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Surjectivity of the Taylor map for complex nilpotent Lie groups

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Abstract

A Hermitian form q on the dual space, $\mathfrak{g}^*$, of the Lie algebra, $\mathfrak{g}$, of a simply connected complex Lie group, G, determines a sub-Laplacian, Δ, on G. Assuming Hörmander's condition for hypoellipticity, there is a smooth heat kernel measure, ρ_t, on G associated to $e^{t\Delta/4}$. In a companion paper [**6**], we proved the existence of a unitary "Taylor" map from the space of holomorphic functions in $L^2(G, \rho_t)$ onto J_t^0 (a subspace of) the dual of the universal enveloping algebra of $\mathfrak{g}$. Here we give a very different proof of the surjectivity of the Taylor map under the assumption that G is nilpotent. This proof provides further insight into the structure of the Taylor map. In particular we show that the finite rank tensors are dense in J_t^0 when the Lie algebra is graded and the Laplacian is adapted to the gradation. We also show how the Fourier–Wigner transform produces a natural family of holomorphic functions in $L^2(G, \rho_t)$, for appropriate t, when G is the complex Heisenberg group.

1. *Introduction*

Let G be a complex Lie group with a given a Hermitian inner product on its Lie algebra, $\mathfrak{g}$. The Laplace operator, Δ, associated to the left-invariant extension of the real part of the inner product, determines a heat kernel ρ_t on G, defined by the identity $e^{t\Delta/4} =$ right convolution by $\rho_t(x)dx$. For each $t > 0$ the heat kernel decays fast enough at infinity so that the Hilbert space consisting of holomorphic functions in $L^2(G, \rho_t(x)dx)$ is non-empty and is in fact a quite substantial space. For a holomorphic function f in this space we will, in a very strong sense, determine the growth rate of its Taylor coefficients at the identity element of G: if $\xi_1, \ldots, \xi_k$ are in $\mathfrak{g}$ then the map $(\xi_1, \ldots, \xi_k) \to (\xi_1 \cdots \xi_k f)(e)$ is a multilinear map into the complex numbers and is consequently represented by a unique element of the dual space of $\mathfrak{g}^{\otimes k}$. Allowing k to vary and putting these elements together now yields an element $\hat{f}$ of the algebraic dual space T', wherein T denotes the tensor algebra over $\mathfrak{g}$. Thus the element $\hat{f}$ is the set of Taylor coefficients of f at the identity element of G. There is a remarkable

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identity of norms involving f and its Taylor coefficients. On the one hand one has the norm $\|f\|_{L^2(G,\rho_t(x)dx)}$, while on the other hand one has the norm of $\hat{f}$ as an element of T' with its natural (t dependent) norm induced by the original Hermitian inner product on $\mathfrak{g}$. These norms are equal. In fact the map $f \to \hat{f}$ is unitary if G is simply connected. See [**5**] for a precise description of these results and some history.

It is not actually necessary, for this type of theorem, that the operator Δ be elliptic. In [**6**] we have shown that indeed such a unitarity theorem holds if the subLaplacian Δ is merely subelliptic. The inner product on $\mathfrak{g}$ must, properly speaking, be defined on the dual space $\mathfrak{g}^*$ in this case and is allowed to be degenerate to the extent that Hörmander's theorem for hypoellipticity of the subLaplacian Δ permits. The fact that the map $f \to \hat{f}$ is an isometry does not require G to be simply connected and, moreover, the proof of isometry in the degenerate case is not substantially different from that in the non-degenerate case. See [**6**, section 4] for this proof. The proof of surjectivity, however, (for simply connected G) is, and has always been, for all cases, by far the most difficult part of unitarity to prove.

This paper is devoted to the special case wherein the group G is nilpotent. In Section 3 we will give a proof of surjectivity in this case. It is a very different proof from the general surjectivity proof in [**6**]. Moreover, along the way it produces a proof that the finite rank tensors are dense in the tensor-side Hilbert space if G is graded nilpotent and the Laplacian is adapted to the gradation. Such denseness is false for a semisimple group. Section 2 gives needed background concerning the Taylor map.

In Section 4 we will show how the Fourier-Wigner transform generates, in a natural way, holomorphic functions in $L^2(H_3^{\mathbb{C}}, \rho_t)$, where ρ_t is the natural subelliptic heat kernel on the three dimensional complex Heisenberg group, $H_3^{\mathbb{C}}$. Moreover the time parameter t is related in an interesting way to Planck's constant and the physical time appearing in the Fourier-Wigner transform.

2. *Notation and background*

In this section we will review some notation and basic results from [**6**]. We will use angle brackets, $\langle\cdot,\cdot\rangle$, to denote the pairing of a vector space, V, and its algebraic dual, V', i.e. $\langle\alpha, v\rangle := \alpha(v)$ for all $v \in V$ and $\alpha \in V'$. Let G be a complex connected Lie group equipped with its right Haar measure dx and let $\mathcal{H} = \mathcal{H}(G)$ denote the space of complex valued holomorphic functions on G. Given $A \in \mathfrak{g} := \mathrm{Lie}(G)$ (the complex Lie algebra of G), let $\tilde{A}$ denote the unique left invariant vector field acting on $C^\infty(G)$ such that $\tilde{A}(e) = A$.

Denote by $T(\mathfrak{g})$ the tensor algebra over $\mathfrak{g}$. An element of $T(\mathfrak{g})$ is a finite sum:

$$\beta = \sum_{k=0}^{N} \beta_k \qquad \beta_k \in \mathfrak{g}^{\otimes k}. \tag{2·1}$$

We may and will identify $T(\mathfrak{g})'$ with the direct product $\prod_{k=0}^{\infty}(\mathfrak{g}^*)^{\otimes k}$ via the pairing,

$$\langle\alpha, \beta\rangle = \sum_{k=0}^{\infty}\langle\alpha_k, \beta_k\rangle, \tag{2·2}$$

where

$$\alpha = \sum_{k=0}^{\infty} \alpha_k \qquad \alpha_k \in (\mathfrak{g}^*)^{\otimes k}. \tag{2·3}$$

Notation 2·1 (Left Invariant Differential Operators). We define a **real** linear map ($\beta \to \tilde{\beta}$) from $T(\mathfrak{g})$ to left invariant differential operators on G determined by: (1) for $\beta = A_1 \otimes \ldots \otimes A_k \in \mathfrak{g}^{\otimes k}$, $\tilde{\beta} f := \tilde{A}_1 \ldots \tilde{A}_k f$ and (2) $\tilde{1} f = f$ for $f \in C^\infty(G)$.

If f is a C^∞ function on G, the **Taylor coefficient of** f at $x \in G$ is the element, $\hat{f}(x)$, in $T(\mathfrak{g})'_{\mathrm{Re}}$ (the real linear functionals on $T(\mathfrak{g})$) defined by

$$\langle \hat{f}(x), \beta \rangle = (\tilde{\beta} f)(x) \text{ for all } \beta \in T(\mathfrak{g}). \tag{2·4}$$

If we further assume $f \in \mathcal{H}$, then $\beta \to (\tilde{\beta} f)(x)$ is complex linear and in this case $\hat{f}(x) \in T(\mathfrak{g})'$. In either case, $\hat{f}(x)$ annihilates the two sided ideal, $J \subset T(\mathfrak{g})$, generated by

$$\{\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] : \xi, \eta \in \mathfrak{g}\}. \tag{2·5}$$

So if $f \in \mathcal{H}$ and $x \in G$, then $\hat{f}(x) \in J^0$ where

$$J^0 := \{\alpha \in T(\mathfrak{g})' : \langle \alpha, J \rangle = \{0\}\}. \tag{2·6}$$

The space J^0 is complex isomorphic to $\mathcal{U}'$ where $\mathcal{U} := T(\mathfrak{g})/J$ is the *universal enveloping algebra* of $\mathfrak{g}$.

Notation 2·2. Let q be a nonnegative quadratic (respectively Hermitian) form on the dual space $\mathfrak{g}^*$. Thus

$$q(a) = (a, a)_q \tag{2·7}$$

for some, possibly degenerate, nonnegative bilinear (respectively sesquilinear) form $(,)_q$ on $\mathfrak{g}^*$.

As is shown in [**6**, lemma 2·2], there exists a linearly independent (over $\mathbb{C}$) subset, $\{X_j\}_{j=1}^m \subset \mathfrak{g}$, such that

$$q(a) = (a, a)_q = \sum_{j=1}^m |\langle a, X_j \rangle|^2 \text{ for all } a \in \mathfrak{g}^*. \tag{2·8}$$

The space, $H := \mathrm{span}(X_1, \ldots, X_m)$ equipped with the unique Hermitian inner product, $(\cdot, \cdot)_H$, for which $\{X_j\}_{j=1}^m$ is an orthonormal basis, is called the **Hörmander subspace** associated to q. H is the backwards annihilator of the kernel of q. See, e.g., [**6**, equation (2·3)]. Also associated to q is the second order left invariant differential operator,

$$\Delta = \sum_{j=1}^m \left(\tilde{X}_j^2 + \widetilde{(iX_j)}^2\right). \tag{2·9}$$

It can be shown that Δ and $(H, (\cdot, \cdot)_H)$ only depend on q and not on the choice of $\{X_j\}_{j=1}^m \subset \mathfrak{g}$ for which (2·8) holds, see [**6**].

