Accepted for publication in the Tohoku Mathematical Journal, Vol. 62, No. 3 (September, 2010).

GROWTH OF TAYLOR COEFFICIENTS OVER COMPLEX HOMOGENEOUS SPACES

BRUCE K. DRIVER†, LEONARD GROSS, AND LAURENT SALOFF-COSTE*

Abstract. Given a non-negative Hermitian form on the dual of the Lie algebra of a complex Lie group, one can associate to it a (possibly degenerate) Laplacian on the Lie group. Under Hörmander's condition on the Laplacian there exists a smooth time-dependent measure, convolution by which gives the semigroup generated by the Laplacian. Fixing a positive time, we may form the Hilbert space of holomorphic functions on the group which are square integrable with respect to this "heat kernel" measure. At the same time, under Hörmander's condition, the given Hermitian form extends to a time dependent norm on the dual of the universal enveloping algebra.

In previous work we have shown that, for each positive time, the Taylor map, which sends a holomorphic function to its set of Taylor coefficients at the identity element, is a unitary map from the previous Hilbert space of square integrable holomorphic functions onto a Hilbert space contained in the dual of the universal enveloping algebra.

The present paper is concerned with the behavior of these two families of Hilbert spaces when the Lie group is replaced by a product of complex Lie groups or by a quotient by a not necessarily normal subgroup. We obtain thereby the first example of unitarity of the Taylor map for a complex manifold which is not a Lie group. In addition, we determine the behavior of these spaces as the given Hermitian form varies.

Contents

2000 *Mathematics Subject Classification.* Primary 32W30; Secondary 35H20, 32C15, 43A15.

Key words and phrases. Subelliptic, heat kernel, complex groups, universal enveloping algebra, Taylor map.

†Research was supported in part by NSF Grants DMS-0504608 and DMS-0804472.

*Research was supported in part by NSF Grants DMS 0102126 and DMS 0603886.

1. Introduction

Let G be a complex Lie group with Lie algebra $\mathfrak{g}$. Denote by e the identity element of G. Let $\widetilde{\xi}$ be the left-invariant vector field on G associated with $\xi \in \mathfrak{g}$. For a holomorphic function f on G, the map

$$\mathfrak{g}^k \ni (\xi_1, \ldots, \xi_k) \mapsto (\widetilde{\xi}_1 \cdots \widetilde{\xi}_k f)(e)$$

is a multilinear map into the complex numbers and is consequently represented by a unique element of the dual space of $\mathfrak{g}^{\otimes k}$. Allowing k to vary and putting these elements together yields an element $\hat{f}$ of the algebraic dual space T', wherein T denotes the tensor algebra over $\mathfrak{g}$. We refer to $\hat{f}$ as "the Taylor coefficient" of f at the identity element of G. Because of the Lie algebra relations, $\hat{f}$ belongs, in fact, to a subspace J^0 of T' that is naturally isomorphic to the dual $\mathcal{U}'$ of the universal enveloping algebra $\mathcal{U}$ of $\mathfrak{g}$.

Let q be a Hermitian form on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. The Hermitian form q naturally yields a family of seminorms indexed by $t > 0$ on T'. Namely, if $\alpha \in T'$ is given by $\alpha = \sum \alpha_k$, α_k being an element of the dual space $(\mathfrak{g}^*)^{\otimes k}$ of $\mathfrak{g}^{\otimes k}$, then define

$$\|\alpha\|_{q,t}^2 = \sum \frac{t^k}{k!} q_k(\alpha_k)$$

where q_k is the Hermitian form induced by q on $(\mathfrak{g}^*)^{\otimes k}$. Let $J^0_{q,t}$ be the subspace of J^0 on which $\|\cdot\|_{q,t}$ is finite. Then $\|\cdot\|_{q,t}$ is a seminorm on $J^0_{q,t}$.

The left-invariant extension of the real part of the Hermitian form q determines a sub-Laplacian Δ on G. In this context, Hörmander's condition for the hypoellitcity of this sub-Laplacian can be understood as a condition on the Hermitian form q. It is proved in [5] that Hörmander's condition is equivalent to the property that each of the semi-norms $\|\cdot\|_{q,t}$ is actually non-degenerate and, consequently, the space $J^0_{q,t}$ is a Hilbert space.

