_s)\,\mathscr{R}_{//_s}^{\mathrm{tr}}\xi\,] \\
&= (V_t^{\ell})^{\mathrm{tr}}\,\xi,
\end{aligned}$$

where in the last equality recall that $R^{E^*}(v, w) = -(R^E(v, w))^{\mathrm{tr}}$. This computation along with Eq. (4.5) of Theorem 4.3 gives:

If $\ell_T = 0$ and $\ell_0 = v \in T_x M$ then

$$\begin{aligned}\langle \nabla_v e^{TL/2}\alpha(x), \xi\rangle &= \langle \nabla e^{TL/2}\alpha(x), v \otimes \xi\rangle \\ &= -\mathbb{E}[\langle Q_T //_T^{-1} a_T(X_T(x)), (V_T^\ell)^{\mathrm{tr}}\, \xi\rangle]\end{aligned}$$

which implies Eq. (5.24) since $\xi \in E_x^*$ is arbitrary. Similarly if $\ell_T = v \in T_x M$ and $\ell_0 = 0$, then by Eq. (4.6) of Theorem 4.3

$$\mathbb{E}[\langle //_T^{-1} \nabla\alpha(X_T(x)), v \otimes \xi\rangle] = \mathbb{E}[\langle Q_T //_T^{-1} \alpha(X_T(x)), (V_T^\ell)^{\mathrm{tr}}\, \xi\rangle].$$

By Eq. (4.15) of Proposition 4.5,

$$\begin{aligned}\langle (e^{T\tilde{\Box}/2}\,\nabla\alpha)_v, \xi\rangle &:= \langle (e^{T\tilde{\Box}/2}\,\nabla\alpha)(x), v \otimes \xi\rangle \\ &= \mathbb{E}[\langle //_T^{-1} \nabla\alpha(X_T(x)), v \otimes \xi\rangle].\end{aligned}$$

Combining the last two equations proves Eq. (5.25). ▮

COROLLARY 5.17. *Let $\alpha \in L^2(E)$ and $T > 0$ and $\mathcal{Q}_t$ be the* End$(T_x M)$-*valued process defined as the solution to the ordinary differential equation*

$$\dot{\mathcal{Q}} + \tfrac{1}{2}\,\mathrm{Ric}_{//_t}\, \mathcal{Q}_t = 0 \qquad \text{with} \quad \mathcal{Q}_0 = \mathrm{id}_{T_x M}.$$

If $h_T = 0$ and $h_0 = v \in T_x M$ then

$$\nabla_v e^{T\Box/2}\alpha = -\mathbb{E}\left[\left\{\int_0^T [(\mathcal{Q}_t \dot{h}_t, dB_t) + R^E_{//_t}(\delta B_t, \mathcal{Q}_t h_t)]\right\} //_T^{-1} \alpha(X_T)\right]. \quad (5.26)$$

Assume in addition that $\alpha \in \Gamma(E)$. If $h_T = \mathcal{Q}_T^{-1} v \in T_x M$ and $h_0 = 0$, then

$$(e^{T\tilde{\Box}/2}\,\nabla\alpha)_v = \mathbb{E}\left[\left\{\int_0^T [(\mathcal{Q}_t \dot{h}_t, dB_t) + R^E_{//_t}(\delta B_t, \mathcal{Q}_t h_t)]\right\} //_T^{-1} \alpha(X_T)\right] \quad (5.27)$$

where $\tilde{E} = T^*M \otimes E$.

Proof. This follows from an application of Theorem 5.16 with $\mathcal{R} = 0$ and $\ell_t := \mathcal{Q}_t h_t$. ▮

As an application of Theorem 5.16 we may recover the following result in [19] (see Theorem 4.1 and Corollary 4.3). A better version of this formula has already appeared in Theorem 5.10 above.

COROLLARY 5.18. *Suppose that Y is a C^1 vector field on M and h is an L^1-finite energy path with values in $\mathbb{R}$ such that $h_0 = 0$ and $h_T = 1$, then*

$$\begin{aligned}&\mathbb{E}[(\nabla \cdot Y)(X_T(x))] \\ &\quad = \mathbb{E}\left(//_T^{-1} Y(X_T(x)), \int_0^T (h_s'\, dB_s - \tfrac{1}{2} h_s\, \mathrm{Ric}_{//_s}\, d\overleftarrow{B}_s)\right) \qquad (5.28).\end{aligned}$$

Proof. By Theorem 5.16 with $E = TM$, $\mathcal{R} = 0$ (and hence $Q = \mathrm{id}$) and $\ell_0 = 0$, we have

$$\mathbb{E}[//_T^{-1}(\nabla_{//_T \ell_T}\alpha)(X_T(x))] = \mathbb{E}[V_T^{\ell} //_T^{-1}\alpha(X_T(x))], \tag{5.29}$$

where

$$V_t^{\ell} := \int_0^t (\ell'_s + \tfrac{1}{2}\,\mathrm{Ric}_{//_s}\,\ell_s, dB_s) - \int_0^t R_{//_s}^{TM}(\ell_s, \delta B_s).$$

Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for T_xM. Replace ℓ in Eq. (5.29) by $\ell_s^i := h_s e_i$, take the inner product with e_i and then sum on i to find:

$$\begin{aligned}
\mathbb{E}[(\nabla \cdot Y)(X_T(x))] &= \sum_{i=1}^n \mathbb{E}[(\nabla_{//_T e_i} Y(X_T(x)), //_T e_i)] \\
&= \sum_{i=1}^n \mathbb{E}[(//_T^{-1}\,\nabla_{//_T \ell_T^i} Y(X_T(x)), e_i)] \\
&= \sum_{i=1}^n \mathbb{E}[(V_T^{\ell^i} //_T^{-1}\, Y(X_T(x)), e_i)] \\
&= \sum_{i=1}^n \mathbb{E}[(//_T^{-1}\, Y(X_T(x)), (V_T^{\ell^i})^*\, e_i)].
\end{aligned}$$

This finishes the proof since

$$\begin{aligned}
\sum_{i=1}^n (V_T^{\ell^i})^*\, e_i &= \int_0^T \sum_{i=1}^n [(h'_s e_i + \tfrac{1}{2} h_s\,\mathrm{Ric}_{//_s}\, e_i, dB_s)\, e_i \\
&\qquad + R_{//_s}^{TM}(h_s e_i, \delta B_s)\, e_i] \\
&= \int_0^T ((h'_s + \tfrac{1}{2} h_s\,\mathrm{Ric}_{//_s})\, dB_s - h_i\,\mathrm{Ric}_{//_s}\,\delta B_s)
\end{aligned}$$

and

$$\begin{aligned}
&\tfrac{1}{2} h_s\,\mathrm{Ric}_{//_s}\, dB_s - h_s\,\mathrm{Ric}_{//_s}\,\delta B_s \\
&\quad = \tfrac{1}{2} h_s\,\mathrm{Ric}_{//_s}\, dB_s - h_s\,\mathrm{Ric}_{//_s}\, dB_s - \tfrac{1}{2} h_s\, d\,\mathrm{Ric}_{//_s}\, dB_s \\
&\quad = -\tfrac{1}{2} h_s\,\mathrm{Ric}_{//_s}\, dB_s - \tfrac{1}{2} h_s\, d\,\mathrm{Ric}_{//_s}\, dB_s \\
&\quad = -\tfrac{1}{2} h_s\,\mathrm{Ric}_{//_s}\, \overleftarrow{dB}_s. \quad \blacksquare
\end{aligned}$$

Remark 5.19. The backwards Itô differential in Eq. (5.28) may be expressed as

$$h_s \operatorname{Ric}_{//_s} \overleftarrow{dB}_s = h_s \operatorname{Ric}_{//_s} dB_s + \tfrac{1}{2} h_s //_s^{-1} \vec{\nabla} \operatorname{scal}(X_s(x))\, ds, \tag{5.30}$$

where $\vec{\nabla}$ scal denotes the gradient of the scalar curvature of M. The proof proceeds as follows,

$$\begin{aligned} h_s \operatorname{Ric}_{//_s} \overleftarrow{dB}_s &= h_s \operatorname{Ric}_{//_s} dB_s + h_s\, d \operatorname{Ric}_{//_s} dB_s \\ &= h_s \operatorname{Ric}_{//_s} dB_s + \sum_{i=1}^{n} h_s //_s^{-1} (\nabla_{//_s e_i} \operatorname{Ric}) //_s e_i\, ds \\ &= h_s \operatorname{Ric}_{//_s} dB_s + h_s //_s^{-1}\, \nabla \cdot \operatorname{Ric}\, ds, \end{aligned} \tag{5.31}$$

where $\nabla \cdot \operatorname{Ric}$ is the vector field on M given by

$$\nabla \cdot \operatorname{Ric} := \sum_{j=1}^{n} (\nabla_{e_j} \operatorname{Ric})\, e_j,$$

where $\{e_j\}_{j=1}^{n}$ is a local orthonormal frame on M. On the other hand for $v \in T_x M$,

$$\begin{aligned} v \operatorname{scal} &= \sum_{i=1}^{n} v(\operatorname{Ric} e_i, e_i) = \sum_{i=1}^{n} (\nabla_v \operatorname{Ric} e_i, e_i) \\ &= \sum_{i,\, j=1}^{n} ((\nabla_v R^{TM})(e_i, e_j)\, e_j, e_i) \end{aligned}$$

which by the Bianchi identity ($d_\nabla R^{TM} = 0$) may be written as

$$\begin{aligned} v \operatorname{scal} &= -\sum_{i,\, j=1}^{n} ((\nabla_{e_j} R^{TM})(v, e_i)\, e_j, e_i) - \sum_{i,\, j=1}^{n} ((\nabla_{e_i} R^{TM})(e_j, v)\, e_j, e_i) \\ &= \sum_{i,\, j=1}^{n} ((\nabla_{e_j} R^{TM})(v, e_i)\, e_i, e_j) + \sum_{i,\, j=1}^{n} ((\nabla_{e_i} R^{TM})(v, e_j)\, e_j, e_i) \\ &= 2 \sum_{j=1}^{n} ((\nabla_{e_j} \operatorname{Ric})\, v, e_j) = 2 \sum_{j=1}^{n} ((\nabla_{e_j} \operatorname{Ric})\, e_j, v) \\ &= 2(\nabla \cdot \operatorname{Ric}, v). \end{aligned}$$

Hence $\nabla \cdot \operatorname{Ric} = \vec{\nabla} \operatorname{scal}/2$ which combined with Eq. (5.31) proves Eq. (5.30).

5.5. *Formulas for* $\nabla e^{t\Delta/2}$ *on* $\Omega^1(M)$

Using Example 2.11, Theorem 5.13 we may write formulas for $\nabla e^{t\Delta/2}$ on $\Omega^k(M)$ for $0 \leqslant k \leqslant n$. Rather than doing this in general, we will content ourselves with the case $k=1$. So in this section let $E = T^*M$, $\tilde{E} = T^*M \otimes T^*M$, and $\Delta = -d^*d - dd^*$ on $\Omega^1(M)$.

PROPOSITION 5.20. *Suppose that* α *is a bounded measurable differential* 1*-form* (*i.e. section of* T^*M) *and* $\xi \in \tilde{E}^*_x = T_xM \otimes T_xM$. *Let* $\{\ell_t\}_{0 \leqslant t \leqslant T}$ *be an* L^1*-finite energy process with values in* $T_xM \otimes T_xM$ (*see Definition* 3.6) *such that* $\ell_T = 0$ *and* $\ell_0 = \xi$. *Then*

$$\langle \nabla e^{T\Delta/2} \alpha(x), \xi \rangle = -\mathbb{E}[\langle Q_T //_T^{-1} \alpha(X_T(x)), U_T^\ell \rangle], \tag{5.32}$$

where

$$U_t^\ell = \int_0^t \mathcal{Q}_s^{-1} A_{dB_s} \tilde{\mathcal{Q}}_s \ell'_s + \tfrac{1}{2} \int_0^t \mathcal{Q}_s^{-1} \rho^{\mathrm{tr}}_{//_s} \tilde{\mathcal{Q}}_s \ell_s \, ds, \tag{5.33}$$

$\rho^{\mathrm{tr}} \in \Gamma(\mathrm{Hom}(TM \otimes TM, TM))$ *is given by*

$$\rho^{\mathrm{tr}}(v \otimes w) = (\nabla_v \mathrm{Ric})\, w - (\nabla \cdot R^{TM})(v)\, w, \tag{5.34}$$

$\mathcal{Q}$ *and* $\tilde{\mathcal{Q}}$ *are defined by Eqs.* (3.11) *and* (3.12) *with* $\mathcal{R}^{\mathrm{tr}} = \mathrm{Ric}$ *and*

$$\tilde{\mathcal{R}}^{\mathrm{tr}} = \mathrm{Ric} \otimes \mathrm{id}_{TM} + 2 \sum_{i=1}^n e_i \otimes R^{TM}(\cdot, e_i) + \mathrm{id}_{TM} \otimes \mathrm{Ric}, \tag{5.35}$$

where (*see Lemma* 5.15)

$$e_i \otimes R^{TM}(\cdot, e_i)(v \otimes w) = e_i \otimes R^{TM}(v, e_i)\, w$$

for $v, w \in T_xM$.