The form q induces a degenerate Hermitian form $q_k := q^{\otimes k}$ whose inner product, $(\cdot, \cdot)_{q_k}$, on $(\mathfrak{g}^*)^{\otimes k}$ is determined by

$$(a_1 \otimes \cdots \otimes a_k, b_1 \otimes \cdots \otimes b_k)_{q_k} = \prod_{j=1}^k (a_j, b_j)_q \quad a_i, b_i \in \mathfrak{g}^*, \ i = 1, \ldots, k \tag{2·10}$$

for $k \geqslant 1$. If $\alpha \in (\mathfrak{g}^*)^{\otimes k}$, we will write $q_k(\alpha)$ or $|\alpha|^2_{q_k}$ for $(\alpha, \alpha)_{q_k}$. By convention, $V^{\otimes 0} = \mathbb{C}$ and we define q_0 on $(\mathfrak{g}^*)^{\otimes 0}$ so that $q_0(1) = 1$. For $t > 0$ define

$$\|\alpha\|_t^2 := \sum_{k=0}^{\infty} \frac{t^k}{k!} |\alpha_k|^2_{q_k} \tag{2·11}$$

when α is given by (2·3).

The function, $\|\cdot\|_t$, defines a seminorm in the subspace of $T(\mathfrak{g})'$ on which $\|\alpha\|_t^2$ is finite. But we will, by restriction, always consider $\|\cdot\|_t$ to be a semi-norm on

$$J_t^0 := \{\alpha \in J^0 : \|\alpha\|_t^2 < \infty\}. \tag{2·12}$$

Definition 2·3. We say that Hörmander's condition holds for q if the smallest Lie subalgebra, Lie(H), containing H is $\mathfrak{g}$. (It is permissible here to view Lie(H) as the Lie algebra generated by $H \subset \mathfrak{g}$ with $\mathfrak{g}$ thought of being either a complex or a real Lie algebra.)

The significance of Hörmander's condition is twofold. 1) By [**6**, theorem 2·7], Lie$(H) = \mathfrak{g}$ iff for some $t > 0$ (hence for all $t > 0$), $\|\cdot\|_t$ is a norm on J_t^0. 2. By Hörmander's theorem [**16**], Lie$(H) = \mathfrak{g}$ iff Δ is hypoelliptic, see the end of Section 1 in [**6**] for a more detailed discussion on this last point. So under Hörmander's condition on q, the operator, Δ, in (2·9) admits a smooth heat kernel, $\rho_t : G \to (0, \infty)$, satisfying

$$\left(e^{t\bar{\Delta}/4} f\right)(e) = \int_G f(x)\rho_t(x)dx \text{ for all } f \in L^2(G, dx), \tag{2·13}$$

where $\bar{\Delta}$ denotes the $L^2(G, dx)$ closure of $\Delta|_{C_c^\infty(G)}$. We call the measure $\rho_t(x)dx$ **the heat kernel measure** on G associated to the sub-Laplacian Δ, see [**6**, section 3] for more details of this construction.

Notation 2·4. We denote by $\mathcal{H}$ the space of holomorphic functions on G and define

$$\mathcal{H}L^2(G, \rho_t(x)dx) = \mathcal{H} \cap L^2(G, \rho_t(x)\, dx). \tag{2·14}$$

(For any complex matrix group the matrix entries and polynomials in these entries lie in this space for any such subelliptic Laplacian.)

We may now summarize some of the main theorems from [**6**].

THEOREM 2·5 ([**6**, theorem 4·2]). *Let G be a connected complex Lie group. Suppose that q is a non-negative Hermitian form on the dual space $\mathfrak{g}^*$ and assume that Hörmander's condition holds. Let ρ_t denote the heat kernel associated to q. Then the* ***Taylor map****,*

$$f \longrightarrow \hat{f}(e), \tag{2·15}$$

is an isometry from $\mathcal{H}L^2(G, \rho_t(x)\, dx)$ into J_t^0.

PROPOSITION 2·6 ([**6**, proposition 4·3]). *Let $f \in \mathcal{H}(G)$ and assume that $\hat{f}(e) \in J_t^0$ (see* (2·4)*) for some $t > 0$. Then $f \in \mathcal{H}L^2(G, \rho_t(x)\, dx)$.*

THEOREM 2·7 ([**6**, theorem 6·1]). *Let G be a connected, simply connected complex Lie group. Suppose that q is a non-negative Hermitian form on the dual space $\mathfrak{g}^*$ and assume that Hörmander's condition holds, (cf. Definition.* 2·3*). Then the Taylor map, $f \to \hat{f}(e)$ is a unitary map from $\mathcal{H}L^2(G, \rho_t(x)dx)$ onto J_t^0.*

One of the main objectives of this paper is to provide an alternate proof of the surjectivity portion of Theorem 2·7 under the restrictive hypothesis that G is nilpotent.

3. *Taylor expansion over complex nilpotent groups*

In this section we are going to give a proof of the surjectivity of the isometry described in Theorem 2·7 when G is nilpotent. This proof is simpler than and very different from the proof given in [**6**] for the general case. It also yields more detailed information on the structure of the Taylor map.

THEOREM 3·1. *Let G be a connected, simply connected, nilpotent complex Lie group. Suppose that q is a nonnegative Hermitian form on the dual space $\mathfrak{g}^*$ of the complex Lie algebra of G. Assume that q satisfies Hörmander's condition (cf. Definition 2·3.) Let $t > 0$. If f is in $\mathcal{H}L^2(G, \rho_t(x)dx)$ then $\hat{f}(e)$ is in J_t^0 and the map*

$$(f \longrightarrow \hat{f}(e)) : \mathcal{H}L^2(G, \rho_t(x)dx) \longrightarrow J_t^0 \tag{3·1}$$

is unitary.

The proof of Theorem 3·1 will follow the proof of Lemma 3·6, which asserts that Theorem 3·1 holds under the additional assumption that $\mathfrak{g}$ is a "graded" Lie algebra and q is nicely related to the gradation.

Notation 3·2. A Lie algebra $\mathfrak{g}$ is *graded* if it is representable as a direct sum:

$$\mathfrak{g} = \oplus_{j=1}^{\infty} V_j \tag{3·2}$$

where all but finitely many of the the subspaces $\{V_j\}_{j=1}^{\infty}$ equal $\{0\}$ and

$$[V_i, V_j] \subset V_{i+j}, \qquad i, j = 1, 2, \ldots. \tag{3·3}$$

A graded algebra is necessarily nilpotent.

Notation 3·3. A Lie algebra $\mathfrak{g}$ is *stratified* if it is graded and V_1 generates $\mathfrak{g}$. In this case, we have

$$[V_1, V_k] = V_{k+1}, \qquad k = 1, \ldots, \infty, \tag{3·4}$$

and there exists an integer r such that $V_r \neq \{0\}$, $V_{r+1} = \{0\}$, and

$$\mathfrak{g} = \oplus_{j=1}^{r} V_j. \tag{3·5}$$

If $\mathfrak{g}$ is stratified and r is as in Notation 3·3 then $\mathfrak{g}$ is r-step nilpotent.

An important example illustrating these definitions comes from the complex Heisenberg algebra $\mathfrak{h}_3^{\mathbb{C}}$. This is a 3-dimensional complex vector space with basis X, Y, Z equipped with a Lie bracket satisfying $[X, Y] = Z$, $[X, Z] = [Y, Z] = 0$. In this case, V_1 is the vector subspace spanned by X, Y and V_2 is spanned by Z. Obviously, $\mathfrak{h}_3^{\mathbb{C}}$ is graded and, in fact, stratified.

Notation 3·4 (Dilations). Let $\mathfrak{g}$ be a graded Lie algebra with $\mathfrak{g} = \oplus_{j=1}^{\infty} V_j$ as in (3·2). For $\lambda \in \mathbb{C}$ and $v = \sum_1^{\infty} v_i \in \mathfrak{g}$, $v_j \in V_j$, $j = 1, \ldots, \infty$, define

$$\delta_\lambda(v) = \sum_{k=1}^{\infty} \lambda^k v_k. \tag{3·6}$$

It is straightforward to verify that

$$\delta_{\lambda\mu} = \delta_\lambda \delta_\mu \qquad \lambda, \mu \in \mathbb{C}. \tag{3·7}$$

and that, for $\lambda \neq 0$, δ_λ is an automorphism of the Lie algebra $\mathfrak{g}$. See [**7**, chapter 1] for details.