Let dx be a fixed right-invariant Haar measure on G. The sub-Laplacian Δ defined on $\mathcal{C}_c^\infty(G) \subset L^2(G, dx)$ is essentially self-adjoint. Abusing notation, we will also denote by Δ its unique self-adjoint extension. The associated heat semigroup $e^{t\Delta/4}$ commutes with left multiplication in G, and thus determines a unique family of probability measures $\{\rho_t\}$ defined by the identity $e^{t\Delta/4} =$ (right convolution by ρ_t), that is,

$$e^{t\Delta/4} f(x) = \int_G f(xy)\rho_t(dy), \ x \in G, \ f \in L^2(G, dx).$$

We call ρ_t the heat kernel measure on G. Under Hörmander's condition, ρ_t admits a smooth positive density w.r.t. the right Haar measure. Somewhat abusively, we will denote this density by $x \mapsto \rho_t(x)$. Hence, we have $\rho_t(dx) = \rho_t(x)dx$. Note that $e^{t\Delta/4}$ also admits a transition kernel $h_t(x, y)$, the heat kernel, so that $e^{t\Delta/4} = \int_G h_t(x, y)f(y)dy$. The function $\rho_t(x)$ and the heat kernel $h_t(x, y)$ are related by the formula $h_t(x, y) = \rho_t(x^{-1}y)m(x)$ where m is the modular function on G. See [5] for more details. For each $t > 0$, $x \mapsto \rho_t(x)$ decays fast enough at infinity so that the Hilbert space $\mathcal{H}L^2(G, \rho_t)$, consisting of holomorphic functions in $L^2(G, \rho_t)$, is non-empty and is in fact a quite substantial space.

There is a remarkable identity of norms involving f and its Taylor coefficients. On the one hand one has the norm $\|f\|_{L^2(G,\rho_t)}$ while on the other hand one has the norm $\|\hat{f}\|_{q,t}$ as an element of $J^0 \subset T'$. These norms are equal. In fact the map $f \mapsto \hat{f}$ is unitary if G is simply connected. This theorem has a long history, starting with the classical case, $G = \mathbb{C}$. When the form q is positive definite and thus induces a left-invariant Riemannian metric on G, the unitarity of the Taylor map $f \mapsto \hat{f}$ was proved in [4]. We refer the reader to [4] for a precise description of the results and some history. The case when q is degenerate but satisfies Hörmander's condition is treated in [5].

These theorems concerning the unitarity of the Taylor map have two parts. First, one proves that the Taylor map is an isometry from $\mathcal{H}L^2(G, \rho_t)$ into $J^0_{q,t}$. Next one shows that this map is surjective. This part can be thought of as the problem of constructing a holomorphic function from a set of Taylor coefficients using some sort of Taylor series (either explicitly or implicitly). Both parts are technically involved and use refined heat kernel estimates. But the surjectivity of the Taylor map has always been the most difficult issue. Moreover, the proof of surjectivity given in [5] for the general case when q satisfies Hörmander's condition is substantially different and more complicated than the proof given in [4] in the positive definite case.

In this context, it is natural to ask how the families of Hilbert spaces associated with two distinct Hermitian forms q^1, q^2 compare. Given two (monotone) families of Hilbert spaces H^i_t, $t > 0$, $i = 1, 2$, all contained in a common underlying vector space, T', we say that the second family controls the first if for each $t > 0$ there is an $s > 0$ such that

$H_t^2 \subset H_s^1$. In terms of norms, this means that for each $t > 0$ there is $C \in (0, \infty)$ and $s > 0$ such that $\|f\|_{H_s^1} \le C\|f\|_{H_t^2}$. A particularly interesting question is whether the family $J^0_{q,t}$ associated with a positive definite Hermitian form can be controlled by the family associated with a form that merely satisfies Hörmander's condition. This is one of the main problems that motivates the results presented in this paper.