Proof. By Eq. (A.17) of Lemma A.9, we have $\Delta = \square - \mathcal{R}$ where $\mathcal{R} = \mathrm{Ric}^{\mathrm{tr}}$. Let $E = T^*M$, $\tilde{E} = T^*M \otimes T^*M$ and m, ρ and $\tilde{\mathcal{R}}$ be as in Proposition 2.15. By Lemma 5.15 above, $m_v^{\mathrm{tr}} = A_v$, $\tilde{\mathcal{R}}^{\mathrm{tr}}$ is given by Eq. (5.35) and

$$\rho^{\mathrm{tr}} = (\nabla \cdot R^{T^*M})^{\mathrm{tr}} + (\nabla^{\mathrm{End}(T^*M)} \mathrm{Ric}^{\mathrm{tr}})^{\mathrm{tr}}.$$

Since $R^{T^*M}(v, w) = -(R^{TM}(v, w))^{\mathrm{tr}}$ and in general $\nabla_v(A^{\mathrm{tr}}) = (\nabla_v A)^{\mathrm{tr}}$, the previous equation is the same as Eq. (5.34) above. Finally Eq. (5.32) follows from Corollary 5.3. ∎

We may get another formula by using Theorem 5.16 in place of Corollary 5.3.

PROPOSITION 5.21. *Suppose* α *is a bounded measurable* 1-*form on* M *and* ℓ_s *is a* T_xM-*valued* L^1-*finite energy process such that* $\ell_0 = v \in T_xM$ *and* $\ell_T = 0$. *Then*

$$\nabla_v e^{T\Delta/2}\alpha(x) = -\mathbb{E}[\, V_T^\ell Q_T //_T^{-1} \alpha(X_T(x))],$$

where

$$V_t^\ell := \int_0^t [(\ell_s' + \tfrac{1}{2}\,\mathrm{Ric}_{//_s}\, \ell_s, dB_s) - R_{//_s}^{T^*M}(\ell_s, \delta B_s) + \tfrac{1}{2}(\ell_s, \overleftarrow{dB}_s)\,\mathrm{Ric}_{//_s}^{\mathrm{tr}}]\, Q_s^{-1}$$

and Q *solves*

$$\frac{dQ_s}{ds} = -Q_s\,\mathrm{Ric}_{//_s}^{\mathrm{tr}} \qquad \textit{with} \quad Q_0 = \mathrm{id}_{T_x^*M}.$$

See Theorem 5.16 *for the meaning of* δB_s *and* $\overleftarrow{dB}_s$.

Remark 5.22. The expression for V_t^ℓ may be written solely in terms of Itô differentials using

$$\begin{aligned} R_{//_s}^{T^*M}(\ell_s, \delta B_s) &= R_{//_s}^{T^*M}(\ell_s, dB_s) + \tfrac{1}{2} \sum_{i=1}^n (\nabla_{//_s e_i} R^{T^*M})_{//_s} (\ell_s, e_i)\, ds \\ &= R_{//_s}^{T^*M}(\ell_s, dB_s) - \tfrac{1}{2}(\nabla \cdot R^{T^*M})_{//_s}(\ell_s)\, ds \end{aligned}$$

and

$$\begin{aligned} (\ell_s, \overleftarrow{dB}_s)\,\mathrm{Ric}_{//_s}^{\mathrm{tr}} &= (\ell_s, dB_s)\,\mathrm{Ric}_{//_s}^{\mathrm{tr}} + \sum_{i=1}^n (\ell_s, e_i)(\nabla_{//_s e_i}\mathrm{Ric})_{//_s}^{\mathrm{tr}}\, ds \\ &= (\ell_s, dB_s)\,\mathrm{Ric}_{//_s}^{\mathrm{tr}} + (\nabla_{//_s \ell_s}\mathrm{Ric})_{//_s}^{\mathrm{tr}}\, ds. \end{aligned}$$

6. APPLICATIONS FOR NON-COMPACT M

For $E = \Lambda T^*M$ let again $\Delta = \square - \mathscr{R}$ be the Rham–Hodge Laplacian on $\Gamma(E)$ where $\mathscr{R} \in \Gamma(\mathrm{End}\, E)$ denotes the Weitzenböck curvature term, see Proposition A.7 of Appendix A. Further, let $\underline{\mathscr{R}} = \min \mathrm{Spec}\, \mathscr{R}$, i.e.

$$\underline{\mathscr{R}}(x) = \min\{\langle \mathscr{R}_x v, v\rangle : v \in E_x,\, |v| = 1\}, \tag{6.1}$$

and consider the scalar semigroup $P_t^{\underline{\mathscr{R}}}$ as defined in Appendix B Sect. B.1.

Now let $\hat{\Delta}$ be the Friedrichs extension of $\Delta \mid \Gamma_c(E)$ and $P_t a = e^{t/2\hat{\Delta}} a$ be the semigroup on $L^2(E)$ generated by $\hat{\Delta}/2$ and Q_t denote the solution to Eq. (3.1). Then,

$$P_t a(x) = \mathbb{E}[\, Q_t //_t^{-1} a(X_t(x))\, 1_{\{t<\zeta(x)\}}] \tag{6.2}$$

for all $a \in L^2(E)$ with $P_t^{\mathcal{R}}\, |a|(x) < \infty$, see Appendix B Theorem B.4.

On a complete manifold, by the spectral theorem, one has $dP_t\, a = P_t\, da$, and dual to this, $d^*P_t\, a = P_t\, d^*a$, see Section B.2 in Appendix B below. If we drop completeness then these equations are no longer true, even if M is BM-complete, see [59]. But we will show that there always exist Bismut type formulas for $dP_t\, a$ and $d^*P_t\, a$, not involving derivatives of a, independently whether a is smooth or not.

THEOREM 6.1. *Let M be a Riemannian manifold, $a \in L^2(\Lambda T^*M)$ and $x \in M$ such that $P_T^{\mathcal{R}}\, |a|\,(x) < \infty$ for some $T > 0$, further let τ be the first exit time of $X(x)$ from some relatively compact neighborhood of x and $T^* = (T-\varepsilon) \wedge \tau$ for some arbitrary small $\varepsilon > 0$. Then for any $v \in \Lambda T_x M$ the following formulas hold*:

$$\langle (dP_T\, a)_x, v\rangle = -\mathbb{E}\left[\left\langle //_T^{-1} a(X_T(x))\, 1_{\{T<\zeta(x)\}},\, \mathcal{Q}_T \int_0^T \mathcal{Q}_t^{-1}(A_{dB_t} \mathcal{Q}_t \ell'_t)\right\rangle\right]$$

$$\langle (d^*P_T\, a)_x, v\rangle = -\mathbb{E}\left[\left\langle //_T^{-1} a(X_T(x))\, 1_{\{T<\zeta(x)\}},\, \mathcal{Q}_T \int_0^T \mathcal{Q}_t^{-1}(dB_t \wedge \mathcal{Q}_t \ell'_t)\right\rangle\right]$$

for any bounded finite energy process ℓ with values in $\Lambda T_x M$ such that $\ell_0 = v$, $\ell_t = 0$ for all $t \geqslant T^*$, and the property that $\mathbb{E}[(\int_0^{T^*} |\ell'_s|^2\, ds)^{1/2}] < \infty$. If, in addition, $a \in L^2(\Lambda T^*M)$ is bounded on this neighborhood, one can take $\varepsilon = 0$.

Proof. Recall that $\mathcal{Q}_t$ is defined by Eq. (5.10) and let $Q_t := \mathcal{Q}_t^{\mathrm{tr}}$. We fix a relatively compact neighborhood U of x. Then $|(P_{T-t} a)(X_t)|$, $|(dP_{T-t}\, a)(X_t)|$ and $|(d^*P_{T-t}\, a)(X_t)|$, where $X \equiv X_\bullet(x)$ denotes our Brownian motion starting from x, as well as Q_t and $\mathcal{Q}_t$, are all bounded on the stochastic interval $[0, T^*]$. This shows that the local martingales in Proposition 3.2 and in Theorem 3.7 are uniformly integrable martingales when stopped at $t = T^*$. Taking expectations at time 0 and T^* leads to

$$\langle (dP_T\, a)_x, v\rangle = -\mathbb{E}\left[\left\langle Q_{T^*} //_{T^*}^{-1} (P_{T-T^*} a)(X_{T^*}),\, \int_0^{T^*} \mathcal{Q}_t^{-1}(A_{dB_t} Q_t^{\mathrm{tr}} \ell'(t))\right\rangle\right].$$

Note that

$$Q_{T^*}//_{T^*}^{-1}(P_{T-T^*}a)(X_{T^*}) = \mathbb{E}^{\mathscr{F}_{T^*}}[Q_T//_T^{-1}a(X_T)\,1_{\{T<\zeta(x)\}}]$$

which is by definition a bounded $\mathscr{F}_{T^*}$-measurable random variable. This gives the first formula of the Theorem, the second one is derived in a completely analogous way.

If a is bounded, eventually by modifying ℓ, we assume first that $\ell \equiv 0$ already on $[T \wedge \tau, T] \cap [T-\varepsilon, T]$ for some small $\varepsilon > 0$. Finally, this restriction can again be removed in the resulting formulas by letting ε tend to 0, see [58] for technical details. ∎

6.1. *Bismut's Formula*

The following example is taken from [58]. There are similar formulas for $(du)_x$ if u is harmonic on some domain about x, see [58, 60].

THEOREM 6.2. *Let $f: M \to \mathbb{R}$ be a bounded measurable function, $x \in M$ and $v \in T_xM$. Then, for any bounded finite energy process $\{\ell_t\}_{t \in [0,\infty[}$ with values in T_xM such that $\mathbb{E}[(\int_0^{\tau\wedge T} |\ell'_s|^2\,ds)^{1/2}] < \infty$, and the property that $\ell_0 = v$, $\ell_s = 0$ for all $s \geqslant \tau \wedge T$, the following formula holds,*

$$\langle (dP_T f)_x, v\rangle = -\mathbb{E}\left[f(X_T(x))\,1_{\{T<\zeta(x)\}} \int_0^{\tau\wedge T} (\mathscr{Q}_s \ell'_s, dB_s)\right], \quad (6.3)$$

where τ is the first exit time of $X(x)$ from some relatively compact open neighborhood D of x and $\mathscr{Q}$ is the process defined in Eq. (5.13) *of Corollary* 5.9.

On the other hand showing, for instance, that

$$\langle (dP_T f)_x, v\rangle = \mathbb{E}[\langle Q_T//_T^{-1}(df)_{X_T(x)}, v\rangle\,1_{\{T<\zeta(x)\}}]$$

is a quite different matter: it requires the martingale property of

$$Q_t//_t^{-1}(dP_{T-t}f)_{X_t(x)}\,1_{\{t<\zeta(x)\}}, \qquad 0 \leqslant t \leqslant T,$$

which comes down to a question of differentiation under the expectation. In particular, it is necessary for f to be differentiable.

6.2. *Dirichlet Problem for Harmonic Forms*

We conclude this section by specializing our results in case of the Dirichlet problem for harmonic forms on bounded domains. In particular, we present stochastic formulas for differentials and co-differential of harmonic forms on manifolds with boundary. These formulas can be used to prove local Harnack type estimates for harmonic forms in the same way as has been done for harmonic functions in [60].