LEMMA 3·5. *Let $\mathfrak{g}$ be a complex graded Lie algebra. Let q be a nonnegative Hermitian form on $\mathfrak{g}^*$ satisfying Hörmander's condition (Definition 2·3). Assume that q is invariant under the action of the transposed dilations $(\delta_{e^{i\theta}})'$. Then the finite rank tensors in J_t^0 are dense in J_t^0 for each $t > 0$.*

Proof. Let $\Gamma_\theta : T(\mathfrak{g}) \to T(\mathfrak{g})$ be the automorphism of the tensor algebra over $\mathfrak{g}$ induced by the automorphism $\delta_{e^{i\theta}}$ of $\mathfrak{g}$, i.e.

$$\Gamma_\theta = \overbrace{\delta_{e^{i\theta}} \otimes \cdots \otimes \delta_{e^{i\theta}}}^{k\text{-times}} \text{ on } \mathfrak{g}^{\otimes k} .$$

For any ξ and η in $\mathfrak{g}$ we have

$$\begin{aligned}\Gamma_\theta(\xi \wedge \eta - [\xi, \eta]) &= (\delta_{e^{i\theta}}\xi) \wedge (\delta_{e^{i\theta}}\eta) - \delta_{e^{i\theta}}[\xi, \eta] \\ &= (\delta_{e^{i\theta}}\xi) \wedge (\delta_{e^{i\theta}}\eta) - [\delta_{e^{i\theta}}\xi, \delta_{e^{i\theta}}\eta].\end{aligned}$$

Thus Γ_θ takes J into, and in fact onto, J. The transpose, Γ'_θ, on $T(\mathfrak{g})'$ therefore takes J^0 onto itself. Since

$$\Gamma'_\theta = \overbrace{(\delta_{e^{i\theta}})' \otimes \cdots \otimes (\delta_{e^{i\theta}})'}^{k\text{-times}} \text{ on } (\mathfrak{g}^*)^{\otimes k},$$

it follows that

$$q_k(\Gamma'_\theta u) = q_k(u) \text{ for all } u \in (\mathfrak{g}^*)^{\otimes k}. \tag{3·8}$$

Since $\delta_{e^{i\theta}}\xi$ is continuous in θ for any norm on $\mathfrak{g}$, $\Gamma_\theta\beta$ is continuous in θ for all k tensors β and for any product norm on $\mathfrak{g}^{\otimes k}$. Similarly $\Gamma'_\theta\alpha$ is continuous in θ for any element α in $(\mathfrak{g}^*)^{\otimes k}$.

Let

$$F_n(\theta) = \frac{1}{2\pi n}\sum_{k=0}^{n-1}\sum_{\ell=-k}^{k} e^{i\ell\theta} = \frac{1}{2\pi n}\frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)}$$

denote Fejer's kernel [**25**, p. 413]. Then $\int_{-\pi}^{\pi} F_n(\theta)\, d\theta = 1$ for all n and

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} F_n(\theta)\varphi(\theta)\, d\theta = \varphi(0)$$

for all continuous functions, $\varphi : [-\pi, \pi] \to \mathbb{C}$.

If $\beta = \xi_1 \otimes \cdots \otimes \xi_k$ with $\xi_p \in V_{j_p}$ for $p = 1, \ldots, k$ then

$$\Gamma_\theta\beta = \left(e^{i\sum_{p=1}^{k} j_p\theta}\right)\beta.$$

So

$$\int_{-\pi}^{\pi} F_n(\theta)\Gamma_\theta\beta d\theta = 0 \quad \text{if} \quad \sum_{j=1}^{k} j_p > n.$$

Since all $j_p \geqslant 1$ we have

$$\int_{-\pi}^{\pi} F_n(\theta)\Gamma_\theta d\theta = 0 \quad \text{on} \quad \mathfrak{g}^{\otimes k} \quad \text{if} \quad k > n.$$

Consequently

$$\int_{-\pi}^{\pi} F_n(\theta)\Gamma'_\theta\alpha d\theta = 0 \quad \text{if} \quad \alpha \in (\mathfrak{g}^*)^{\otimes k} \quad \text{and} \quad k > n.$$

Now an elementary argument using (3·8) and the strong continuity of $\theta \mapsto \Gamma'_\theta$ on each $(\mathfrak{g}^*)^{\otimes k}$ shows that $\theta \mapsto \Gamma'_\theta$ is strongly continuous on J_t^0 in the norm (2·11). Hence if $\alpha \in J_t^0$ then

$$\gamma_n := \int_{-\pi}^{\pi} F_n(\theta)\Gamma'_\theta \alpha \, d\theta$$

is also in J_t^0 and is zero in all ranks $> n$. Moreover

$$\begin{aligned} \|\gamma_n - \alpha\|_t &= \left\| \int_{-\pi}^{\pi} F_n(\theta)(\Gamma'_\theta\alpha - \alpha) d\theta \right\|_t \\ &\leqslant \int_{-\pi}^{\pi} F_n(\theta)\|\Gamma'_\theta\alpha - \alpha\|_t d\theta \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \end{aligned}$$

LEMMA 3·6 (Theorem 3·1–Graded case). *In addition to the hypotheses of Theorem* 3·1*, assume that* $\mathfrak{g}$ *is a graded algebra and that for any complex number* λ *with* $|\lambda| = 1$*,* q *is invariant under the transposes,* $(\delta_\lambda)'$*, of the dilations introduced in Notation* 3·4. *Then the conclusions of Theorem* 3·1 *hold.*

Proof. By Theorem 2·5 the map $f \to \hat{f}(e)$ is isometric from $\mathcal{H}L^2(G, \rho_t)$ into J_t^0. To prove the surjectivity it suffices therefore to prove that the image is dense. To this end it suffices, by Lemma 3·5, to show that if α is a finite rank tensor in J_t^0 then there exists a function $u \in \mathcal{H}L^2(G, \rho_t)$ such that $\alpha = \hat{u}$. Since, in our case, the exponential map is a holomorphic diffeomorphism onto G we may identify G with $\mathbb{C}^N$ and define u as the holomorphic function on G given by

$$u(\exp \xi) = \sum_{n=0}^{\infty} \left\langle \alpha_n, \xi^{\otimes n} \right\rangle / n!$$

This is a finite sum because α is of finite rank.

One now easily concludes (see [**4**, proposition 6·3]) that $\hat{u} = \alpha$. Indeed, for any $\xi \in \mathfrak{g}$,

$$\begin{aligned} \left\langle \hat{u}(e), \xi^{\otimes k} \right\rangle &= \left(\frac{d}{dt}\right)^k \bigg|_{t=0} u(e^{t\xi}) = \left(\frac{d}{dt}\right)^k \bigg|_{t=0} \sum_{0}^{\infty} \frac{1}{n!} \left\langle \alpha_n, (t\xi)^{\otimes n} \right\rangle \\ &= \left\langle \alpha_k, \xi^{\otimes k} \right\rangle. \end{aligned} \tag{3·9}$$

By polarization, the linear span of $\{\xi^{\otimes k} : \xi \in \mathfrak{g}\}$ is the set of all symmetric $\mathbb{R}$-tensors, $\mathcal{S}$. It follows that $\hat{u}(e) = \alpha$ on $\mathcal{S}$. But, by the Poincaré–Birkhoff–Witt theorem, [**26**, lemma 3·3·3], we know that $T(\mathfrak{g}) = \mathcal{S} \oplus J$, and, since $\hat{u} - \alpha$ annihilates J, we conclude that $\hat{u}(e) = \alpha$ on $T(\mathfrak{g})$.

Since u is a holomorphic function such that $\hat{u}(e) = \alpha \in J_t^0$, it follows from Proposition 2·6 that $u \in \mathcal{H}L^2(G, \rho_t)$.

Alternatively, one may conclude that $u \in L^2(G, \rho_t)$ (or in fact that $u \in L^p(G, \rho_t)$ for all $0 < p < \infty$) on the grounds that any polynomial is in $L^p(G, \rho_t)$. The latter assertion is proved using the heat kernel upper bound in [**6**, theorem 3·4] and the fact that for any polynomial P on G (i.e., $P(x) = \widetilde{P} \circ \exp^{-1}(x)$ where $\widetilde{P}$ is a polynomial on $\mathfrak{g}$) there exist $C, \alpha \geqslant 0$ (for example, see [**27**, section IV·5]) such that $|P(x)| \leqslant C(1 + d(e, x))^\alpha$ with $d(x, y)$ being the sub-Riemannian distance associated with q as in [**6**, equation (3·1)].