Section 2 reviews the definition of the spaces $J^0_{q,t}$, $t > 0$, which all lie in $J^0 \subset T'$ and which are associated with a Hermitian form q on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. It also describes the main results of [5] regarding the Taylor map.

Section 3.2 shows how known heat kernel estimates and the fact that the Taylor map is unitary (the main result of [5]) imply that the family of Hilbert spaces $J^0_{q^2,t}$ associated with any given Hermitian form q^2 satisfying Hörmander's condition controls the family of Hilbert spaces $J^0_{q^1,t}$ associated with any other such Hermitian form q^1. Section 3.1 points out that, if one can prove such a control a priori, then the surjectivity of the Taylor map for the form q^1 actually implies the surjectivity of the Taylor map for q_2. In particular, this opens the door to the possibility of deducing the main result of [5] (i.e., the surjectivity of the Taylor map for Hermitian forms satisfying Hörmander's condition) from the main result of [4] (i.e., the surjectivity of the Taylor map for positive definite Hermitian forms). This raises the question of whether this control between families of Hilbert spaces can be deduced directly on the tensor side (i.e., Taylor coefficients side) by algebraic or combinatorial means. Section 4 shows that this can indeed be done for the three dimensional (complex) Heisenberg group $H_3^{\mathbb{C}}$ and, more generally, for $H_{2n+1}^{\mathbb{C}}$.

In Sections 5, we consider the situation wherein the group G is the direct product of two simply connected complex Lie groups $G_a \times G_b$ (i.e., the Lie algebra $\mathfrak{g}$ is the direct sum of two Lie algebras $\mathfrak{g}_a, \mathfrak{g}_b$) and $q = q_a \oplus q_b$, with q_a, q_b Hermitian forms on $\mathfrak{g}_a^*, \mathfrak{g}_b^*$ satisfying Hörmander's condition. We show that there is a unique natural surjective isometry from $J^o_{q_a,t} \otimes J^0_{q_b,t}$ onto $J^0_{q,t}$. As an application, we observe that this allows us to extend the comparison of families obtained for the Heisenberg groups $H_{2n+1}^{\mathbb{C}}$ in Section 4 to products of such groups.

In section 6, we discuss the case of homogeneous spaces $M = K \setminus G$ where K is a closed connected Lie subgroup of G. When K is normal, $M = K \setminus G$ is a simply connected Lie group and we stay in the same framework as in the previous sections of this paper. In this case, the results of Section 6 show that the comparison of two families of Taylor coefficient Hilbert spaces on G descends to $M = K \setminus G$. However, when K is not a normal subgroup, we obtain a host of new complex manifolds for which we can produce a unitary Taylor map between an L^2 space of holomorphic functions on M and a Hilbert space of Taylor coefficients. These are the first examples of such a result for complex manifolds that are not complex Lie groups. In Section 7, we discuss in some detail two simple examples: the Grushin two dimensional complex space and a quotient of the group of holomorphic affine motions on the complex plane.

2. Notation and Background

In this section we will review some notation and basic results from [5]. We will use angle brackets $\langle \cdot, \cdot \rangle$ to denote the pairing of a vector space V and its algebraic dual V', i.e., $\langle \alpha, v \rangle := \alpha(v)$ for all $v \in V$ and $\alpha \in V'$. Let G be a complex connected Lie group equipped with its right Haar measure dx and let $\mathcal{H} = \mathcal{H}(G)$ denote the space of complex valued holomorphic functions on G. Given $A \in \mathfrak{g} := \mathrm{Lie}(G)$ (the complex Lie algebra of G), let $\tilde{A}$ denote the unique left invariant vector field acting on $C^\infty(G)$ such that $\tilde{A}(e) = A$. We let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$ (we use * instead of $'$ in this case because of the possible confusion with the derived subalgebra).