THEOREM 6.3. *Let M be a compact Riemannian manifold with non-empty boundary ∂M, $a \in \Gamma(\Lambda T^*M)$, $x \in M\backslash\partial M$ and τ be a positive bounded stopping time which is dominated by the first time the Brownian motion $X(x)$ hits the boundary ∂M. (For simplicity, a is assumed to be smooth up to the boundary of M.) Let $\mathcal{Q}_t$ denote the solution to Eq. (5.10) of Theorem 5.7 defined on the stochastic interval $[0, \tau]$ and $Q_t = \mathcal{Q}_t^{\mathrm{tr}}$. If a is harmonic (i.e. $\Delta a \equiv -(d^*d + dd^*)\,a = 0$) on $M\backslash\partial M$, then*

$$a(x) = \mathbb{E}[\,Q_\tau //_\tau^{-1} a(X_\tau(x))] \tag{6.4}$$

and for any $v \in \Lambda T_x M$ the following formulas hold,

$$\langle (da)_x, v\rangle = -\mathbb{E}\left[\left\langle Q_\tau //_\tau^{-1} a(X_\tau(x)), \int_0^\tau \mathcal{Q}_t^{-1}(A_{dB_t}\mathcal{Q}_t \ell'_t)\right\rangle\right] \tag{6.5}$$

$$\langle (d^*a)_x, v\rangle = -\mathbb{E}\left[\left\langle Q_\tau //_\tau^{-1} a(X_\tau(x)), \int_0^\tau \mathcal{Q}_t^{-1}(dB_t \wedge \mathcal{Q}_t \ell'_t)\right\rangle\right], \tag{6.6}$$

where $\{\ell\}_{s \in [0,\infty[}$ is a bounded L^1-finite energy process taking values in $\Lambda T_x M$ such that $\ell_0 = v$, $\ell_\tau = 0$, and the property that $(\int_0^\tau |\ell'_s|^2\,ds)^{1/2} \in L^1$.

Proof. Note that a, da, d^*a extend as bounded sections to M since M is compact. The proof is now essentially the same as the proof of Theorem 5.7 above with $P_t a$ replaced by a and T by τ. The key point is that the local martingales of Proposition 3.2 and Theorem 3.7 are easily seen to be martingales up to the first exit time τ. Hence the optional sampling theorem applies to give stopped versions of Proposition 4.5, Corollary 5.3 and Corollary 5.5 from which Eqs. (6.4), (6.5) and (6.6) follow. ∎

Remark 6.4. The formulas (6.4), (6.5), and (6.6) in Theorem 6.3 hold as well when τ is the first time the Brownian motion $X(x)$ hits the boundary, ∂M, provided that

$$\mathbb{E}\left[\exp\left(-\tfrac{1}{2}\int_0^\tau \mathcal{R}(X_s(x)) \wedge 0\,ds\right)\right] < \infty, \tag{6.7}$$

where $\mathcal{R}$ is given by Eq. (6.1) and the L^1-finite energy process $\{\ell\}_{s\in[0,\infty[}$ in Eqs. (6.5) and (6.6) is chosen with the additional restriction that $\ell_s = 0$ if $s \geqslant \tau \wedge t_0$ for some $t_0 \in (0, \infty)$. Indeed, if τ denotes the first hitting time of the boundary ∂M then the formulas (6.4), (6.5), and (6.6) hold with $\tau \wedge T$ instead of τ for any $T \geqslant t_0$, and condition (6.7) gives the existence of an L^1-dominating function for $\{Q_{\tau\wedge T}\}_{T>0}$ (see estimate (B.11) of Appendix B) which allows by the dominated convergence theorem to pass to the limit as $T \to \infty$.

The method used in Theorem 6.3 can easily be adapted to other situations, for instance, if a is smooth on $M\backslash\partial M$ and extends only continuously to ∂M.

For technical details about how to construct finite energy processes ℓ satisfying the conditions in Theorems 6.1, 6.2 and 6.3 the reader is referred to [58] and [60].

7. HIGHER DERIVATIVE FORMULAS

We can get higher derivative formulas by iterating our previous formulas following the ideas of Elworthy and Li in [24] and [26]. In order to carry this out, we will need a minor extension of the results in Section 4. Let $\{\mathcal{F}_t\}$ denote the filtration associated to the Brownian motion $X_t(x)$ and $\mathbb{E}^{\mathcal{F}_t}$ denote conditional expectation relative to the σ-field $\mathcal{F}_t$. As usual we will assume that E, $\tilde{E}$, L, $\tilde{L}$, m satisfy Assumption 2 at the beginning of Section 4. For simplicity, let us assume that M is a compact manifold. The next theorem is the conditioned version of Theorem 3.7 and Corollaries 5.3 and 5.5.

THEOREM 7.1. *Let $0\leqslant\tau<T$, α be a bounded measurable section of E,*

$$a_t := e^{(T-t)\,L/2}\alpha,$$

$\{\ell_t\}_{\tau\leqslant t\leqslant T}$ be an L^1-finite energy process with values in $\tilde{E}^*_x$ (see Definition 3.6), and N and $\tilde{N}$ be as in Eqs. (3.4) and (3.5). For $0\leqslant\tau\leqslant t\leqslant T$ let

$$U^{\ell}_{\tau,\,t} := \int_\tau^t \mathcal{Q}_s^{-1} m^{\mathrm{tr}}_{dB_s} \tilde{\mathcal{Q}}_s \tilde{\mathcal{Q}}_\tau^{-1} \ell'_s + \tfrac{1}{2}\int_\tau^t \mathcal{Q}_s^{-1} \rho^{\mathrm{tr}}_{//_s} \tilde{\mathcal{Q}}_s \tilde{\mathcal{Q}}_\tau^{-1} \ell_s \, ds \tag{7.1}$$

and

$$Z^{\ell}_{\tau,\,t} := \langle \tilde{N}_t, \tilde{\mathcal{Q}}_\tau^{-1}\ell_t\rangle - \langle N_t, U^{\ell}_{\tau,\,t}\rangle. \tag{7.2}$$

Then $\{Z^{\ell}_{\tau,\,t}\}_{\tau\leqslant t\leqslant T}$ is an $\{\mathcal{F}_t\}$-martingale with

$$\begin{aligned} dZ^{\ell}_{\tau,\,t} &= \langle \tilde{Q}_t //_t^{-1} \nabla_{//_t\, dB_t} Da_t(X_t(x)), \tilde{\mathcal{Q}}_\tau^{-1}\ell_t\rangle \\ &\quad - \langle Q_t //_t^{-1} \nabla_{//_t\, dB_t} a_t(X_t(x)), U^{\ell}_{\tau,\,t}\rangle \\ &\quad - \langle N_t, \mathcal{Q}_t^{-1} m^{\mathrm{tr}}_{dB_t} \tilde{\mathcal{Q}}_t \tilde{\mathcal{Q}}_\tau^{-1} \ell'_t\rangle \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} \langle //_\tau^{-1} Da_\tau(X_\tau(x)), \ell_\tau\rangle &= \mathbb{E}^{\mathcal{F}_\tau}[\langle \tilde{Q}_T //_T^{-1} Da_T(X_T(x)), \tilde{\mathcal{Q}}_\tau^{-1}\ell_T\rangle] \\ &\quad - \mathbb{E}^{\mathcal{F}_\tau}[\langle Q_T //_T^{-1} a_T(X_T(x)), U^{\ell}_{\tau,\,T}\rangle]. \end{aligned} \tag{7.4}$$

Therefore,

1. *if $\ell_T=0$ then*

$$\begin{aligned} &\langle //_\tau^{-1} De^{(T-\tau)\,L/2}\alpha(X_\tau(x)), \ell_\tau\rangle \\ &\quad = -\mathbb{E}^{\mathcal{F}_\tau}[\langle Q_T //_T^{-1} \alpha(X_T(x)), U^{\ell}_{\tau,\,T}\rangle], \end{aligned} \tag{7.5}$$

2. *or if* $\ell_\tau = 0$ *then*

$$\mathbb{E}^{\mathscr{F}_\tau}[\langle \tilde{Q}_T //_T^{-1} D\alpha(X_T(x)), \tilde{\mathscr{Q}}_T^{-1} \ell_T \rangle] = \mathbb{E}^{\mathscr{F}_\tau}[\langle Q_T //_T^{-1} \alpha(X_T(x)), U^{\ell}_{\tau,T} \rangle]. \tag{7.6}$$

Remark 7.2. The results in this theorem are direct analogues of Theorems 3.7, and Corollaries 5.3 and 5.5. In fact this theorem could be deduced using these results along with the strong Markov property of the Brownian motion $X_t(x)$. (If H: $C(\mathbb{R}_+; M) \to \mathbb{R}_+$ is bounded measurable and τ a finite stopping time, then

$$\mathbb{E}^{\mathscr{F}_\tau}[H \circ X_{\tau+\cdot}(x)] = \mathbb{E}[H \circ X_\bullet(y)]|_{y = X_\tau(x)} \text{ a.s.}) \tag{7.7}$$

We will sketch a proof here using the methods already developed in Section 3.

Proof. Notice that $dU^{\ell}_{\tau,t} = \mathscr{Q}_t^{-1} m^{\mathrm{tr}}_{//_t dB_t} \tilde{\mathscr{Q}}_t \tilde{\mathscr{Q}}_\tau^{-1} \ell'_t + \frac{1}{2} \mathscr{Q}_t^{-1} \rho^{\mathrm{tr}}_{//_t} \tilde{\mathscr{Q}}_t \tilde{\mathscr{Q}}_\tau^{-1} \ell_t \, dt$ and hence by Theorem 3.4, $U^{\ell}_{\tau,t}$ and $\tilde{\mathscr{Q}}_\tau^{-1} \ell_t$ for $\tau \leqslant t \leqslant T$ are a dual pair, i.e. $\{Z^{\ell}_{\tau,t}\}_{\tau \leqslant t \leqslant T}$ is an $\{\mathscr{F}_t\}$-martingale. The same computations leading to Eq. (3.16) in Theorem 3.7 proves Eq. (7.3). (In fact Eq. (7.3) is Eq. (3.16) with ℓ_t replaced by $\tilde{\mathscr{Q}}_\tau^{-1} \ell_t$ and the lower limits in the integrals defining $U_t^{\tilde{\mathscr{Q}}_\tau^{-1} \ell}$ being changed from 0 to τ.)

Since $Z^{\ell}_{\tau,t}$ is an $\{\mathscr{F}_t\}$-martingale, $Z^{\ell}_{\tau,\tau} = \mathbb{E}^{\mathscr{F}_\tau}[Z^{\ell}_{\tau,T}]$. This identity is the same as Eq. (7.4) because

$$\begin{aligned} Z^{\ell}_{\tau,\tau} &= \langle \tilde{N}_\tau, \tilde{\mathscr{Q}}_\tau^{-1} \ell_\tau \rangle = \langle \tilde{Q}_\tau //_\tau^{-1} Da_\tau(X_\tau(x)), \tilde{\mathscr{Q}}_\tau^{-1} \ell_\tau \rangle \\ &= \langle //_\tau^{-1} Da_\tau(X_\tau(x)), \ell_\tau \rangle \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}^{\mathscr{F}_\tau}[Z^{\ell}_{\tau,T}] &= \mathbb{E}^{\mathscr{F}_\tau}[\langle \tilde{N}_T, \tilde{\mathscr{Q}}_\tau^{-1} \ell_T \rangle] - \mathbb{E}^{\mathscr{F}_\tau}[\langle N_T, U^{\ell}_{\tau,T} \rangle] \\ &= \mathbb{E}^{\mathscr{F}_\tau}[\langle \tilde{Q}_T //_T^{-1} Da_T(X_T(x)), \tilde{\mathscr{Q}}_\tau^{-1} \ell_T \rangle] \\ &\quad - \mathbb{E}^{\mathscr{F}_\tau}[\langle Q_T //_T^{-1} a_T(X_T(x)), U^{\ell}_{\tau,T} \rangle]. \end{aligned}$$

Finally, Eqs. (7.5) and (7.6) are immediate consequences of Eq. (7.4). ∎

The following Corollary is the conditioned analogue of Eq. (4.10) of Corollary 4.4.