Remark 3·7. Let $\mathfrak{g}$ be a graded algebra with decomposition $\mathfrak{g} = \oplus_{i=1}^{\infty} V_i$ and equipped with the dilations introduced in Notation 3·4. Let q be a Hermitian form on $\mathfrak{g}^*$ satisfying

Hörmander's condition. Let H be the Hörmander subspace of $\mathfrak{g}$ equipped with its scalar product $(\cdot,\cdot)_H$ induced by q. See the discussion after (2·8). It is not hard to check that a necessary and sufficient condition for q to be invariant under the dilation $(\delta_\lambda)'$, $|\lambda| = 1$, is that H be the orthogonal direct sum of the non-trivial $H \bigcap V_i$, $i = 1, \ldots,$ under $(\cdot,\cdot)_H$. This is equivalent to saying that there exists an orthonormal basis $\{X_j\}_{j=1}^m$ of $(H, (\cdot,\cdot)_H)$ such that each X_j belongs to V_i for some $i = i(j)$. In particular, if $\mathfrak{g}$ is stratified and $H = V_1$, the Hermitian form q is invariant under the dilations $(\delta_\lambda)'$ with $|\lambda| = 1$. But this is far from the only example. For instance, in the Heisenberg algebra $\mathfrak{h}_3^{\mathbb{C}}$ described above, consider the following two cases:

(a) The model subelliptic case where $H = V_1 = \operatorname{span}(X, Y)$ with X, Y being an orthonormal basis (this is equivalent to a description of q);
(b) The non-degenerate case where $H = \mathfrak{h}_3^{\mathbb{C}} = \operatorname{span}(X, Y, Z)$ with X, Y, Z being an orthonormal basis.

Note that the dilatations δ_λ on $\mathfrak{h}_3^{\mathbb{C}}$ are given by

$$\delta_\lambda(X) = \lambda X, \quad \delta_\lambda(Y) = \lambda Y, \quad \delta_\lambda(Z) = \lambda^2 Z.$$

Although only structure (a) above is "homogeneous" with respect to all dilations δ_λ, $\lambda \in \mathbb{C}$, the structures (a) and (b) are both invariant under these dilations when $|\lambda| = 1$. Thus Lemma 3·5 applies to both and shows that the finite rank tensors are dense in J_t^0, $t > 0$, for the Hermitian forms in both cases (a) and (b).

Remark 3·8. Lemma 3·5 asserts that the finite rank tensors in J^0 are dense in J_t^0 for each $t > 0$ if $\mathfrak{g}$ is graded nilpotent and the Hermitian form q is automorphism invariant, as in the hypotheses of Lemma 3·5. We don't know whether such density holds if $\mathfrak{g}$ is nilpotent but not graded or even if $\mathfrak{g}$ is graded but q is not invariant under δ_λ, $|\lambda| = 1$. On the other hand, in view of [**14**, theorem 4·15], we know that when q is nondegenerate the finite rank tensors cannot be dense in J_t^0 for any $t > 0$ unless $\mathfrak{g}$ is nilpotent.

Proof of Theorem 3·1. Let G be a connected, simply connected, nilpotent complex Lie group with Lie algebra $\mathfrak{g}$ whose dual is equipped with a quadratic form q satisfying Hörmander's condition. Let $X_1, \ldots, X_m$ be an orthonormal basis of $(\ker q)^0$. Choose $r \in \mathbb{N}$ sufficiently large so that $\mathfrak{g}$ is nilpotent of step r. Let $\mathfrak{n}(m, r)$ denote the step r free nilpotent complex Lie algebra on m generators $\eta_1, \ldots, \eta_m$. (see [**23**], [**9**, p. 37] and also [**1**, chapter 2 section 2]). By definition of $\mathfrak{n}(m, r)$, there exists a Lie algebra homomorphism

$$\pi : \mathfrak{n}(m, r) \longrightarrow \mathfrak{g}$$

such that $\pi(\eta_i) = X_i$. (This property holds for any step r nilpotent Lie algebra $\mathfrak{g}$ generated by m elements $X_1, \ldots, X_m$.) The algebra $\mathfrak{n}(m, r)$ is a stratified Lie algebra with

$$\mathfrak{n}(m, r) = V_1 + \cdots + V_r$$

where V_1 is the linear span of $\eta_1, \ldots, \eta_m$ and $V_i = [V_1, V_{i-1}]$, $i = 2, \ldots, r$. For a description of a basis of V_i, see [**1**, **9**]. The natural dilation structure on $\mathfrak{n}(m, r)$ is defined by setting $\delta_\lambda(\xi) = \lambda^i \xi$ for $\xi \in V_i$, $i = 1, 2, \ldots, r$, $\lambda \in \mathbb{C}$.

On the dual $\mathfrak{n}(m, r)^*$ of $\mathfrak{n}(m, r)$, set

$$\widetilde{q}(a) = \sum_1^m |\langle a, \eta_i \rangle|^2. \tag{3·10}$$

By construction, we have

$$(\ker \widetilde{q})^0 = \mathrm{span}(\eta_1, \ldots, \eta_m) = V_1.$$

Hence $\widetilde{q}$ satisfies the Hörmander condition and is invariant with respect to the dilations $\delta_{e^{i\theta}}$. Moreover, by (2·8) and (3·10), $\widetilde{q} = \pi^* q = q \circ \pi$.

Suppose now that $t > 0$ and that $\alpha \in J_t^0(\mathfrak{g})$. The surjective homomorphism π extends to a surjective homomorphism from $T(\mathfrak{n}(m, r))$ onto $T(\mathfrak{g})$. We denote the extension again by π. Then π^* maps from $T(\mathfrak{g})'$ into $T(\mathfrak{n}(m, r))'$. Moreover $\pi(J(\mathfrak{n}(m, r)) = J(\mathfrak{g})$ and so $\pi^*(J^0(\mathfrak{g})) \subset J^0(\mathfrak{n}(m, r))$. Let $\alpha' = \pi^*\alpha = \alpha \circ \pi$. It follows from (3·10) and (2·8) that $\alpha' \in J_t^0(\mathfrak{n}(m, r))$ and

$$\|\alpha'\|_t = \|\alpha\|_t < \infty. \tag{3·11}$$

Let $N(m, r)$ be the simply connected nilpotent Lie group whose Lie algebra is $\mathfrak{n}(m, r)$. An application of Lemma 3·6 allows us to conclude that there exists a holomorphic function, v, on $N(m, r)$ such that $\hat{v} = \alpha'$. We further know that v is square integrable relative to the time t heat kernel measure associated to $\tilde{q}$, but we will not need this fact here.

Let φ be the unique complex surjective Lie group homomorphism from $N(m, r)$ to G such that $\varphi_{*e} = \pi$. It is known that

$$G_0 := \ker \varphi \subset N(m, r)$$

is connected if and only if G is simply connected, see [**10**, theorem 4·8]. Since we have assumed that G is simply connected, G_0 is connected in our case. Moreover, G_0 is a complex Lie group because $\mathrm{Lie}(G_0) = \ker \pi$ is a complex Lie algebra.

We assert that the function v is right (and therefore left) invariant under the normal subgroup G_0 and consequently factors through a holomorphic function f on G, i.e. $v = f \circ \varphi$. To prove this assertion, it suffices to show that $\tilde{\eta} v \equiv 0$ on $N(m, r)$ for any vector $\eta \in T_e(G_0)$. Since $\tilde{\eta} v$ is holomorphic on $N(m, r)$ it is enough to show that $(\tilde{\beta}\tilde{\eta} v)(e) = 0$ for all $\beta \in \mathfrak{n}(m, r)$. But because $\pi\eta = 0$,

$$(\tilde{\beta}\tilde{\eta} v)(\tilde{e}) = \langle \alpha', \beta \otimes \eta \rangle = \langle \alpha, \pi(\beta \otimes \eta) \rangle = \langle \alpha, (\pi\beta) \otimes (\pi\eta) \rangle = 0$$

and the assertion is proved.

To each $A_i \in \mathfrak{g}$, we may use the surjectivity of π to find a $B_i \in \mathfrak{n}(m, r)$ such that $\pi B_i = A_i$. It is well known and easy to show that $\tilde{B}_i(F \circ \varphi) = (\tilde{A}_i F) \circ \varphi$ for any smooth function F on G. By repeated use of this identity, we find

$$\begin{aligned} \langle \hat{v}(e), B_1 \otimes \cdots \otimes B_n \rangle &= (\tilde{B}_1 \ldots \tilde{B}_n v)(e) = (\tilde{B}_1 \ldots \tilde{B}_n (f \circ \varphi))(e) \\ &= (\tilde{A}_1 \ldots \tilde{A}_n f)(\varphi(e)) = \langle \hat{f}(e), A_1 \otimes \cdots \otimes A_n \rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle \hat{v}\,(e)\,, B_1 \otimes \cdots \otimes B_n \rangle &= \left\langle \alpha', B_1 \otimes \cdots \otimes B_n \right\rangle = \left\langle \pi^*\alpha, B_1 \otimes \cdots \otimes B_n \right\rangle \\ &= \langle \alpha, \pi B_1 \otimes \cdots \otimes \pi B_n \rangle = \langle \alpha, A_1 \otimes \cdots \otimes A_n \rangle\,. \end{aligned}$$

Comparing the previous two equations allows us to conclude that f is a holomorphic function on G such that $\hat{f}(e) = \alpha$. In light of Theorem 2·5 and Proposition 2·6, this fact is sufficient to complete the proof of Theorem 3·1.