Denote by $T(\mathfrak{g})$ the tensor algebra over $\mathfrak{g}$. An element of $T(\mathfrak{g})$ is a finite sum:

$$\beta = \sum_{k=0}^{N} \beta_k \qquad \beta_k \in \mathfrak{g}^{\otimes k}. \tag{2.1}$$

We may and will identify $T(\mathfrak{g})'$ with the direct product $\prod_{k=0}^{\infty} (\mathfrak{g}^*)^{\otimes k}$ via the pairing,

$$\langle \alpha, \beta \rangle = \sum_{k=0}^{\infty} \langle \alpha_k, \beta_k \rangle, \tag{2.2}$$

where

$$\alpha = \sum_{k=0}^{\infty} \alpha_k \qquad \alpha_k \in (\mathfrak{g}^*)^{\otimes k}. \tag{2.3}$$

Notation 2.1 (Left Invariant Differential Operators). We define a *real* linear map $\left(\beta \to \tilde{\beta}\right)$ from $T(\mathfrak{g})$ to left invariant differential operators on G determined by

(1) $\tilde{1} f = f$ and
(2) for $\beta = A_1 \otimes \cdots \otimes A_k \in \mathfrak{g}^{\otimes k}$, $\tilde{\beta} f := \tilde{A}_1 \cdots \tilde{A}_k f$ for $f \in C^\infty(G)$.

If f is a C^∞ function on G, the *Taylor coefficient of* f at $x \in G$ is the element $\hat{f}(x)$ in $T(\mathfrak{g})'_{\mathrm{Re}}$ (the real linear functionals on $T(\mathfrak{g})$) defined by

$$\langle \hat{f}(x), \beta \rangle = (\tilde{\beta} f)(x) \text{ for all } \beta \in T(\mathfrak{g}). \tag{2.4}$$

If we further assume $f \in \mathcal{H}$, then $\beta \to (\tilde{\beta} f)(x)$ is complex linear and in this case $\hat{f}(x) \in T(\mathfrak{g})'$. In either case, $\hat{f}(x)$ annihilates the two-sided ideal $J \subset T(\mathfrak{g})$ generated by

$$\{\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta]; \xi, \eta \in \mathfrak{g}\}. \tag{2.5}$$

So if $f \in \mathcal{H}$ and $x \in G$, then $\hat{f}(x) \in J^0$ where

$$J^0 := \{\alpha \in T(\mathfrak{g})'; \langle \alpha, J \rangle = \{0\}\}. \tag{2.6}$$

The space J^0 is complex isomorphic to $\mathcal{U}'$ where $\mathcal{U} := T(\mathfrak{g})/J$ is the *universal enveloping algebra* of $\mathfrak{g}$.

NOTATION 2.2. Let q be a nonnegative Hermitian form on the dual space $\mathfrak{g}^*$. Thus

$$q(a) = (a,a)_q \tag{2.7}$$

for some, possibly degenerate, nonnegative sesquilinear form $(\ ,\ )_q$ on $\mathfrak{g}^*$.

As is shown in [5, Lemma 2.2], there exists a linearly independent (over $\mathbb{C}$) subset $\{X_j\}_{j=1}^m \subset \mathfrak{g}$ such that

$$q\,(a) = (a,a)_q = \sum_{j=1}^m |\langle a, X_j\rangle|^2 \text{ for all } a \in \mathfrak{g}^*. \tag{2.8}$$

The space, $H := \operatorname{span}(X_1, \ldots, X_m)$ equipped with the unique Hermitian inner product $(\cdot,\cdot)_H$, for which $\{X_j\}_{j=1}^m$ is an orthonormal basis, is called the *Hörmander subspace* associated to q. H is the backwards annihilator of the kernel of q. See, e.g., [5, Equation (2.3)]. Also associated to q is the second order left invariant differential operator

$$\Delta = \sum_{j=1}^m (\tilde{X}_j^2 + \widetilde{(iX_j)}^2). \tag{2.9}$$

It can be shown that Δ and $(H, (\cdot,\cdot)_H)$ depend only on q and not on the choice of basis $\{X_j\}_{j=1}^m \subset \mathfrak{g}$ for which Eq. (2.8) holds (see [5]).