COROLLARY 7.3. *Let* $0 \leqslant \tau < T$, α *be a bounded measurable section of* E, $\{\bar{\ell}_t\}_{\tau \leqslant t \leqslant T}$ *be an* L^1*-finite energy process with values in* $\mathbb{R}$ *such that* $\bar{\ell}_T = 0$. *For* $0 \leqslant \tau \leqslant t \leqslant T$ *let*

$$V^{\bar{\ell}}_{\tau,t} = \int_\tau^t \tilde{Q}_\tau^{-1} \tilde{Q}_s (\bar{\ell}'_s m_{dB_s} + \tfrac{1}{2} \rho_{//_s} \bar{\ell}_s \, ds) \, Q_s^{-1}. \tag{7.8}$$

Then

$$//_\tau^{-1} De^{(T-\tau)L/2}\alpha(X_\tau(x))\,\bar{\ell}_\tau = -\mathbb{E}^{\mathscr{F}_\tau}[V^{\bar{\ell}}_{\tau,T}Q_T//_T^{-1}\alpha(X_T(x))]. \quad (7.9)$$

Proof. Let ξ be a bounded $\mathscr{F}_\tau$-measurable random variable with values in $\tilde{E}^*_x$ and let $\ell_t = \bar{\ell}_t\xi$. In this case

$$\begin{aligned} U^\ell_{\tau,T} &= \int_\tau^T \mathscr{Q}_s^{-1} m^{\mathrm{tr}}_{dB_s}\tilde{\mathscr{Q}}_s\tilde{\mathscr{Q}}_\tau^{-1}\bar{\ell}'_s\xi + \tfrac{1}{2}\int_\tau^T \mathscr{Q}_s^{-1}\rho^{\mathrm{tr}}_{//_s}\tilde{\mathscr{Q}}_s\tilde{\mathscr{Q}}_\tau^{-1}\bar{\ell}_s\xi\,ds \\ &= (V^{\bar{\ell}}_{\tau,T})^{\mathrm{tr}}\,\xi, \end{aligned}$$

where $V^{\bar{\ell}}_{\tau,T}$ is defined in Eq. (7.8). Hence by Eq. (7.5) above,

$$\begin{aligned} \langle //_\tau^{-1} De^{(T-\tau)L/2}\alpha(X_\tau(x)), \bar{\ell}_\tau\xi\rangle &= -\mathbb{E}^{\mathscr{F}_t}[\langle Q_T//_T^{-1}\alpha(X_T(x)), (V^{\bar{\ell}}_{\tau,T})^{\mathrm{tr}}\,\xi\rangle] \\ &= -\langle \mathbb{E}^{\mathscr{F}_\tau}[V^{\bar{\ell}}_{\tau,T}Q_T//_T^{-1}\alpha(X_T(x))], \xi\rangle. \end{aligned}$$

Since ξ is arbitrary, this proves Eq. (7.9). ∎

7.1. *Higher Derivative Formula for Dirac Operators*

Let $E = \tilde{E} = S \to M$ be a spinor bundle over M, $m = \gamma$ be the Clifford multiplication, $D = D_\gamma$ the Dirac operator, and $L = \tilde{L} = -D^2$. Recall that in this case $\rho = 0$, and $\mathscr{R} = \tilde{\mathscr{R}} = \frac{1}{4}\,\mathrm{scal}$.

THEOREM 7.4. *Suppose that* $0 < T_1 < T$ *and* ℓ *is an* L^1*-finite energy process with values in* $\mathbb{R}$ *such that* $\ell_0 = 2$, $\ell_{T_1} = 1$ *and* $\ell_T = 0$, *then*

$$\begin{aligned} &(D^2 e^{-TD^2/2}\alpha)(x) \\ &\quad = \mathbb{E}[e^{-(1/8)\int_0^T \mathrm{scal}(X_t(x))\,dt}\gamma_{\int_0^{T_1}\ell'_{s_1}dB_{s_1}}\gamma_{\int_{T_1}^T \ell'_{s_2}dB_{s_2}}//_T^{-1}\alpha(X_T(x))] \end{aligned} \quad (7.10)$$

where α *is a bounded measurable section of* S. *For example, if* $\ell'_s = -(T_1^{-1}1_{[0,T_1)} + (T-T_1)^{-1}1_{[T_1,T]})$, *Eq.* (7.10) *becomes*

$$\begin{aligned} &(D^2 e^{-TD^2/2}\alpha)(x) \\ &= \frac{1}{T_1(T-T_1)}\mathbb{E}[e^{-(1/8)\int_0^T \mathrm{scal}(X_t(x))\,dt}\gamma_{B_{T_1}}\gamma_{B_T - B_{T_1}}//_T^{-1}\alpha(X_T(x))]. \end{aligned} \quad (7.11)$$

Proof. Since $\mathscr{R} = \tilde{\mathscr{R}} = \frac{1}{2}\,\mathrm{scal}$ and $\rho = 0$,

$$Q_t = \tilde{Q}_t = e^{-(1/8)\int_0^t \mathrm{scal}(X_s(x))\,ds},$$

$$V^{\bar{\ell}}_{\tau,t} = e^{(1/8)\int_0^\tau \mathrm{scal}(X_s(x))\,ds}\int_\tau^t \bar{\ell}'_s\gamma_{dB_s} = e^{(1/8)\int_0^\tau \mathrm{scal}(X_s(x))\,ds}\gamma_{\int_\tau^t \bar{\ell}'_s\,dB_s}$$

and hence by Corollary 7.3, for $0 \leqslant \tau < t$,

$$//_{\tau}^{-1} De^{-(t-\tau) D^2/2} \alpha(X_\tau(x)) \bar{\ell}_\tau = -\mathbb{E}^{\mathscr{F}_\tau}[e^{-(1/8)\int_\tau^t \mathrm{scal}(X_s(x))\, ds} \gamma_{\int_\tau^t \bar{\ell}'_s\, dB_s} //_t^{-1} \alpha(X_t(x))], \tag{7.12}$$

where $\bar{\ell}$ is an $\mathbb{R}$-valued L^1-finite energy process such that $\bar{\ell}_t = 0$.

Let $h := \ell|_{[0, T_1]} - 1$ and $k := \ell|_{[T_1, T]}$ so that h and k are L^1-finite energy processes on $[0, T_1]$ and $[T_1, T]$ respectively such that $h_0 = 1$, $h_{T_1} = 0$, $k_{T_1} = 1$, and $k_0 = 0$. Using the semigroup property of e^{-tD^2} and the fact that D commutes with e^{-tD^2},

$$D^2 e^{-TD^2/2} \alpha = De^{-T_1 D^2/2} De^{-(T-T_1) D^2/2} \alpha.$$

By Eq. (7.12) with $\tau = 0$ and $t = T_1$,

$$(D^2 e^{-TD^2/2} \alpha)(x) = -\mathbb{E}[e^{-(1/8)\int_0^{T_1} \mathrm{scal}(X_t(x))\, dt} \gamma_{\int_0^{T_1} h'_s\, dB_s} //_{T_1}^{-1} De^{-(T-T_1) D^2/2} \alpha(X_{t_1}(x))]$$

and by Eq. (7.12) with $\tau = T_1$ and $t = T$

$$//_{T_1}^{-1} De^{-(T-T_1) D^2/2} \alpha(X_{T_1}(x)) = -\mathbb{E}^{\mathscr{F}_{T_1}}[e^{-(1/8)\int_{T_1}^T \mathrm{scal}(X_s(x))\, ds} \gamma_{\int_{T_1}^T k'_s\, dB_s} //_T^{-1} \alpha(X_T(x))].$$

Combining these two equations implies that

$$\begin{aligned}(D^2 e^{-TD^2/2} \alpha)(x) &= \mathbb{E}[e^{-(1/8)\int_0^T \mathrm{scal}(X_t(x))\, dt} \gamma_{\int_0^{T_1} h'_{s_1}\, dB_{s_1}} \gamma_{\int_{T_1}^T k'_{s_2}\, dB_{s_2}} //_T^{-1} \alpha(X_T(x))] \\ &= \mathbb{E}[e^{-(1/8)\int_0^T \mathrm{scal}(X_t(x))\, dt} \gamma_{\int_0^{T_1} \ell'_{s_1}\, dB_{s_1}} \gamma_{\int_{T_1}^T \ell'_{s_2}\, dB_{s_2}} //_T^{-1} \alpha(X_T(x))]. \quad \blacksquare\end{aligned}$$

Remark 7.5. The method used in this proof already appears in the work of Elworthy and Li, see [24] in the context of 0-forms on a manifold and [26] for proving formulas for $dd^* P_t a$, $d^*d P_t a$, $\Delta^k P_t a$ where a is a differential form.

Remark 7.6. Let $0 < T_1 < T_2 < \cdots < T_{n-1} < T_n = T$. The previous Theorem may easily be extended to give

$$(D^n e^{-TD^2/2} \alpha)(x) = \mathbb{E}\left[e^{-(1/8)\int_0^T \mathrm{scal}(X_t(x))\, dt} \int_J \ell'_{s_1} \ell'_{s_2} \cdots \ell'_{s_n} \gamma_{dB_{s_1}} \gamma_{dB_{s_2}} \cdots \gamma_{dB_{s_n}} //_T^{-1} a(X_T(x)) \right],$$

where ℓ is an $\mathbb{R}$-valued L^1-finite energy process such that $\ell_0 = n$, $\ell_{T_1} = n-1, ..., \ell_{T_{n-1}} = 1$ and $\ell_{T_n} = 0$ and

$$J = \{(s_1, s_2, ..., s_n): 0 \leqslant s_1 \leqslant T_1 \leqslant s_2 \leqslant T_2 \leqslant \cdots \leqslant T_{n-1} \leqslant s_n \leqslant T_n\}.$$

7.2. *A Hessian Formula*

In this section we will work out a formula for

$$\mathrm{Hess}(e^{T\Delta/2}f) := \nabla\, de^{T\Delta/2}f,$$

where f is a bounded measurable function on M.

Similar formulas hold for $\nabla\, de^{T\Delta/2}\alpha$, where α is a differential form on M. Related Hessian formulas may also be found in Norris [44], Elworthy and Li [24, 26], Stroock and Turetsky [54, 55] and Hsu [35, 34].

THEOREM 7.7. *Let M be a compact Riemannian manifold, $\xi \in T_xM \otimes T_xM$ for some $x \in M$, further $f\colon M \to \mathbb{R}$ be a bounded measurable function on M. Also for $0 < T_1 < T$, let $\{\ell_s\}_{0 \leqslant s \leqslant T_1}$ be an L^1-finite energy process with values in $T_xM \otimes T_xM$ such that $\ell_0 = \xi$ and $\ell_{T_1} = 0$, and let $\{h_s\}_{T_1 \leqslant s \leqslant T}$ be an $\mathbb{R}$-valued L^1-finite energy process such that $h_{T_1} = 1$ and $h_T = 0$. Then*

$$\langle \mathrm{Hess}(e^{T\Delta/2}f)(x), \xi \rangle = \mathbb{E}\left[f(X_T(x)) \int_{T_1}^{T} h'_s(\mathcal{Q}_s U^{\ell}_{T_1}, dB_s) \right], \qquad (7.13)$$

where $\mathcal{Q}$ solves

$$\mathcal{Q}'_s = -\tfrac{1}{2}\,\mathrm{Ric}_{//_s}\, \mathcal{Q}_s \qquad \text{with} \quad \mathcal{Q}_0 = \mathrm{id}_{T_xM} \qquad (7.14)$$

and $U^{\ell}_{T_1}$ is given in Eq. (5.33) *of Proposition* 5.20. (*The process $U^{\ell}_{T_1}$ depends on the curvature tensor and its first derivatives.*)

Proof. Since $d\Delta = \Delta d$,

$$\nabla\, de^{T\Delta/2}f = \nabla\, de^{T_1\Delta/2}e^{(T-T_1)\,\Delta/2}f = \nabla e^{T_1\Delta/2}\, de^{(T-T_1)\,\Delta/2}f.$$

Consequently by Proposition 5.20,

$$\begin{aligned} \langle \nabla\, de^{T\Delta/2}f, \xi \rangle &= -\mathbb{E}[\langle Q_{T_1} //_{T_1}^{-1}\, de^{(T-T_1)\,\Delta/2}f(X_{T_1}(x)), U^{\ell}_{T_1} \rangle] \\ &= -\mathbb{E}[\langle //_{T_1}^{-1}\, de^{(T-T_1)\,\Delta/2}f(X_{T_1}(x)), \mathcal{Q}_{T_1} U^{\ell}_{T_1} \rangle]. \end{aligned} \qquad (7.15)$$