4. *The Fourier–Wigner transform and holomorphic functions*

We are going to show in this section how the harmonic oscillator Hamiltonian produces a natural source of holomorphic functions on the complex three dimensional Heisenberg group, $H_3^{\mathbb{C}}$, which lie in $\mathcal{H}L^2(H_3^{\mathbb{C}}, \rho_t)$, where ρ_t is the heat kernel of the natural subelliptic Laplacian. There is a correspondence between analytic vectors for the quantum mechanical harmonic oscillator Hamiltonian and holomorphic functions on the complex Heisenberg group. The correspondence is induced by the Fourier-Wigner transform and also by the Wigner transform itself. The former seems easier to deal with. We will study only the Fourier-Wigner transform in this paper.

4·1. *Holomorphic Fourier–Wigner Functions*

Notation 4·1. The complex Heisenberg group is $H_3^{\mathbb{C}} = \mathbb{C}^3$ with the group law

$$(z_1, z_2, z_3) \cdot (z_1', z_2', z_3') = (z_1 + z_1', z_2 + z_2', z_3 + z_3' + (1/2)(z_1 z_2' - z_2 z_1')).$$

Let us observe here that if $z_j = x_j + i y_j$ then

$$z_1 z_2' - z_2 z_1' = [x_1 x_2' - x_2 x_1' - (y_1 y_2' - y_2 y_1')] + i[x_1 y_2' - y_2 x_1' + y_1 x_2' - x_2 y_1'].$$

Define vector fields by

$$X_1 = \partial/\partial x_1 - (x_2/2)\partial/\partial x_3 - (y_2/2)\partial/\partial y_3$$

$$X_2 = \partial/\partial x_2 + (x_1/2)\partial/\partial x_3 + (y_1/2)\partial/\partial y_3$$

$$Y_1 = \partial/\partial y_1 + (y_2/2)\partial/\partial x_3 - (x_2/2)\partial/\partial y_3$$

$$Y_2 = \partial/\partial y_2 - (y_1/2)\partial/\partial x_3 + (x_1/2)\partial/\partial y_3.$$

These are the left invariant vector fields which reduce at the origin to $\partial/\partial x_1$, etc. We will use the sub-Laplacian given by

$$\Delta = X_1^2 + X_2^2 + Y_1^2 + Y_2^2. \tag{4·1}$$

Define the kernel ρ_t on $H_3^{\mathbb{C}}$ by the identity $e^{t\Delta/4} f = f * \rho_t$.

Notation 4·2. Let Q denote the operator of multiplication by x on $L^2(\mathbb{R})$ with its natural domain of self-adjointness and let P denote the operator $-i d/dx$ with its natural domain of self-adjointness. Denote by H_0 the operator, $(1/2)(P^2 + Q^2)$ with its natural domain of self-adjointness.

We will show in Lemma 4·4 that for any real numbers u and v the closure of the operator $(uP + vQ)|\mathcal{S}(\mathbb{R})$ is self-adjoint. This is a very well known fact that goes back at least to J. M. Cook [**2**, theorem 10]. See also [**21**, theorem X·41]. These proofs show that these operators are essentially self-adjoint on any domain that contains the Hermite functions. Nevertheless we will give a short self contained proof in Lemma 4·4 because it comes right out of an identity that we will need anyway. We will always interpret the sum $uP + vQ$ as this self-adjoint operator.

The main theorem of this section is the following.

THEOREM 4·3. *Let $s > 0$. Suppose that f is in the domain of e^{sH_0}. Then the Fourier-Wigner transform*

$$W(u, v, w) := e^{iw} \left(e^{i(uP+vQ)} f, f\right), \quad u, v, w \in \mathbb{R}, \tag{4·2}$$

has a unique analytic continuation to an entire function $\tilde{W}$ on $\mathbb{C}^3$. Moreover $\tilde{W}$ is in $\mathcal{H}L^2(H_3^{\mathbb{C}}, \rho_t)$ if

$$t < \frac{\tanh s}{2(1+\tanh s)} = \frac{1}{4}(1 - e^{-2s}). \tag{4·3}$$

The proof depends on the following lemmas, of which the first is in part a precise restatement of a well known identity (cf. (4·5) below) expressing the evolution of the harmonic oscillator in the Heisenberg picture. It can be found in many elementary books on quantum mechanics. See e.g. [**11**, page 257].

LEMMA 4·4 (Rotation in P, Q space). *Let u and v be real, let $r = (u^2 + v^2)^{1/2}$ and let $s > 0$. Then $uP + vQ$ is essentially self-adjoint on $\mathcal{S}$ and*

$$\|e^{uP+vQ}e^{-sH_0}\| = \|e^{rQ}e^{-sH_0}\|. \tag{4·4}$$

Proof. Since $\mathcal{S} = C^\infty(H_0)$ we have, for any real number θ, $e^{i\theta H_0}\mathcal{S} = \mathcal{S}$. Therefore, since $\mathcal{S}$ is a core for Q it is also a core for $Qe^{-i\theta H_0}$ and for $e^{i\theta H_0}Qe^{-i\theta H_0}$. Let $\epsilon > 0$ and note that range $e^{-\epsilon H_0} \subset \mathcal{S}$. Define an operator valued function of θ by

$$T(\theta) := \{e^{-i\theta H_0}e^{-\epsilon H_0}\}\{(P\sin\theta + Q\cos\theta)e^{-\epsilon H_0}\}\{e^{i\theta H_0}e^{-\epsilon H_0}\}.$$

Each of the three operators in braces is a function of θ into the space of bounded operators and each is differentiable with respect to θ with the operator norm on the range. Using the commutation relations $[iH_0, P] = -Q$, $[iH_0, Q] = P$ on $\mathcal{S}$ (and therefore on the range of $e^{-\epsilon H_0}$) it is straightforward to compute that $dT(\theta)/d\theta = 0$ by a computation which is easily justified, given the preceding information. Hence $T(\theta) = T(0)$ for all real θ. That is,

$$e^{-\epsilon H_0}e^{-i\theta H_0}\{P\sin\theta + Q\cos\theta\}e^{i\theta H_0}e^{-2\epsilon H_0} = e^{-\epsilon H_0}Qe^{-2\epsilon H_0}.$$

We may cancel the injective operator $e^{-\epsilon H_0}$ on the left and then multiply by $e^{i\theta H_0}$ on the left and by $e^{-i\theta H_0}$ on the right to find

$$\{P\sin\theta + Q\cos\theta\}e^{-2\epsilon H_0}f = e^{i\theta H_0}Qe^{-i\theta H_0}e^{-2\epsilon H_0}f$$

for all $f \in L^2(\mathbb{R})$ and all $\epsilon > 0$. Let $g \in L^2(\mathbb{R})$ and insert $f := (H_0+1)^{-1}g$ into this equality. Shift the factors $(H_0+1)^{-1}$ to the left of the factors $e^{-2\epsilon H_0}$. We may then let $\epsilon \downarrow 0$ because the product to the left of the operator $e^{-2\epsilon H_0}$ on each side of the equation is a bounded operator. Since any function f in $\mathcal{S}$ may be written in the form $f = (H_0 + 1)^{-1}g$ with $g \in L^2(\mathbb{R})$, we have shown

$$\{P\sin\theta + Q\cos\theta\} = e^{i\theta H_0}Qe^{-i\theta H_0} \tag{4·5}$$

on $\mathcal{S}$. Since $\mathcal{S}$ is a core for the selfadjoint operator on the right, $\{P\sin\theta + Q\cos\theta\}$ is essentially self-adjoint on $\mathcal{S}$ and (4·5) holds on the full domain of the closure of $P\sin\theta + Q\cos\theta$. The functional calculus now shows that

$$\begin{aligned} e^{r(P\sin\theta + Q\cos\theta)}e^{-sH_0} &= e^{e^{i\theta H_0}rQe^{-i\theta H_0}}e^{-sH_0} \\ &= e^{i\theta H_0}e^{rQ}e^{-i\theta H_0}e^{-sH_0} \\ &= e^{i\theta H_0}e^{rQ}e^{-sH_0}e^{-i\theta H_0}, \end{aligned}$$

from which (4·4) follows.