The form q induces a degenerate Hermitian form $q_k := q^{\otimes k}$ whose inner product, $(\cdot,\cdot)_{q_k}$, on $(\mathfrak{g}^*)^{\otimes k}$ is determined by

$$(a_1 \otimes \cdots \otimes a_k, b_1 \otimes \cdots \otimes b_k)_{q_k} = \prod_{j=1}^k (a_j, b_j)_q \quad a_i, b_i \in \mathfrak{g}^*,\ i = 1, \ldots, k \tag{2.10}$$

for $k \geq 1$. If $\alpha \in (\mathfrak{g}^*)^{\otimes k}$, we will write $q_k\,(\alpha)$ or $|\alpha|_{q_k}^2$ for $(\alpha,\alpha)_{q_k}$. By convention, $V^{\otimes 0} = \mathbb{C}$ and we define q_0 on $(\mathfrak{g}^*)^{\otimes 0}$ so that $q_0\,(1) = 1$. For $t > 0$, define

$$\|\alpha\|_{q,t}^2 := \sum_{k=0}^\infty \frac{t^k}{k!} |\alpha_k|_{q_k}^2 \tag{2.11}$$

when α is given by (2.3).

The function $\|\cdot\|_{q,t}$ defines a seminorm in the subspace of $T(\mathfrak{g})'$ on which $\|\alpha\|_{q,t}^2$ is finite. But we will, by restriction, always consider $\|\cdot\|_{q,t}$ to be a semi-norm on

$$J_{q,t}^0 := \{\alpha \in J^0; \|\alpha\|_{q,t}^2 < \infty\}. \tag{2.12}$$

Whenever there is no risk of confusion concerning which form q is under consideration, we will often drop the reference to q and write

$$\|\cdot\|_t = \|\cdot\|_{q,t} \text{ and } J_t^0 = J_{q,t}^0.$$

REMARK 2.3. Similar constructions can be carried out over a real Lie algebra equipped with a non-negative quadratic form. See [5] and Section 5 of the present paper.

Definition 2.4. We say that Hörmander's condition holds for q if the smallest Lie subalgebra, $\mathrm{Lie}\,(H)$, containing H is $\mathfrak{g}$. It is permissible here to view $\mathrm{Lie}\,(H)$ as the Lie algebra generated by $H \subset \mathfrak{g}$ with $\mathfrak{g}$ regarded as either a complex or a real Lie algebra.

The significance of Hörmander's condition is twofold. 1) By [5, Theorem 2.7], $\mathrm{Lie}\,(H) = \mathfrak{g}$ iff for some $t > 0$ (hence for all $t > 0$), $\|\cdot\|_t$ is a norm on J_t^0. 2. By Hörmander's theorem [13], $\mathrm{Lie}\,(H) = \mathfrak{g}$ iff Δ is hypoelliptic, see the end of Section 1 in [5] for a more detailed discussion on this last point. So under Hörmander's condition on q, the operator Δ in Eq. (2.9) induces a heat semigroup, $e^{t\Delta/4}$, with a smooth convolution kernel, $G \ni x \mapsto \rho_t(x) \in (0,\infty)$, satisfying

$$\left(e^{t\Delta/4} f\right)(e) = \int_G f(x)\rho_t(x)dx \text{ for all } f \in L^2(G, dx). \tag{2.13}$$

Recall that $\Delta : \mathcal{C}_c^\infty(G) \to \mathcal{C}_c^\infty(G)$ is essentially self-adjoint in $L^2(G, dx)$ and that, abusing notation, we use the same letter Δ for its unique self-adjoint extension. We call the measure

$$\rho_t(dx) = \rho_t(x)dx$$

the heat kernel measure on G associated to the sub-Laplacian Δ. In what follows, we will refer to $x \mapsto \rho_t(x)$ as the *heat kernel* associated with Δ although, properly speaking, the heat kernel $h_t(x,y)$ is a function of time and two space-variables (x,y) related to ρ_t by $h_t(x,y) = \rho_t(x^{-1}y)m(x)$ where m is the modular function on G. See [5, Section 3] for more details of this construction.