Suppose now that $\{k_s\}_{T_1 \leqslant s \leqslant T}$ is a T_xM-valued L^1-finite energy process such that $k_T = 0$ and $k_{T_1} = \mathcal{Q}_{T_1} U^{\ell}_{T_1}$. By Eq. (7.5) of Theorem 7.1 above applied to the case where $E = M \times \mathbb{R}$, $\tilde{E} = T^*M$, $L = \Delta^{(0)}$, $\tilde{L} = \Delta^{(1)}$, $\mathcal{R} = 0$, $\tilde{\mathcal{R}} = \mathrm{Ric}^{\mathrm{tr}}$ and $m = C$ (as in item 1. of Example 2.11) gives

$$\langle //_{T_1}^{-1}\, de^{(T-T_1)\,\Delta/2}f(X_{T_1}(x)), \mathcal{Q}_{T_1} U^{\ell}_{T_1} \rangle = -\mathbb{E}^{\mathcal{F}_{T_1}}[f(X_T(x))\, U^k_{T_1, T}], \qquad (7.16)$$

where

$$U^k_{T_1,T} = \int_{T_1}^{T} C^{\mathrm{tr}}_{dB_s}\, \mathcal{Q}_s \mathcal{Q}_{T_1}^{-1} k'_s = \int_{T_1}^{T} (\mathcal{Q}_s \mathcal{Q}_{T_1}^{-1} k'_s, dB_s). \tag{7.17}$$

(Notice that when applying Theorem 7.1, $\mathcal{Q} = \mathrm{id}$ and $\tilde{\mathcal{Q}}$ is the $\mathcal{Q}$ defined by Eq. (7.14) above.) Plugging Eq. (7.16) into Eq. (7.15) shows that

$$\langle \nabla\, de^{T\Delta/2} f, \xi \rangle = \mathbb{E}\left[f(X_T(x)) \int_{T_1}^{T} (\mathcal{Q}_s \mathcal{Q}_{T_1}^{-1} k'_s, dB_s) \right].$$

Taking $k_s := h_s \mathcal{Q}_{T_1} U^{\ell}_{T_1}$ in this formula implies Eq. (7.13) which proves the theorem. ∎

Remark 7.8. In a more general setting, it is possible to develop higher derivative formulas in the following situation. Let $D_1: \Gamma(E) \to \Gamma(E')$ be a "Dirac type" operator such that $D_1 L - L' D_1 = 0$, assuming conditions to ensure that $D_1 e^{-tL/2} = e^{-tL'/2} D_1$. Let $D_2: \Gamma(E') \to \Gamma(E'')$ be another Dirac type operator such that $D_2 L' - L'' D_2 = \rho$. Then using the ideas described above, one can derive a stochastic representation formula for $D_2 D_1 e^{-tL/2}$.

APPENDIX A: DIFFERENTIAL GEOMETRIC NOTATION AND IDENTITIES

A.1. *Conventions on Differential Forms*

Let V and W be finite dimensional vector spaces and let $\Lambda V^* := \bigoplus_k \Lambda^k V^*$ be the exterior algebra over V^*. As is usual, we will identify elements of $\Lambda^k V^* \otimes W$ and alternating k-forms on V with values in W. Our convention for doing this is to define, when $\alpha = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k \otimes w$,

$$\alpha(v_1, v_2, ..., v_k) := \det[\{\alpha_i(v_j)\}_{i,j=1}^{k}]\, w \tag{A.1}$$

for all $\{v_i\}_{i=1}^k$ in V. Eq. (A.1) gives rise to the pairing $\langle \cdot, \cdot \rangle: \Lambda^k V^* \times \Lambda^k V \to \mathbb{R}$ determined by

$$\langle \alpha, v_1 \wedge v_2 \wedge \cdots \wedge v_k \rangle = \alpha(v_1, v_2, ..., v_k). \tag{A.2}$$

This pairing allows us to identify $(\Lambda^k V)^*$ with $\Lambda^k V^*$.

Suppose V and W are equipped with inner products $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_W$ respectively. Given this data, we may define the inner product of $\alpha \in \Lambda^k V^* \otimes W$ and $\beta \in \Lambda^\ell V^* \otimes W$ by

$$(\alpha, \beta) = \delta_{k,\ell} \frac{1}{k!} \sum_{v_1, v_2, \ldots, v_k \in \Gamma} (\alpha(v_1, v_2, \ldots, v_k), \beta(v_1, v_2, \ldots, v_k))_W$$
$$= \delta_{k,\ell} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} (\alpha(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}),$$
$$\beta(e_{i_1} \wedge e_{i_1} \cdots \wedge e_{i_k}))_W, \tag{A.3}$$

where $\Gamma = \{e_1, e_2, \ldots, e_n\}$ is any orthonormal basis of V. It may be checked that $(\cdot, \cdot)$ on $\Lambda^k V^* \otimes W$ is the unique inner product with the property that

$$(\alpha_1 \wedge \cdots \wedge \alpha_k \otimes w, \beta_1 \wedge \cdots \wedge \beta_\ell \otimes w')$$
$$= \delta_{k,\ell} \det[\{(\alpha_i, \beta_j)_{V^*}\}_{i,j=1}^k] \, (w, w')_W$$

for all $\alpha_i, \beta_j \in V^*$ and $w, w' \in W$. The dual inner product on ΛV relative to the pairing in Eq. (A.2) is determined by

$$(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_k, \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_\ell) = \delta_{k,\ell} \det[\{(\xi_i, \eta_j)_V\}_{i,j=1}^k]$$

as is easily checked because, by Eq. (A.3),

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} : 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n\}$$

is an orthonormal basis for $\Lambda^k V$.

DEFINITION A.1. For $v \in V$, the creation (exterior product) operator C_v is the linear operator on $\Lambda V \oplus \Lambda V^*$ given by $C_v \xi = v \wedge \xi$ and $C_v \alpha := (v, \cdot)_V \wedge \alpha$ for $\xi \in \Lambda V$ and $\alpha \in \Lambda V^*$. The annihilation (interior product) operator A_v is the linear operator on $\Lambda V \oplus \Lambda V^*$ given by the adjoint of C_v. Notice that $C_v(\Lambda^k V \oplus \Lambda^k V^*) = \Lambda^{k+1} V \oplus \Lambda^{k+1} V^*$ and $A_v(\Lambda^k V \oplus \Lambda^k V^*) = \Lambda^{k-1} V \oplus \Lambda^{k-1} V^*$.

We have the following well known (and easily checked) facts about C and A.

LEMMA A.2. *Let A be as in Definition* A.1, *then*

1. (Formula for A_v) *For $v, v_1, v_2, \ldots, v_k \in V$ and $\alpha \in \Lambda V^*$, $A_v \alpha := \alpha(v, \cdot, \ldots, \cdot)$ and*

$$A_v(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \sum_{i=1}^n (-1)^{i+1} (v, v_i) \, v_1 \wedge v_2 \wedge \cdots \wedge \hat{v}_i \cdots \wedge v_k.$$

2. (Derivation Property) *For each $v \in V$,*

$$A_v(\alpha \wedge \beta) = A_v \alpha \wedge \beta + (-1)^k \alpha \wedge A_v \beta$$

where $\alpha \in \Lambda^k V^*$ and $\beta \in \Lambda V^*$ or $\alpha \in \Lambda^k V$ and $\beta \in \Lambda V$.

3. (Multiplicative Property) *For each* $v \in V$,

$$C_v(\alpha \wedge \beta) = (-1)^k \alpha \wedge C_v \beta$$

for $\alpha \in \Lambda^k V^*$ *and* $\beta \in \Lambda V^*$ *or* $\alpha \in \Lambda^k V$ *and* $\beta \in \Lambda V$.

4. (Transposes) *The pairing in Eq.* (A.2) *allows us to identify* ΛV^* *with* $(\Lambda V)^*$ *and hence* $(\Lambda V^*)^*$ *with* $(\Lambda V)^{**} \cong \Lambda V$. *Under these identifications,* $C_v^{\mathrm{tr}} = A_v$.

5. (Commutation Relations) *For all* v, $w \in V$,

$$C_v C_w + C_w C_v = 0,$$
$$A_v A_w + A_w A_v = 0, \qquad \textit{and}$$
$$A_v C_w + C_w A_v = (v, w)\, \mathrm{id}_{\Lambda V \oplus \Lambda V^*}.$$

Let $\gamma_v := C_v - A_v$, an operator on $E := \Lambda^k V^*$ for all $v \in V$. Then

$$\gamma_v^* = C_v^* - A_v^* = A_v - C_v = -\gamma_v$$

and by property 5 above,

$$\gamma_v \gamma_w + \gamma_w \gamma_v = -2(v, w)_V\, \mathrm{id}. \tag{A.4}$$

Notation A.3. If W is another vector space, we will abuse notation by using A_v and C_v to denote the operators $A_v \otimes \mathrm{id}_W$ and $C_v \otimes \mathrm{id}_W$ on $(\Lambda V \oplus \Lambda V^*) \otimes W$.

DEFINITION A.4 (Clifford Multiplication). Let V and E be inner product spaces, a multiplication map γ: $V^* \otimes E \to E$ is called a Clifford multiplication provided $\gamma_v^* = -\gamma_v$ for all $v \in V$ and

$$\gamma_v \gamma_w + \gamma_w \gamma_v = -2(v, w)_V\, \mathrm{id}_E$$

holds for all v, $w \in V$.

A.2. *Curvature*

Suppose that $E \to M$ is a vector bundle and that TM and E are equipped with covariant derivatives ∇^E and ∇^{TM} respectively. As in Section 2.2, these covariant derivatives induce a covariant derivative on $T^*M \otimes E$ as well. For $a \in \Gamma(E)$ and v, $w \in T_x M$, let $\nabla^2_{v \otimes w} a = (\nabla^{T^*M \otimes E} \nabla^E a)(v, w)$. Using this notation, we have the following useful formula for the curvature of ∇^E.

LEMMA A.5. *For* v, $w \in T_x M$,

$$\nabla^2_{v \otimes w} a - \nabla^2_{w \otimes v} a = R^E(v, w)\, a - \nabla^E_{T(v, w)} a,$$

where R^E is the curvature tensor of ∇^E and T is the torsion tensor of ∇^{TM}. Hence if ∇^{TM} is torsion free, as is the Levi–Civita connection, then

$$\nabla^2_{v\otimes w}a - \nabla^2_{w\otimes v}a = R^E(v, w)\, a.$$

Proof. Let $X, Y \in \Gamma(TM)$, then

$$\nabla^2_{X\otimes Y}a = \nabla^E_X\nabla^E_Y a - \nabla^E_{\nabla^{TM}_X Y}a$$

and hence

$$\begin{aligned}\nabla^2_{X\otimes Y}a - \nabla^2_{Y\otimes X}a &= \nabla^E_X\nabla^E_Y a - \nabla^E_{\nabla^{TM}_X Y}a - \nabla^E_Y\nabla^E_X a + \nabla^E_{\nabla^{TM}_Y X}a \\ &= \nabla^E_X\nabla^E_Y a - \nabla^E_Y\nabla^E_X a - \nabla^E_{[X, Y]}a \\ &\quad - \nabla^E_{(\nabla^{TM}_X Y - \nabla^{TM}_Y X - [X, Y])}a \\ &= R^E(X, Y)\, a - \nabla^E_{T(X, Y)}a. \quad \blacksquare\end{aligned}$$

A.3. *Weitzenböck Formulas for Generalized Dirac Operators*

Suppose that $E \to M$ is a Riemannian vector bundle with metric compatible covariant derivative. Suppose further that γ: $T^*M \otimes E \to E$ is a Clifford multiplication map, i.e. $\gamma_v\gamma_w + \gamma_w\gamma_v = -2(v, w)$ for all $v, w \in T_xM$ and $x \in M$. We also assume that γ is compatible with the covariant derivative on E and that γ_v acts as a skew adjoint operator on E_x for all $x \in M$. This implies by Lemma 2.16 that the formal adjoint of D_γ is D_γ. Let $L := -D_\gamma^2$, then the following Weitzenböck formula holds,

$$L = \square - \tfrac{1}{2}\sum_{i, j}\gamma_{e_i}\gamma_{e_j}R^E(e_i, e_j), \tag{A.5}$$

where $\{e_i\}_{i=1}^n$ is any local orthonormal frame of TM. Fixing x and choosing $\{e_i\}_{i=1}^n$ such that $(\nabla e_i)_x = 0$ for all i, this is verified by the computation

$$\begin{aligned} D_\gamma^2 &= \sum_i \gamma_{e_i}\nabla_{e_i}\gamma\nabla = \sum_i \gamma_{e_i}\gamma\nabla_{e_i}\nabla \\ &= \sum_{i, j}\gamma_{e_i}\gamma_{e_j}\nabla^2_{e_i\otimes e_j} = \tfrac{1}{2}\sum_{i, j}(\gamma_{e_i}\gamma_{e_j}\nabla^2_{e_i\otimes e_j} + \gamma_{e_j}\gamma_{e_i}\nabla^2_{e_j\otimes e_i}) \\ &= \tfrac{1}{2}\sum_{i, j}(-\gamma_{e_j}\gamma_{e_i}\nabla^2_{e_i\otimes e_j} - 2\delta_{i, j}\nabla^2_{e_i\otimes e_j} + \gamma_{e_j}\gamma_{e_i}\nabla^2_{e_j\otimes e_i}) \\ &= -\square + \tfrac{1}{2}\sum_{i, j}\gamma_{e_j}\gamma_{e_i}(\nabla^2_{e_j\otimes e_i} - \nabla^2_{e_i\otimes e_j}) \\ &= -\square + \tfrac{1}{2}\sum_{i, j}\gamma_{e_j}\gamma_{e_i}R^E(e_j, e_i), \end{aligned}$$

where the last equality is a consequence of Lemma A.5. Clearly we also have that $D_\gamma L = L D_\gamma$.