Notation 4·5. The ground state (lowest eigenfunction) for the operator H_0 is the function $\psi_0(x) = \pi^{-1/4}e^{-x^2/2}$. The associated ground state transformation, [**17**, page 71] and

[**3**, page 458], is defined as follows. Define the ground state measure γ by $\gamma(dx) = \psi_0(x)^2 dx = \pi^{-1/2}e^{-x^2}dx$. Under the unitary map $U : f \to f(x)/\psi_0(x)$ from $L^2(\mathbb{R}, dx)$ to $L^2(\mathbb{R}, \gamma)$, the Hamiltonian H_0 transforms to $UH_0U^{-1} = N+(1/2)$ where N is the Dirichlet form operator associated to the measure γ by the formula,

$$(Nf, g)_{L^2(\gamma)} = (1/2)\int_{\mathbb{R}} f'(x)\overline{g}'(x)d\gamma(x).$$

Under the unitary transform U the operator Q goes over to an operator $\hat{Q} := UQU^{-1}$, which again consists of multiplication by x (but on a different domain).

LEMMA 4·6 (Hypercontractive estimates). *Let $s > 0$ and let r be real. Then*

$$\|e^{r\hat{Q}}e^{-sN}\|_{L^2(\gamma)\to L^2(\gamma)} \leqslant e^{r^2/2\tanh s}. \tag{4·6}$$

Moreover

$$\|e^{rQ}e^{-sH_0}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leqslant e^{r^2/2\tanh s}e^{-s/2} \tag{4·7}$$

and

$$\|e^{uP+vQ}e^{-sH_0}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leqslant e^{r^2/2\tanh s}e^{-s/2} \tag{4·8}$$

when $r = (u^2+v^2)^{1/2}$.

Proof. Let $f \in L^2(\gamma)$ and let $g = e^{-sN}f$. Define p by the equation $e^{2s} = 2p-1$. Then e^{-sN} is a contraction from $L^2(\gamma)$ to $L^{2p}(\gamma)$, [**13**]. [A change of variance from [**13**] and a change to $(1/2)\nabla^*\nabla$ cancel.] So $\|g\|_{2p} \leqslant \|f\|_2$. Let $q = p/(p-1) = (\tanh s)^{-1}$. Then

$$\|e^{r\hat{Q}}g\|_2^2 = \int_{\mathbb{R}} e^{2rx}|g(x)|^2 d\gamma(x) \leqslant \|e^{2rx}\|_q\|g\|_{2p}^2 \leqslant \|e^{2rx}\|_q\|f\|_2^2.$$

But $\|e^{2rx}\|_q^q = \int_{\mathbb{R}} e^{2qrx}d\gamma(x) = e^{(2qr)^2/4}$. Hence $\|e^{2rx}\|_q = e^{qr^2}$. So

$$\|e^{r\hat{Q}}e^{-sN}f\|_2 = \|e^{r\hat{Q}}g\|_2 \leqslant e^{qr^2/2}\|f\|_2.$$

Returning now to Lebesgue measure, the inequality (4·7) follows from (4·6) because $UH_0U^{-1} = N+(1/2)$, while $Ue^{rQ}U^{-1} = e^{r\hat{Q}}$. (4·8) now follows from (4·4) and (4·7).

LEMMA 4·7 (Taylor coefficient estimates). *Suppose that $n_1, \ldots, n_{2r}$ are non-negative integers with $n_1+\cdots+n_{2r} = k$. Then*

$$\|P^{n_1}Q^{n_2}\cdots P^{n_{2r-1}}Q^{n_{2r}}f\| \leqslant 2^{k/2}\big\|(H_0+k)^{k/2}f\big\| \tag{4·9}$$

for all $f \in \mathcal{S}(\mathbb{R})$. There is a constant C such that, for $s > 0$,

$$\|P^{n_1}Q^{n_2}\cdots P^{n_{2r-1}}Q^{n_{2r}}e^{-sH_0}\| \leqslant C\sqrt{k!}\Big(\frac{e^{2s}}{s}\Big)^{k/2}\Big/k^{1/4}, \quad k \geqslant 1. \tag{4·10}$$

Proof. We are going to give a proof here for the reader's convenience. But we want to emphasize that the machinery we will use is quite well known in the literature of quantum field theory. See e.g. [**21**, section X·6, example 2] and also [**21**, section X·7].

Let $a = (Q+iP)/\sqrt{2}$, interpreted as the closure of the actual sum. Then $a^* = (Q-iP)/\sqrt{2}$ (closure of sum). Moreover $\mathcal{S}(\mathbb{R})$ is a core for both operators and both leave $\mathcal{S}(\mathbb{R})$ invariant. Let $M = a^*a$. Then M is a non-negative self-adjoint operator with core $\mathcal{S}(\mathbb{R})$ and leaves $\mathcal{S}(\mathbb{R})$ invariant. One can easily verify on $\mathcal{S}(\mathbb{R})$ the identities $aa^* = a^*a+1$, $H_0 = M+(1/2)$, $Ma = a(M-1)$ and $Ma^* = a^*(M+1)$.

Since $Q = (a + a^*)/\sqrt{2}$ and $P = (a - a^*)/i\sqrt{2}$ the product $P^{n_1} Q^{n_2} \cdots P^{n_{2r-1}} Q^{n_{2r}}$ is a sum of products $A_1 \cdots A_k$ with each $A_j = a$ or a^* and with an overall factor of $2^{-k/2}$ in magnitude. Hence the left side of (4·9) is at most $2^{-k/2} \sum \|A_1 \cdots A_k f\|$ where the sum is over all possible choices, $A_j = a$ or a^*, for each $j \in \{1, \ldots, k\}$.

We may now use, on each of these 2^k terms, the inequality

$$\|A_1 \cdots A_k f\| \leqslant \left\|(M + k)^{k/2} f\right\|$$

stated in [**22**, problem 36 on page 178] and proved in [**21**, section X·7]. This proves (4·9).

In order to derive (4·10) note first that the range of $e^{-sH_0} \subset \mathcal{S}(\mathbb{R})$ because $\mathcal{S}(\mathbb{R})$ is exactly the set of C^∞ vectors for H_0. Taking $f = e^{-sH_0} g$ in (4·9) with $\|g\| = 1$, the inequality (4·10) can be deduced from (4·9) by observing that

$$\|P^{n_1} Q^{n_2} \cdots P^{n_{2r-1}} Q^{n_{2r}} f\| \leqslant 2^{k/2} \left\|(H_0 + k)^{k/2} e^{-sH_0} g\right\| \leqslant 2^{k/2} \left\|(H_0 + k)^{k/2} e^{-sH_0}\right\|,$$

while

$$\begin{aligned} 2^{k/2} \|(H_0 + k)^{k/2} e^{-sH_0}\| &\leqslant 2^{k/2} \sup_{u \geqslant 1/2} (u + k)^{k/2} e^{-s(u+k)} e^{sk} \\ &\leqslant 2^{k/2} e^{sk} \sup_{v \geqslant 0} v^{k/2} e^{-sv} \\ &= 2^{k/2} e^{sk} (k/2s)^{k/2} e^{-k/2} \\ &= k^{k/2} \left(\frac{e^{2s}}{es}\right)^{k/2}, \end{aligned}$$

since $v^{k/2} e^{-sv}$ has a maximum on $[0, \infty)$ at $v = k/(2s)$. Stirling's formula, $k^{k/2} \sim (k!)^{1/2} e^{k/2} / (2\pi k)^{1/4}$, now shows that

$$2^{k/2} \left\|(H_0 + k)^{k/2} e^{-sH_0}\right\| \sim (k!)^{1/2} (e^{2s}/s)^{k/2} / (2\pi k)^{1/4}$$

for large k. This proves (4·10).

LEMMA 4·8 (Convergence of power series). *Let $s > 0$ and suppose that $f \in D(e^{sH_0})$. Then the power series expansion of $e^{z_1 P + z_2 Q} f$ in the two complex variables z_1, z_2 converges absolutely. Moreover if z_1 and z_2 are both real then the sum is $e^{z_1 P + z_2 Q} f$, where the exponential is defined by the spectral theorem for the self-adjoint operator $z_1 P + z_2 Q$. Similarly, if $z_1 = i b_1$ and $z_2 = i b_2$ are purely imaginary then the sum is $e^{i(b_1 P + b_2 Q)} f$, where the exponential is defined by the spectral theorem for the self-adjoint operator $b_1 P + b_2 Q$.*

Proof. We may assume that $f = e^{-sH_0} g$ with $\|g\| = 1$. Each term of the series

$$\sum_{k=0}^{\infty} \frac{(z_1 P + z_2 Q)^k}{k!} f$$

is well defined by Lemma 4·7 and has the form

$$(1/k!) \sum_{j=0}^{k} z_1^j z_2^{k-j} E_j f$$

where E_j is a sum of $\binom{k}{j}$ products of k factors of P and Q as in Lemma 4·7. In view of the estimate (4·10) we find

$$\left\|(1/k!)\sum_{j=0}^{k} z_1^j z_2^{k-j} E_j f\right\| \leqslant (1/k!)\sum_{j=0}^{k} |z_1|^j |z_2|^{k-j}\binom{k}{j} C\,(k!)^{1/2}\left(\frac{e^{2s}}{s}\right)^{k/2} \Big/ k^{1/4}$$
$$\leqslant (|z_1|+|z_2|)^k (k!)^{-1/2} C \left(\frac{e^{2s}}{s}\right)^{k/2} \Big/ k^{1/4}.$$

Therefore the series converges absolutely.