Notation 2.5. We denote by $\mathcal{H}$ the space of holomorphic functions on G and define

$$\mathcal{H}L^2(G, \rho_t) = \mathcal{H} \cap L^2(G, \rho_t). \tag{2.14}$$

For any complex matrix group, the matrix entries and polynomials in these entries lie in this space for any such subelliptic Laplacian.

We may now summarize some of the main theorems from [5].

Theorem 2.6 ([5, Theorem 4.2]). *Let G be a connected complex Lie group. Suppose that q is a non-negative Hermitian form on the dual space $\mathfrak{g}^*$ and assume that Hörmander's condition holds, (cf. Definition 2.4). Let ρ_t denote the heat kernel measure associated to q. Then the* Taylor map

$$f \to \hat{f}(e) \tag{2.15}$$

is an isometry from $\mathcal{H}L^2(G, \rho_t)$ into J_t^0.

Proposition 2.7 ([5, Proposition 4.3]). *Let $f \in \mathcal{H}(G)$ and assume that $\hat{f}(e) \in J_t^0$ (see Eq. (2.4)) for some $t > 0$. Then $f \in \mathcal{H}L^2(G, \rho_t)$.*

THEOREM 2.8 ([5, Theorem 6.1]). *Let G be a connected, simply connected complex Lie group. Suppose that q is a non-negative Hermitian form on the dual space $\mathfrak{g}^*$ and assume that Hörmander's condition holds, (cf. Definition 2.4). Then the Taylor map, $f \mapsto \hat{f}(e)$ is a unitary map from $\mathcal{H}L^2(G, \rho_t)$ onto J_t^0.*

3. Comparison of norms

3.1. **Surjectivity via comparison of norms.** Suppose $\mathfrak{g}$ is a complex Lie algebra, G is the simply connected complex Lie group with $\mathfrak{g} = \mathrm{Lie}\,(G)$, and that we are given two nonnegative Hermitian quadratic forms, say q^j, $j = 1, 2$, on the complex vector space, $\mathfrak{g}^*$. Assume that both satisfy Hörmander's condition. By (2.9) and (2.13), second order differential operators Δ^j and heat kernels $x \mapsto \rho_t^j(x)$ on G for $j = 1, 2$ are associated to these Hermitian forms.

By (2.11), each of the Hermitian forms q^j, $j = 1, 2$, induces on J^0 the seminorms (where finite)

$$\|\alpha\|_{q^j,t} = \left(\sum (t^k/k!)|\alpha_k|^2_{q^j_k}\right)^{1/2}, \quad t > 0, \tag{3.1}$$

and yields the family of Hilbert spaces

$$J^0_{q^j,t} = \{\alpha \in J^0; \|\alpha\|_{q^j,t} < \infty\}, \quad t > 0.$$

We will say that the q^2 family (partially) controls the q^1 family if, for each $t > 0$, there is a $s > 0$ and a constant C such that

$$\|\alpha\|_{q^1,s} \le C\|\alpha\|_{q^2,t}. \tag{3.2}$$

Our interest in this definition comes from the following proposition.

PROPOSITION 3.1. *Suppose that the q^2 family controls the q^1 family as in (3.2). Suppose also that the Taylor map from $\mathcal{H}L^2(G, \rho_t^1)$ to $J^0_{q^1,t}$ is surjective for all $t > 0$. Then the Taylor map from $\mathcal{H}L^2(G, \rho_t^2)$ to $J^0_{q^2,t}$ is also surjective for all $t > 0$.*

PROOF. If $\alpha \in J^0_{q^2,t}$ then we learn from (3.2) that $\alpha \in J^0_{q^1,s}$ for some $s > 0$, and consequently there exists a function $f \in \mathcal{H}(G)$ such that $\hat{f}\,(e) = \alpha$. It now follows from Proposition 2.7 that $f \in \mathcal{H}L^2(G, \rho_t^2)$. Therefore, the Taylor map is also surjective from $\mathcal{H}L^2(G, \rho_t^2)$ onto $J^0_{q^2,t}$. □

REMARK 3.2. If the q^2 family controls the q^1 family and q^1 is positive definite, then [4, Theorem 2.6] shows that the hypothesis of Proposition 3.1 holds, that is, the Taylor map from $\mathcal{H}L^2(G, \rho_t^1)$ to $J^0_{q^1,t}$ is surjective for all $t > 0$. Therefore Proposition 3.1 implies that the Taylor map from $\mathcal{H}L^2(G, \rho_t^2)$ to $J^0_{q^2,t}$ is surjective for all $t > 0$. This surjectivity in the Hörmander case is the main result of [5] (see Theorem 2.8 above). But the comparison inequalities (3.2) and Proposition 3.1 provide an alternate proof based on the result for the positive definite case obtained earlier in [4].