A.3.1. *Dirac operator on an spinor bundle.* If M is a spin manifold, $E = S$ is a spinor bundle over M, and ∇^S is the spin connection on S, then the previous formula reduces to $D_\gamma^2 = -\Box + \frac{1}{4}\,\mathrm{scal}$, where scal is the scalar curvature of M, see p. 126 in [4].

A.3.2. *Vector-valued differential forms.* Let M be a Riemannian manifold with Levi–Civita connection and $E \to M$ be a Riemannian vector bundle over M, endowed with a Riemannian linear connection. Let $\mathscr{A}^p(E) = \Gamma(\Lambda^p T^*M \otimes E)$ be the space of p-forms on M with values in the vector bundle E and let

$$\mathscr{A}(E) = \Gamma(\Lambda T^*M \otimes E) = \bigoplus_{p \geqslant 0} \mathscr{A}^p(E).$$

The same symbol ∇ will be used to denote various covariant derivations induced naturally from the Riemannian metric on M and the metric connection in E. Let C: $T^*M \otimes (\Lambda T^*M \otimes E) \to \Lambda T^*M \otimes E$ be the creation multiplication operator and define d_∇: $\mathscr{A}(E) \to \mathscr{A}(E)$ by $d_\nabla := D_C = C\nabla$. The explicit formula for d_∇ is

$$d_\nabla a(v_1, ..., v_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} (\nabla_{v_i} a)(v_1, ..., \hat{v}_i, ..., v_{p+1}),$$

where $a \in \mathscr{A}^p(E)$ and $v_1, ..., v_{p+1} \in T_x M$ for some $x \in M$.

Since

$$\begin{aligned} \nabla_X(C(\alpha \otimes a)) &= \nabla_X(\alpha \wedge a) = \nabla_X \alpha \wedge a + \alpha \wedge \nabla_X a \\ &= C(\nabla_X \alpha \otimes a + \alpha \otimes \nabla_X a) \end{aligned} \tag{A.6}$$

the exterior multiplication C is compatible with ∇. The following properties of d_∇ (see [20, 21]) follow from the product rule for ∇, Eq. (A.6), and basic properties of C:

1. $d_\nabla a = \nabla a$ for $a \in \mathscr{A}^0(E) = \Gamma(E)$.

2. $d_\nabla(\alpha \wedge a) = d\alpha \wedge a + (-1)^{\deg(\alpha)} \alpha \wedge d_\nabla a$ for all homogeneous differential forms α on M and $a \in \mathscr{A}(E)$.

3. $d_\nabla^2(\alpha \wedge a) = \alpha \wedge d_\nabla^2 a$ for all homogeneous differential forms α on M and $a \in \mathscr{A}(E)$.

4. If $a \in \mathscr{A}^0(E) := \Gamma(E)$, then $d_\nabla^2 a = R^E a$ because

$$\begin{aligned} d_\nabla^2 a &= C\nabla C\nabla a \\ &= \sum_{i,\, j=1}^{n} C_{e_i} C_{e_j} \nabla^2_{e_i \otimes e_j} a \\ &= \tfrac{1}{2} \sum_{i,\, j=1}^{n} [C_{e_i} C_{e_j} (\nabla^2_{e_i \otimes e_j} a - \nabla^2_{e_j \otimes e_i} a)] \\ &= \tfrac{1}{2} \sum_{i,\, j=1}^{n} C_{e_i} C_{e_j} R(e_i, e_j)\, a = R^E a, \end{aligned}$$

wherein we have used Lemma A.5.

5. Property 4 may be extended to all $a \in \mathscr{A}(E)$ to read $d_\nabla^2 a = R^E \wedge a$, where $R^E \wedge$ is the linear operator on $\Lambda(T^*M) \otimes E$ determined by $R^E \wedge (\alpha \otimes b) = \alpha \wedge R^E b$ for $\alpha \in \Lambda(T^*M)$ and $b \in \mathscr{A}^0(E)$. Alternatively,

$$d_\nabla^2 a = \tfrac{1}{2} \sum_{i,\, j=1}^{n} C_{e_i} C_{e_j} R(e_i, e_j)\, a \tag{A.7}$$

where now R is the curvature tensor for the induced connection on $\Lambda(T^*M) \otimes E$.

Remark A.6. If $E = M \times \mathbb{R}$ is the trivial vector bundle so that $R^E = 0$ and hence $d_\nabla^2 = 0$, then d_∇ is precisely the exterior differential d on differential forms on M.

Relative to the Riemannian structures on E and TM, the co-differential operator $d_\nabla^*: \mathscr{A}^p(E) \to \mathscr{A}^{p-1}(E)$ is characterized as the adjoint of d_∇ via

$$\int_M (d_\nabla a(x), b(x))_{\Lambda^p(T^*M) \otimes E} \operatorname{vol}(dx) = \int_M (a(x), d_\nabla^* b(x))_{\Lambda^{p-1}(T^*M) \otimes E} \operatorname{vol}(dx) \tag{A.8}$$

for $a \in \mathscr{A}^{p-1}(E)$ of compact support and $b \in \mathscr{A}^p(E)$. By Lemma 2.16 and the fact that $C_v^* = A_v$,

$$d_\nabla^* a = -\sum_i A_{e_i} \nabla_{e_i} a \qquad \text{for} \quad a \in \mathscr{A}^p(E) \tag{A.9}$$

see [21], p. 8. Since d_∇^2 is a zero order operator, see Eq. (A.7), it follows that d_∇^{*2} is the zero order operator:

$$\begin{aligned} d_\nabla^{*2} b &= \tfrac{1}{2} \sum_{i,\, j=1}^{n} R^E(e_i, e_j)\, A_{e_i} A_{e_j} b \\ &= \sum_{1 \leqslant i < j \leqslant n} R^E(e_i, e_j)\, A_{e_i} A_{e_j} b, \end{aligned} \tag{A.10}$$

because

$$\begin{aligned} (R^E \wedge a, b) &= \tfrac{1}{2} \sum_{i,\, j=1}^{n} (C_{e_i} C_{e_j} R^E(e_i, e_j)\, a, b) \\ &= \tfrac{1}{2} \sum_{i,\, j=1}^{n} (R^E(e_i, e_j)\, a, A_{e_j} A_{e_i} b) \\ &= -\tfrac{1}{2} \sum_{i,\, j=1}^{n} (a, R^E(e_i, e_j)\, A_{e_j} A_{e_i} b) \\ &= \tfrac{1}{2} \sum_{i,\, j=1}^{n} (a, R^E(e_i, e_j)\, A_{e_i} A_{e_j} b). \end{aligned}$$

There are now different Laplacians on E-valued differential forms, e.g.,

$$\Delta = -(d_\nabla d_\nabla^* + d_\nabla^* d_\nabla) \qquad \text{and} \qquad \Delta' = -(d_\nabla + d_\nabla^*)^2 = -D_\gamma^2, \tag{A.11}$$

where γ is the Clifford multiplication defined by $\gamma_v = C_v - A_v$. The Laplacians Δ and

$$\Delta' = \Delta - d_\nabla^2 - d_\nabla^{*2} = \Delta - \sum_{1 \leqslant i < j \leqslant n} R^E(e_i, e_j)(A_{e_i} A_{e_j} + C_{e_i} C_{e_j}) \tag{A.12}$$

do not coincide on $\mathscr{A}^\bullet(E)$, except in the case of flat bundles. Both Δ and Δ' are elliptic, negative, and essentially self-adjoint on complete manifolds. The operator Δ is the most popular choice since it has the advantage of being homogeneous, i.e. Δ: $\mathscr{A}^p(E) \to \mathscr{A}^p(E)$. For our purposes however, Δ' seems to be more natural.

Proposition A.7 (Weitzenböck's Formula). *The relation between Δ and $\square$ is $\Delta = \square - \mathscr{R}$, where*

$$\mathscr{R} = -\sum_{i,\, j=1}^{n} C_{e_i} A_{e_j} R(e_i, e_j). \tag{A.13}$$

Alternatively, $\mathcal{R}$ may be described by requiring

$$(\mathcal{R}a)(v_1, ..., v_p) = \sum_{k=1}^{p} \sum_{j=1}^{n} (-1)^k (R(v_k, e_j)\, a)(e_j, v_1, ..., \hat{v}_k, ..., v_p)$$

for all $a \in \mathcal{A}^p(E)$, $v_1, ..., v_p \in T_xM$, $x \in M$ and $p \geqslant 1$ and $\mathcal{R}a = 0$ if $a \in \mathcal{A}^0(E)$. Similarly (using Eq. (A.12)), $\Delta' = \square - \mathcal{R}'$, where

$$\mathcal{R}' = \mathcal{R} + \sum_{1 \leqslant i < j \leqslant n} R^E(e_i, e_j)(A_{e_i} A_{e_j} + C_{e_i} C_{e_j}) \tag{A.14}$$

Proof. By the definition of Δ,

$$\begin{aligned}
\Delta &= C\nabla A\nabla + A\nabla C\nabla \\
&= C_{e_i} \nabla_{e_i} A\nabla + A_{e_i} \nabla_{e_i} C\nabla \\
&= C_{e_i} A\nabla_{e_i} \nabla + A_{e_i} C\nabla_{e_i} \nabla \\
&= C_{e_i} A_{e_j} \nabla^2_{e_i \otimes e_j} + A_{e_j} C_{e_i} \nabla^2_{e_j \otimes e_i} \\
&= C_{e_i} A_{e_j} \nabla^2_{e_i \otimes e_j} + (\delta_{ij} - C_{e_i} A_{e_j})\, \nabla^2_{e_j \otimes e_i} \\
&= \square + C_{e_i} A_{e_j} (\nabla^2_{e_i \otimes e_j} - \nabla^2_{e_j \otimes e_i}) \\
&= \square + C_{e_i} A_{e_j} R(e_i, e_j),
\end{aligned}$$

where we are summing on repeated indices. This proves $\Delta = \square - \mathcal{R}$ where $\mathcal{R}$ is given as in Eq. (A.13). For $v_1, ..., v_p \in T_xM$,

$$\begin{aligned}
&-(\mathcal{R}a)(v_1, ..., v_p) \\
&\quad = ((e_i, \cdot) \wedge A_{e_j} R(e_i, e_j)\, a)(v_1, ..., v_p) \\
&\quad = \sum_{k=1}^{p} (-1)^{k+1} (e_i, v_k)(A_{e_j} R(e_i, e_j)\, a)(v_1, ..., \hat{v}_k, ..., v_p) \\
&\quad = \sum_{k=1}^{p} (-1)^{k+1} (R(v_k, e_j)\, a)(e_j, v_1, ..., \hat{v}_k, ..., v_p)
\end{aligned}$$

which proves Eq. (A.14). ∎

Remark A.8. Because Δ, Δ', and $\square$ are symmetric operators on compactly supported smooth sections, it follows that $\mathcal{R}$ and $\mathcal{R}'$ are fiberwise symmetric operators as well. In particular it follows that

$$\begin{aligned}
\mathcal{R} = \mathcal{R}^* &= -\sum_{i,\, j=1}^{n} (R(e_i, e_j))^*\, A^*_{e_j} C^*_{e_i} \\
&= \sum_{i,\, j=1}^{n} R(e_i, e_j)\, C_{e_j} A_{e_i} = -\sum_{i,\, j=1}^{n} R(e_j, e_i)\, C_{e_j} A_{e_i}.
\end{aligned} \tag{A.15}$$

The derivative formulas that appear in the body of this article often involve the operator $\mathscr{R}^{\mathrm{tr}}$ rather than $\mathscr{R}$.