In particular if z_1 and z_2 are real then f is an analytic vector for the self-adjoint operator $z_1 P + z_2 Q$. Hence the series converges to the exponential defined by the spectral theorem. A similar observation applies to $e^{i(b_1 P + b_2 Q)} f$. This proves Lemma 4·8.

LEMMA 4·9 (Power series vs. spectral theorem). *Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$. Then, for $s > 0$,*

$$e^{i(z_1 P + z_2 Q)} e^{-sH_0} = e^{(a_2 b_1 - a_1 b_2)/2} e^{i(a_1 P + a_2 Q)} e^{-(b_1 P + b_2 Q)} e^{-sH_0} \tag{4·11}$$

wherein the left-hand side is defined as a power series as in Lemma 4·8 *while the operators on the right-hand side are all defined by the spectral theorem.*

Proof. Let $f = e^{-sH_0} g$. If u_1, u_2, z_1, z_2 are all real then the Weyl form of the canonical commutation relations, cf. [**8**, equation (1·24)], implies

$$e^{i((u_1+z_1)P + (u_2+z_2)Q)} f = e^{i(u_1 z_2 - u_2 z_1)/2} e^{i(u_1 P + u_2 Q)} e^{i(z_1 P + z_2 Q)} f. \tag{4·12}$$

Since $e^{i(u_1 P + u_2 Q)}$ is unitary, both sides of (4·12) are analytic functions of $z_1, z_2 \in \mathbb{C}^2$. Hence (4·12) holds for all complex z_1, z_2. Choose $z_1 = iv_1$ and $z_2 = iv_2$ purely imaginary. Then (4·12) reduces to

$$e^{i((u_1+iv_1)P + (u_2+iv_2)Q)} f = e^{(u_2 v_1 - v_2 u_1)/2} e^{i(u_1 P + u_2 Q)} e^{-(v_1 P + v_2 Q)} f.$$

LEMMA 4·10 (Operator bounds for complex exponents). *For $s > 0$ and any two complex numbers $z_j = a_j + ib_j$ we have the operator bound*

$$\left\| e^{i(z_1 P + z_2 Q)} e^{-sH_0} \right\| \leqslant e^{(a_2 b_1 - a_1 b_2)/2} e^{(|z_1|^2 + |z_2|^2)/2 \tanh s} e^{-s/2}. \tag{4·13}$$

Proof. If $z_j = a_j + ib_j$ for $j = 1, 2$ then, by (4·11) and (4·8),

$$\left\| (e^{i(z_1 P + z_2 Q)} e^{-sH_0} \right\| = e^{(a_2 b_1 - a_1 b_2)/2} \left\| e^{i(a_1 P + a_2 Q)} e^{-(b_1 P + b_2 Q)} e^{-sH_0} \right\|$$
$$= e^{(a_2 b_1 - a_1 b_2)/2} \left\| e^{-(b_1 P + b_2 Q)} e^{-sH_0} \right\|$$
$$\leqslant e^{(a_2 b_1 - a_1 b_2)/2} e^{(b_1^2 + b_2^2)/2 \tanh s} e^{-s/2}.$$

LEMMA 4·11 (Form bounds for complex exponents). *If $(z_1, z_2) \in \mathbb{C}^2$ and f is in the domain of e^{sH_0} then*

$$\left| \left(e^{i(z_1 P + z_2 Q)} f, f \right) \right| \leqslant e^{m^2/(4 \tanh s)} \| e^{sH_0} f \|^2 e^{-s}$$

where $m^2 = |z_1|^2 + |z_2|^2$.

Proof. If z_1 and z_2 are real, then

$$\left(e^{i(z_1 P + z_2 Q)} f, f \right) = \left(e^{(i/2)(z_1 P + z_2 Q)} f, e^{-(i/2)(\bar{z}_1 P + \bar{z}_2 Q)} f \right). \tag{4·14}$$

As both sides of this equation are entire functions of z_1 and z_2, it follows that (4·14) also holds for all $z_1, z_2 \in \mathbb{C}$. Therefore

$$\begin{aligned}\left|\left(e^{i(z_1P+z_2Q)}f, f\right)\right| &\leqslant \left\|e^{(i/2)(z_1P+z_2Q)}f\right\|\,\left\|e^{-(i/2)(\bar z_1P+\bar z_2Q)}f\right\| \\ &\leqslant \left\|e^{(i/2)(z_1P+z_2Q)}e^{-sH_0}\right\|\,\left\|e^{-(i/2)(\bar z_1P+\bar z_2Q)}e^{-sH_0}\right\|\,\|e^{sH_0}f\|^2 \\ &\leqslant \left\{e^{(a_2b_1-a_1b_2)/8}e^{m^2/(8\tanh s)}e^{-s/2}\|e^{sH_0}f\|\right\}\cdot \\ &\qquad \left\{e^{-(a_2b_1-a_1b_2)/8}e^{m^2/(8\tanh s)}e^{-s/2}\|e^{sH_0}f\|\right\} \\ &\leqslant e^{m^2/(4\tanh s)}e^{-s}\|e^{sH_0}f\|^2.\end{aligned}$$

In the second from last line we have used (4·13) twice, but with opposite signs for the a_j.

Proof of Theorem 4·3. Consider the functions

$$\varphi(z) := k(|z_1|^2+|z_2|^2)^2+|z_3|^2 \text{and } \psi(z) := \varphi(z)^{1/4}.$$

We will choose a number $k>0$ later. Writing $m^2 = |z_1|^2+|z_2|^2$, a computation shows that

$$\begin{aligned}X_1\varphi &= 4kx_1m^2 - x_2x_3 - y_2y_3\\ X_2\varphi &= 4kx_2m^2 + x_1x_3 + y_1y_3\\ Y_1\varphi &= 4ky_1m^2 + y_2x_3 - x_2y_3\\ Y_2\varphi &= 4ky_2m^2 - y_1x_3 + x_1y_3.\end{aligned}$$

Another computation then shows that, for $z \neq 0$,

$$|\nabla\varphi(z)|^2 = m^2[16k^2m^4+|z_3|^2].$$

If we choose $k = 1/16$ then we find

$$|\nabla\varphi(z)|^2 = m^2\varphi$$

and therefore

$$|\nabla\psi| = \frac{1}{4}\frac{|\nabla\varphi|}{\varphi^{3/4}} = \frac{m}{4\varphi^{1/4}} \leqslant \frac{1}{2}.$$

The intrinsic distance d is defined by

$$d(x,y) = \sup\{f(x)-f(y)\colon f\in C^1(H_3^{\mathbb{C}}),\ |\nabla f|\leqslant 1\}.$$

For further information about this distance function the reader is referred to the discussion preceding [**6**, definition 3·1]. Thus the distance, $d(z)$, from the origin to (z_1, z_2, z_3) satisfies

$$d(z) \geqslant 2\psi = 2[2^{-4}(|z_1|^2+|z_2|^2)^2+|z_3|^2]^{1/4} = [(|z_1|^2+|z_2|^2)^2+16|z_3|^2]^{1/4}.$$

It follows that $m^2 \leqslant d^2(z)$ and $4|z_3| \leqslant d^2(z)$.

Now by Lemma 4·11

$$\begin{aligned}|\tilde W(z_1,z_2,z_3)| &\leqslant e^{|z_3|}e^{m^2/(4\tanh s)}\|e^{sH_0}f\|^2e^{-s},\\ &\leqslant e^{d^2(z)\frac{1+\tanh s}{4\tanh s}}\|e^{sH_0}f\|^2e^{-s}.\end{aligned}$$

Hence

$$|\tilde W(z_1,z_2,z_3)|^2 \leqslant e^{d^2(z)\frac{1+\tanh s}{2\tanh s}}\|e^{sH_0}f\|^4e^{-2s}. \tag{4·15}$$

By [**27**, theorem IV·4·2], for any $\varepsilon>0$ there exists a finite constant C_ε such that

$$\rho_t(z) \leqslant C_\varepsilon t^{-4}\exp\left(-\frac{d^2(z)}{4(1+\varepsilon)t}\right).$$

Here the 4 in t^{-4} is 8/2 where $8 = 4 \times 1 + 2 \times 2$ is the volume growth exponent of $H_3^{\mathbb{C}}$. Moreover, for any $\eta > 0$,

$$t^{-4} \int_{H_3^C} e^{-\eta d^2(z)} dz \leqslant A_\eta < \infty.$$

Thus $\widetilde{W}$ belongs to $L^2(H_3^C, \rho_t)$ if $1/(4t) > (1 + \tanh s)/2 \tanh s$. That is, if (4·3) holds then $\tilde{W} \in L^2(H_3^{\mathbb{C}}, \rho_t)$.