LEMMA A.9. *The transpose* $\mathscr{R}^{\mathrm{tr}}$ *of* $\mathscr{R} \in \Gamma(\mathrm{End}(\Lambda(T^*M) \otimes E))$ *in Eq.* (A.13) *is formally given by the same formula as* $\mathscr{R}$,

$$\mathscr{R}^{\mathrm{tr}} = -\sum_{i,\,j=1}^{n} C_{e_i} A_{e_j} R(e_i, e_j) = -\sum_{i,\,j=1}^{n} R(e_j, e_i)\, C_{e_j} A_{e_i}, \tag{A.16}$$

but now acting on $\Lambda(TM) \otimes E^*$. *In particular when* $E = M \times \mathbb{R}$ (*so* $\Lambda(T^*M) \otimes E = \Lambda(T^*M)$), *if* $v \in TM = \Lambda^1 TM \subset \Lambda TM$, *then*

$$\mathscr{R}^{\mathrm{tr}} v = \sum_{i,\,j=1}^{n} R(e_i, e_j)\, e_j(v, e_i) = \mathrm{Ric}\, v. \tag{A.17}$$

Proof. Since R is the curvature tensor induced by the covariant derivatives on E and TM, it follows that $R(v, w)^{\mathrm{tr}} \mid (\Lambda(TM) \otimes E^*) = -R(v, w) \mid (\Lambda(T^*M) \otimes E)$. Hence starting with Eq. (A.13) and using Lemma A.2 we find,

$$\begin{aligned}\mathscr{R}^{\mathrm{tr}} &= -\sum_{i,\,j=1}^{n} R(e_i, e_j)^{\mathrm{tr}} A_{e_j}^{\mathrm{tr}} C_{e_i}^{\mathrm{tr}} \\ &= \sum_{i,\,j=1}^{n} R(e_i, e_j)\, C_{e_j} A_{e_i} = -\sum_{i,\,j=1}^{n} R(e_j, e_i)\, C_{e_j} A_{e_i}.\end{aligned}$$

An analogous computation starting with Eq. (A.15) shows

$$\mathscr{R}^{\mathrm{tr}} = -\sum_{i,\,j=1}^{n} C_{e_i} A_{e_j} R(e_i, e_j). \quad \blacksquare$$

PROPOSITION A.10. *Let* $a \in \mathscr{A}^0(E) = \Gamma(E)$ *and* $v \in TM$, *then*

$$[\square, \nabla]_v\, a = \nabla_{\mathrm{Ric}\, v} a + (\nabla \cdot R^E)_v\, a - 2 \sum_{i=1}^{n} R^E(v, e_i)\, \nabla_{e_i} a, \tag{A.18}$$

where $\mathrm{Ric}\, v \equiv \sum_{i=1}^n R^{TM}(v, e_i)\, e_i$ *and* $(\nabla \cdot R^E)_v \equiv \sum_{i=1}^n (\nabla_{e_i} R^E)(e_i, v)$.

Proof. We start with the relation,

$$\begin{aligned}\Delta d_\nabla - d_\nabla \Delta &= d_\nabla(d_\nabla^* d_\nabla + d_\nabla d_\nabla^*) - (d_\nabla^* d_\nabla + d_\nabla d_\nabla^*)\, d_\nabla \\ &= d_\nabla^2 d_\nabla^* - d_\nabla^* d_\nabla^2 = R \wedge d_\nabla^* - d_\nabla^* R \wedge .\end{aligned}$$

Applying this relation to $a \in \mathscr{A}_0(E)$ gives

$$\begin{aligned}(\Box - \mathscr{R})\,\nabla a - \nabla\,\Box a &= \varDelta d_\nabla a - d_\nabla \varDelta a = -d_\nabla^*(R \wedge a)\\ &= \sum_{i=1}^n A_{e_i}\nabla_{e_i}(Ra) = \sum_{i=1}^n A_{e_i}(\nabla_{e_i} Ra + R\nabla_{e_i} a)\\ &= \sum_{i=1}^n \{(\nabla_{e_i} R)(e_i, \cdot)\,a + R(e_i, \cdot)\,\nabla_{e_i} a\}\\ &= (\nabla \cdot R)\,a - R \cdot \nabla a,\end{aligned}$$

where $\mathscr{R}$ is defined in Proposition A.7 above. Adding this equation to the formula for $\mathscr{R}\nabla a$,

$$\begin{aligned}(\mathscr{R}\nabla a)(v) &= -\sum_{i=1}^n (R(v, e_i)\,\nabla a)(e_i)\\ &= -\sum_{i=1}^n R^E(v, e_i)\,\nabla_{e_i} a + \sum_{i=1}^n \nabla_{R(v, e_i)\,e_i} a\\ &= -R \cdot \nabla a + \nabla_{\mathrm{Ric}\,v} a,\end{aligned}$$

completes the proof. ∎

APPENDIX B: SEMIGROUP RESULTS

B.1. *Some Spectral Theory for Vector-Valued Schrödinger Operators*

Let M be a Riemannian manifold and $\pi: E \to M$ a Riemannian vector bundle over M, endowed with a Riemannian connection. Recall that $\Gamma(E)$ denotes the smooth sections, $\Gamma_c(E)$ the compactly supported smooth sections, and L^2-$\Gamma(E)$ the smooth square-integrable sections of E. Finally, $L^2(E)$ is the Hilbert space of square-integrable sections of E with the inner product

$$(a, b)_{L^2(E)} = \int_M (a(x), b(x))_{E_x}\,\mathrm{vol}(dx). \tag{B.1}$$

LEMMA B.1. *The operator* $\Box = \mathrm{tr}\,\nabla^2$ *is non-positive and formally self-adjoint. More precisely,*

$$(\Box a, b)_{L^2(E)} = -(\nabla a, \nabla b)_{L^2(T^*M \otimes E)} \tag{B.2}$$

for $a, b \in \Gamma(E)$ *with* a *or* b *of compact support.*

Proof. Let m be the identity multiplication map from E to $\tilde{E} := T^*M \otimes E$. Then $D_m = \nabla^E$, $m_v^* S = S(v)$ for $S \in \Gamma(\tilde{E}) \cong \Gamma(\mathrm{Hom}(TM, E))$ and $\square = D_{m^*} D_m$ and hence by Lemma 2.16,

$$\begin{aligned}(\square a, b)_{L^2(E)} &= (D_{m^*} D_m a, b)_{L^2(E)}\\ &= -(D_m a, D_m b)_{L^2(T^*M \otimes E)}\\ &= -(\nabla a, \nabla b)_{L^2(T^*M \otimes E)}. \quad \blacksquare\end{aligned}$$

B.1.1. *Self-adjoint extensions and elliptic regularity.* Let $L = \square - \mathscr{R}$ where $\mathscr{R} \in \Gamma(\mathrm{End}\, E)$ is assumed to be symmetric, i.e. $\mathscr{R}_x$ is a symmetric linear transformation of E_x for each $x \in M$. Suppose that $(\square - \mathscr{R}) \mid \Gamma_c(E)$ is bounded above, i.e.,

$$\lambda_0(\mathscr{R}) := \sup \left\{ \frac{((\square - \mathscr{R})\, a, a)_{L^2}}{(a, a)_{L^2}} : 0 \neq a \in \Gamma_c(E) \right\} < \infty. \tag{B.3}$$

In this situation, there is a canonical self-adjoint extension of $\square - \mathscr{R}$, the so-called *Friedrichs extension*, cf. [45]. We briefly sketch the construction: Defining $\mathscr{E}(a, b) := -(\nabla a, \nabla b)_{L^2} - (\mathscr{R} a, b)_{L^2}$ for $a, b \in \mathscr{D}(\mathscr{E}) := \Gamma_c(E)$, then for any $c > \lambda_0(\mathscr{R})$,

$$q(a, b) := -\mathscr{E}(a, b) + c(a, b)_{L^2} \tag{B.4}$$

in a positive quadratic form on $\mathscr{D}(\mathscr{E})$. On completing $\mathscr{D}(\mathscr{E})$ in the q-norm to $\bar{\mathscr{D}}(\mathscr{E})$ and extending $\mathscr{E}$ by continuity to a closed quadratic form $\bar{\mathscr{E}}$ on $\bar{\mathscr{D}}(\mathscr{E})$, we get

$$\bar{\mathscr{E}}(a, b) = ((\square - \mathscr{R})^\wedge\, a, b)_{L^2} \tag{B.5}$$

for some self-adjoint operator $(\square - \mathscr{R})^\wedge$ with form domain $\bar{\mathscr{D}}(\mathscr{E}) \subset L^2(E)$. The operator $(\square - \mathscr{R})^\wedge$ is called the Friedrichs extensions of $(\square - \mathscr{R}) \mid \Gamma_c(E)$.

Remark B.2. If the manifold M is complete, then $(\square - \mathscr{R}) \mid \Gamma_c(E)$ is essentially self-adjoint and $(\square - \mathscr{R})^\wedge = (\square - \mathscr{R})^-$ where $(\square - \mathscr{R})^-$ denotes the closure of $(\square - \mathscr{R}) \mid \Gamma_c(E)$. See for example Strichartz [51] and Davies [11].

In the following we are going to deal with the L^2 semigroup

$$P_t a = e^{t(\square - \mathscr{R})^\wedge/2} a, \qquad a \in L^2(E), \tag{B.6}$$

defined by the spectral theorem. First, we note some consequences from standard elliptic theory.

Remark B.3 (Elliptic regularity). For $a \in L^2(E)$ the following properties hold:

(i) If $a \in \ker(\Box - \mathscr{R})^\wedge$ then $a \in L^2\text{-}\Gamma(E)$.

(ii) The map $(t, x) \mapsto P_t a(x)$ is smooth on $]0, \infty[\times M$, for $a \in L^2\Gamma(E)$ even on $[0, \infty[\times M$. In addition, there exists a kernel $(t, x, y) \mapsto p(t, x, y) \in \mathrm{Hom}(E_y, E_x)$ which is smooth on $]0, \infty[\times M \times M$, such that

$$P_t a(x) = \int_M p(t, x, y)\, a(y)\, \mathrm{vol}(dy) \tag{B.7}$$

for the C^∞ version of $P_t a$, see [11].

B.1.2. *Semigroup domination and Feynman–Kac identities.* Given ρ: $M \to \mathbb{R}$ continuous and a measurable function f on M, let

$$P_t^\rho f(x) = \mathbb{E}\left[\exp\left(-\tfrac{1}{2}\int_0^t \rho(X_s(x))\, ds\right) f(X_t(x))\, 1_{\{t<\zeta(x)\}}\right] \tag{B.8}$$

when the right-hand side is well-defined.

Consider again $L = \Box - \mathscr{R}$ where $\mathscr{R} \in \Gamma(\mathrm{End}\, E)$ is assumed to be symmetric, and let

$$\underline{\mathscr{R}}(x) = \min\{(\mathscr{R}_x v, v) \colon v \in E_x, |v| = 1\}. \tag{B.9}$$

By uniform continuity, $\underline{\mathscr{R}}$ is a continuous function on M. If (Q_t) is defined by

$$\frac{d}{dt} Q_t = -\frac{1}{2} Q_t \mathscr{R}_{//_t} \qquad \text{with} \quad Q_0 = \mathrm{id}_{E_x}, \tag{B.10}$$

then

$$|Q_t|_{\mathrm{op}} \leqslant \exp\left(-\tfrac{1}{2}\int_0^t \underline{\mathscr{R}}(X_s(x))\, ds\right). \tag{B.11}$$

which can be seen, for instance, by representing the solution to (B.10) in terms of a product integral ([13], p. 28) or by Gronwall's inequality.

THEOREM B.4. *Let $\Box - \mathscr{R}$ be as above where $\mathscr{R} \in \Gamma(\mathrm{End}\, E)$ is a symmetric field of endomorphisms. Suppose that $(\Box - \mathscr{R}) \mid \Gamma_c(E)$ is bounded from above. For $a \in L^2(E)$ let*

$$P_t a = e^{t/2(\Box - \mathscr{R})^\wedge} a \tag{B.12}$$

be the C^∞ version of the L^2 semigroup. Then the formula

$$P_t a(x) = \mathbb{E}[\,Q_t //_t^{-1} a(X_t(x))\, 1_{\{t<\zeta(x)\}}\,] \tag{B.13}$$

holds for all $a \in L^2(E)$ with $P_t^{\mathscr{R}}\, |a|\, (x) < \infty$.

Proof. Since $P_t a$ has an integral kernel, using a monotone class argument, it is sufficient to check (B.13) for $a \in L^2(E) \cap \Gamma_c(E)$. Further, note that any (connected) manifold M can be exhausted by a sequence of relatively compact open domains D_n with smooth boundary. For instance, let $(\varphi_\ell)_{\ell \in \mathbb{N}}$ be a partition of unity such that $0 \leqslant \varphi_\ell \in C_c^\infty(M)$. Consider $\phi_n := \sum_{\ell=1}^n \varphi_\ell$. Choosing numbers $\varepsilon_n \searrow 0$ such that $\{\phi_n = \varepsilon_n\}$ are smooth submanifolds of M (which is possible by Sard's theorem), then

$$D'_n := \{\phi_n > \varepsilon_n\} \nearrow M$$

gives a smooth exhaustion of M, since $\phi_n \nearrow 1$ pointwise. Finally, by fixing a point x_0 in M and defining D_n as the component of D'_n containing x_0, we get a sequence of connected sets D_n with the desired properties.

Now, fix an exhausting sequence $D_n \nearrow M$ as above, and let $\hat{L}_n$ denote the Friedrichs extensions of $(\square - \mathscr{R}) \mid \Gamma_c(E/D_n)$. Then, by monotone convergence of the corresponding quadratic forms, see e.g. [36], Theorem VIII-3.11, we get

$$P_t^{(n)} a := e^{t\hat{L}_n/2} a \to P_t a \qquad \text{in } L^2. \tag{B.14}$$

We use the following two properties of $P_t^{(n)} a$:

1. the map $(t, x) \mapsto P_t^{(n)} a(x)$ is smooth (in particular bounded) on $[0, T] \times D_n$,
2. the semigroup $P_t^{(n)} a$ vanishes on ∂D_n.

Recall that for each $t > 0$,

$$N_s^{(n)} := Q_s //_s^{-1} P_{t-s}^{(n)} a(X_s(x)) \tag{B.15}$$

is a local martingale with lifetime $t \wedge \tau_n(x)$ where $\tau_n(x)$ denotes the first exit time of $X(x)$ from D_n. Since $N_s^{(n)}$ in Eq. (B.15) is bounded, we may conclude $\mathbb{E}[N_0^{(n)}] = \mathbb{E}[N_{t \wedge \tau_n(x)}^{(n)}]$. But note that

$$\begin{aligned}\mathbb{E}[N_{t \wedge \tau_n(x)}^{(n)}] &= \mathbb{E}[N_t^{(n)} 1_{\{t < \tau_n(x)\}}] + \mathbb{E}[N_{\tau_n(x)}^{(n)} 1_{\{t \geqslant \tau_n(x)\}}] \\ &= \mathbb{E}[N_t^{(n)} 1_{\{t < \tau_n(x)\}}] = \mathbb{E}[\,Q_t //_t^{-1} a(X_t(x))\, 1_{\{t < \tau_n(x)\}}\,]\end{aligned}$$

and

$$|Q_t //_t^{-1} a(X_t(x))\, 1_{\{t<\tau_n(x)\}}| \leqslant \exp\left(-\tfrac{1}{2}\int_0^t \mathscr{R}(X_s(x))\, ds\right) |a|\,(X_t(x))\, 1_{\{t<\zeta(x)\}}.$$

By assumption, the right-hand side is in L^1, thus by dominated convergence,

$$P_t^{(n)} a(x) = \mathbb{E}[N^{(n)}_{t\wedge\tau_n(x)}] \to \mathbb{E}[Q_t //_t^{-1} a(X_t(x))\, 1_{\{t<\zeta(x)\}}]$$

which combined with (B.14) gives the claim. ∎

Note that the above proof shows in particular semigroup domination, see Theorem 4.3 of Donnelly and Li [14]:

$$|P_t a|\,(x) \leqslant P_t^{\mathscr{R}}\, |a|\,(x), \tag{B.16}$$

see [3, 50] for a general account on this. Before discussing this point, let us specialize Theorem B.4 to a Feynman–Kac identity on functions. The following Corollary is well-known at least in the case $\rho=0$, e.g. [12].

Corollary B.5. *Let M be a Riemannian manifold, Δ_M its Laplace–Beltrami operator, and $\Delta_M^\rho := \Delta_M - \rho$ where $\rho\colon M\to\mathbb{R}$ is continuous. Suppose that $\Delta_M^\rho \mid C_c^\infty(M)$ is bounded from above, i.e.,*

$$\lambda_0(\rho) := \sup\left\{\frac{(\Delta_M^\rho \varphi, \varphi)}{(\varphi, \varphi)} : 0\neq\varphi\in C_c^\infty(M)\right\} < \infty. \tag{B.17}$$

Then $P_t^\rho\,|f|(x)<\infty$ for any $f\in L^2(M)$ and $x\mapsto P_t^\rho f(x)$ is continuous for $t>0$. Let $\hat{\Delta}_M^\rho$ be the Friedrichs extension of $\Delta_M^\rho \mid C_c^\infty(M)$. Then,

$$e^{t\hat{\Delta}_M^\rho/2} f = P_t^\rho f \tag{B.18}$$

for the L^2 semigroup given by the spectral theorem.

Proof. Take the trivial bundle $E=M\times\mathbb{R}$, then $\Gamma(E)=C^\infty(M)$ and $\square f = \Delta_M f$. Theorem B.4 gives the claim at least for smooth ρ, a restriction which can easily be removed. ∎

Theorem B.6 (Semigroup Domination). *For a field $\mathscr{R}\in\Gamma(\operatorname{End} E)$ of symmetric endomorphisms, let $L^{\mathscr{R}} = \square - \mathscr{R}$ and $\Delta_M^{\underline{\mathscr{R}}} = \Delta_M - \underline{\mathscr{R}}$. Then*

$$\lambda_0(\mathscr{R}) \leqslant \lambda_0(\underline{\mathscr{R}}). \tag{B.19}$$

In particular, if $\Delta^{\mathscr{R}}_{\bar M} \mid C_c^\infty(M)$ is bounded from above, then $L^{\mathscr{R}} \mid \Gamma_c(E)$ is also bounded from above, and moreover the following estimate holds:

$$|P_t a| \leqslant P_t^{\mathscr{R}} |a|. \tag{B.20}$$

Proof. We may assume that $\lambda_0(\underline{\mathscr{R}}) < \infty$. If $\lambda_0(\mathscr{R}) < \infty$, then the Friedrichs extension $\hat{L}^{\mathscr{R}}$ of $L^{\mathscr{R}} \mid \Gamma_c(E)$ is well-defined, and by Eq. (B.16),

$$\begin{aligned} e^{\lambda_0(\mathscr{R})/2} &= \sup_{a \in \Gamma_c(E) \setminus \{0\}} \left\{ \frac{(a, e^{\hat{L}^{\mathscr{R}}/2} a)}{(a, a)} \right\} \\ &\leqslant \sup_{a \in \Gamma_c(E) \setminus \{0\}} \left\{ \frac{(|a|, e^{\hat{\Delta}^{\mathscr{R}}_{\bar M}/2} |a|)}{(|a|, |a|)} \right\} \leqslant e^{\lambda_0(\underline{\mathscr{R}})/2}. \end{aligned}$$

To see that $\lambda_0(\underline{\mathscr{R}}) < \infty$ implies $\lambda_0(\mathscr{R}) < \infty$, we note that the above argument can be applied first to give $\lambda_0(\mathscr{R} \mid D_n) \leqslant \lambda_0(\underline{\mathscr{R}})$ for each D_n of a sequence of smoothly bounded, relatively compact open domains $D_n \nearrow M$. From this the claim follows obviously. ∎

B.2. *A Commutativity Result*

In this subsection, we investigate conditions under which a Dirac type operator D commutes with the L^2 semigroup generated by the Friedrichs extension of D^2. In particular, we shall recover the fact that d and d^* commute with $e^{t\bar{\Delta}}$ when M is complete. The precise statement is given in Remark B.9 below. Similar discussions may be found in Brüning and Lesch [7], Xue-Mei Li [40, 41] and in Bueler [8].

THEOREM B.7. *Let D be a closable densely defined operator on a Hilbert space H. Then $D^*\bar{D}$ and $\bar{D}D^*$ are densely defined, self-adjoint and $\geqslant 0$. Furthermore*

$$\begin{aligned} D^* e^{-t\bar{D}D^*} \mid \mathscr{D}(D^*) &= e^{-tD^*\bar{D}} D^* \qquad \textit{and} \\ \bar{D} e^{-tD^*\bar{D}} \mid \mathscr{D}(\bar{D}) &= e^{-t\bar{D}D^*} \bar{D}. \end{aligned} \tag{B.21}$$

Proof (By "Nelson's trick", as in [57], Section 5.2). The operator

$$Q := \begin{pmatrix} 0 & D^* \\ \bar{D} & 0 \end{pmatrix}$$

on $H \oplus H$ is self-adjoint (see [57], Lemma 5.3). Thus, by the spectral theorem,

$$Q^2 = \begin{pmatrix} D^*\bar{D} & 0 \\ 0 & \bar{D}D^* \end{pmatrix}$$

is densely defined, self-adjoint and non-negative and hence are its components $D^*\bar{D}$ respectively $\bar{D}D^*$. By the spectral theorem, $Qe^{-tQ^2} \mid \mathscr{D}(Q) = e^{-tQ^2}Q$. It makes no difference applying the spectral theorem to Q^2 as a whole or to its components. Hence

$$\begin{aligned}\begin{pmatrix} 0 & D^*e^{-t\bar{D}D} \\ \bar{D}e^{-tD^*\bar{D}} & 0 \end{pmatrix} &= \begin{pmatrix} 0 & D^* \\ \bar{D} & 0 \end{pmatrix}\begin{pmatrix} e^{-tD^*\bar{D}} & 0 \\ 0 & e^{-t\bar{D}D} \end{pmatrix} \\ &= \begin{pmatrix} e^{-tD^*\bar{D}} & 0 \\ 0 & e^{-t\bar{D}D^*} \end{pmatrix}\begin{pmatrix} 0 & D^* \\ \bar{D} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & e^{-tD^*\bar{D}}D^* \\ e^{-t\bar{D}D^*}\bar{D} & 0 \end{pmatrix}. \quad \blacksquare\end{aligned}$$

Remark B.8. In Theorem B.7, the operator $D^*\bar{D}$ is the Friedrichs extension of D^2, e.g. Reed–Simon [45]. In particular, if D is essentially self-adjoint, i.e. $\bar{D} = D^*$, then $\bar{D}$ commutes (on the domain of $\bar{D}$) with the semigroup generated by the Friedrichs extension of $-D^2$.

Remark B.9. Let $E = \Lambda T^*M$ and $D = d + \delta$ on compactly supported smooth sections of E, where d and δ denote the exterior differential and its formal adjoint, both restricted to smooth sections of compact support. Under the assumption that M is complete, $D = d + \delta$ and all its powers are known to be essentially self-adjoint on $\Gamma_c(E) \subset L^2(E)$, see [9], also [51], Theorem 2.4. An immediate consequence is

$$\overline{d+\delta} = \bar{d} + \bar{\delta} = \delta^* + d^*. \tag{B.22}$$

Hence, the Hodge–de Rham Laplacian $L = -D^2$ is essentially self-adjoint on $\Gamma_c(E) \subset L^2(E)$. Thus, on a complete manifold,

$$\bar{d}P_t = P_t\bar{d} \qquad \text{on the domain on } \bar{d}, \qquad \text{and}$$

$$d^*P_t = P_t d^* \qquad \text{on the domain of } d^*,$$

where P_t is the semigroup generated by $\bar{L} = \hat{L} = -D^*\bar{D} = -\bar{D}D^*$. Indeed, this follows from Theorem B.7, together with Eq. (B.22), by taking into account that P_t, d, d^* are homogeneous respectively of degree 0, 1, -1.

Remark B.10. For non-complete manifolds M there are in general several different self-adjoint extensions of $-D^2 = -(d+\delta)^2 \mid \Gamma_c(E)$, see [10, 31, 32] for details.

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