Remark 4·12. In applying Theorem IV·4·2 of [**27**] a reader might notice that the definition of the distance used there differs slightly from that given above. It is however a well known fact that these definitions coincide, [**18**].

THEOREM 4·13 (Insertion of Planck's constant). *Let h be a strictly positive real number. Define $P_h = -ihD$ and let*

$$H = \frac{1}{2}\big(P_h^2 + Q^2\big). \tag{4·16}$$

If f is in the domain of e^{sH} then the function

$$\tilde{W}_h(u, v, w) = e^{ihw}\left(e^{i(uP_h + vQ)} f, f\right) \tag{4·17}$$

on the real Heisenberg group H_3 has a holomorphic extension to all of H_3^C. Moreover, if

$$t < (1 - e^{-2sh})/(4h) \tag{4·18}$$

then the extension is in $\mathcal{H}L^2(H_3^C, \rho_t)$

Proof. The scale transformation $(Sf)(x) = h^{-1/4} f(x/h^{1/2})$ is a unitary operator on $L^2(\mathbb{R})$ and one can compute easily that $S^{-1}P_hS = h^{1/2}P$ and $S^{-1}QS = h^{1/2}Q$. Consequently $S^{-1}(uP_h + vQ)S = h^{1/2}(uP + vQ)$ and $S^{-1}HS = hH_0$. Therefore

$$\begin{aligned}\left\| e^{(uP_h+vQ)} e^{-sH} \right\| &= \left\| e^{h^{1/2}(uP+vQ)} e^{-shH_0} \right\| \\ &\leqslant e^{h(|u|^2+|v|^2)/2\tanh sh} e^{-sh/2}.\end{aligned}$$

The same argument leading to (4·15) now gives

$$|\tilde{W}_h(z_1, z_2, z_3)|^2 \leqslant e^{h\left(\frac{1+\tanh sh}{2\tanh sh}\right)d^2(z)} \|e^{sH} f\|^4 e^{-2sh}. \tag{4·19}$$

Consequently $\tilde{W}_h \in L^2(H_3^{\mathbb{C}}, \rho_t)$ if $1/(4t) > h((1 + \tanh sh)/2 \tanh sh)$. That is , if (4·18) holds.

Remark 4·14. The artificial relation (4·3) between t and s should be attributed to the fact that we are analytically continuing the Fourier-Wigner transform in (4·2) rather than the Wigner transform itself, [**8**]. The Wigner transform will be studied from the point of view of coherent states elsewhere. We expect a more perspicuous relation between t and s in that case. An analytic continuation of the Wigner transform has already been discussed in [**19**] using a description of the range space which is not based on the heat kernel measure that we are using in this paper.

Remark 4·15. We might point out, however, that the condition (4·18) suggests some kind of "semiclassical limit": as $h \downarrow 0$ the relation (4·18) goes over to $t < s/2$. On the other hand, keeping h fixed and letting $s \to \infty$, the relation (4·18) goes over to $th < 1/4$. This limit can be loosely interpreted to suggest that even for the "most" regular functions f the Fourier–Wigner transform associated to Planck'sconstant h will be in $\mathcal{H}L^2(H_3^{\mathbb{C}}, \rho_t)$ for only a bounded set of t, depending on h. In this sense Theorem 4·13 seems analogous to [**15**, theorem 4·6], according to which, the matrix elements of an irreducible unitary representation of a compact Lie group K have holomorphic extensions to the complexification of K lying in a certain L^2 space over the complexification if and only if the Casimir operator for the representation is appropriately related to the measure.

Remark 4·16. Extension of our results from the lowest dimensional Heisenberg group to higher dimensional Heisenberg groups is routine. The relation between analytic vectors for the harmonic oscillator Hamiltonian H and real analytic functions was discussed systematically in E. Nelson's paper [**20**]. Such a connection between the domain of e^{sH} and analytic functions was also discussed for Hamiltonians in infinitely many variables in [**12**].

4·2. *Alternative proofs*

Our proof of Theorem 4·3, in the previous subsection, was based on functional analytic methods. In this subsection we are going to reprove some parts of this theorem by use of the known explicit kernel of the key integral operator.

Suppose $f \in \mathcal{D}(e^{sH_0})$ and let $F := e^{sH_0} f \in L^2(\mathbb{R})$. We are going sketch an alternative proof to the fact that the function,

$$V(u, v) := \left(e^{i(uP+vQ)} f, f\right)$$

has an extension to an analytic function on $\mathbb{C}^2$ which satisfies the bounds in Lemma 4·11. We begin by using Mehler's formula, see for example [**24**, p. 38], which shows that $f = e^{-sH_0} F$ may be represented as

$$f(z) = \sqrt{\frac{1}{2\pi \sinh s}} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\coth s \cdot (z^2 + w^2) + \frac{1}{\sinh s} zw\right\} F(w)\, dw. \tag{4·20}$$

It is now evident that f has an analytic continuation to the complex plane given by the the right-hand side of (4·20). Moreover, an application of the Cauchy-Schwarz inequality along with an explicit Gaussian integration shows,

$$|f(x+iy)| \leqslant \left(\frac{1}{4\pi \sinh s \cdot \cosh s}\right)^{1/4} \|F\|_2 \exp\left(-\frac{x^2}{2}\tanh s + \frac{y^2}{2}\coth s\right). \tag{4·21}$$

According to Folland [**8**, p. 30],

$$V(u, v) = \int_{\mathbb{R}} e^{ivx} f(x + u/2)\, \overline{f(x - u/2)} dx.$$

Using this representation along with properties of f just described, it is easily seen that V also has an analytic continuation to $\mathbb{C}^2$ given by

$$V(z_1, z_2) = \int_{\mathbb{R}} e^{iz_2 x/2} f(x + z_1/2) e^{iz_2 x/2} \overline{f(x - \bar{z}_1/2)} dx.$$

Let $z_l = a_l + ib_l$. Using the Cauchy–Schwarz inequality and the translation invariance of Lebesgue measure, we find

$$\begin{aligned} |V(z_1, z_2)| &\leqslant \left\|e^{-b_2(\cdot)/2} f(\cdot + z_1/2)\right\|_2 \left\|e^{-b_2(\cdot)/2} f(\cdot - \bar{z}_1/2)\right\|_2 \\ &= \left\|e^{-b_2(\cdot)/2} f(\cdot + ib_1/2)\right\|_2^2. \end{aligned} \tag{4·22}$$

Another application of the bound in (4·21) along with an explicit Gaussian integration, shows

$$|V(z_1, z_2)| \leqslant \left(\frac{1}{4\pi \sinh s \cdot \cosh s}\right)^{1/2} \|F\|_2^2 \, e^{\frac{b_1^2}{4}\coth s} \int_{\mathbb{R}} e^{-x^2 \tanh s} e^{-b_2 x} dx$$
$$= \frac{1}{2\sinh s} \|F\|_2^2 \, e^{\frac{1}{4}\coth s\left(b_1^2+b_2^2\right)}. \tag{4·23}$$

This is the same bound appearing in Lemma 4·11 except that e^{-s} has been replaced by $(2\sinh s)^{-1} \geqslant e^{-s}$.

We can improve on the estimate (4·23) if we allow ourselves to use the hyper-contractivity estimate in Lemma 4·6. Indeed, it is simple to verify from (4·20) that

$$|f(x+iy)| \leqslant e^{\coth s \cdot y^2/2} e^{-sH_0} |F|(x) \quad \text{for all } x, y \in \mathbb{R}.$$

Using this estimate in (4·22) along with the hyper-contractivity estimate in (4·7) then shows

$$|V(z_1, z_2)| \leqslant \left\| e^{-b_2(\cdot)/2} f(\cdot + ib_1/2)\right\|_2^2 \leqslant e^{\frac{b_1^2}{4}\coth s} \left\| e^{-b_2(\cdot)/2} e^{-sH_0} |F| \right\|_2^2$$
$$\leqslant e^{\frac{b_1^2}{4}\coth s} e^{\frac{b_2^2}{4}\coth s} e^{-s} \, \||F|\|_2^2 = e^{\coth s \cdot \left(b_1^2+b_2^2\right)/4} e^{-s} \|F\|_2^2 .$$

This is is precisely the estimate appearing in Lemma 4·11.

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