In Section 3.2 we will show (see Theorem 3.3) that, for any two quadratic forms satisfying Hörmander's condition, each family controls the other. The proof depends on Theorem 2.8 and, in particular, on the already known surjectivity of the Taylor map. But in Section 4 we will give a direct combinatorial proof of the inequalities (3.2) in case $\mathfrak{g}$ is the three dimensional complex Heisenberg Lie algebra.

3.2. **Comparison of norms via heat kernels.** In this section we will show that, as a consequence of the unitarity theorem, Theorem 2.8, the family of inequalities (3.2) holds.

THEOREM 3.3. *Let $\mathfrak{g}$ be a complex Lie algebra and let q^1, q^2 be nonnegative Hermitian forms on the dual space $\mathfrak{g}^*$ which satisfy Hörmander's condition (cf. Definition 2.4.). Then there exists $\varepsilon \in (0,1)$ such that for any $0 < s \leq \varepsilon t < \infty$ and $\alpha \in J^0$, we have*

$$\|\alpha\|_{q^1,s} \leq C(s,t)\|\alpha\|_{q^2,t} \tag{3.3}$$

with $C(s,t) \in (0,\infty)$.

The proof of Theorem 3.3 depends on the following lemma which compares the size of two heat kernels.

LEMMA 3.4. *Assume that q^1, q^2 satisfy Hörmander's condition. Then there exists $\varepsilon \in (0,1)$ such that for all $0 < s \leq \varepsilon t < \infty$ we have*

$$\sup_{x \in G} \left\{ \frac{\rho_s^1(x)}{\rho_t^2(x)} \right\} = R(s,t) < \infty.$$

PROOF. This follows from [18, Prop. III.4.2], which states that any two proper left invariant length distances are comparable on a large scale on G, and from the heat kernel bounds in [5, Theorem 3.4]. These yield

$$R(s,t) \leq C_1(1 + 1/s)^{c_1} \exp(c_2 t).$$

□

REMARK 3.5. For nilpotent groups (and, more generally, for Lie groups with polynomial volume growth), the better heat kernel estimates of [18, Chap. IV] give

$$R(s,t) \leq C_1(1 + 1/s)^{c_1}(1 + t/s)^{c_2}.$$

The positive finite constants ε, c_1, c_2 and C_1 depend on the group and the forms q^1, q^2 but not on s and t as long as $0 < s \leq \varepsilon t$. In the nilpotent (or polynomial volume growth) case, under the additional assumption that the form q^2 is dominated by the form q^1 in the sense that there exists $\kappa \in (0,\infty)$ such that $q^2 \leq \kappa q^1$, the heat kernel estimates of [18, Chap. IV] give the improved estimate

$$R(s,t) \leq C_1(1 + t/s)^{c_2}.$$

As an immediate corollary of Lemma 3.4 we have the following proposition.

Proposition 3.6. *Let G be a connected, complex Lie group. Suppose that q^1, q^2 are nonnegative Hermitian forms on the dual space $\mathfrak{g}^*$ of the complex Lie algebra of G. Assume that q^1, q^2 satisfy Hörmander's condition. Then there exists $\varepsilon \in (0,1)$ such that, for any $0 < s < \varepsilon t < \infty$ and for any $f \in \mathcal{H}(G)$, we have*

$$\|\hat{f}(e)\|_{q^1,s} \leq C(s,t)\|\hat{f}(e)\|_{q^2,t}$$

with $C(s,t) \in (0,\infty)$.

Proof. By Lemma 3.4, there exists $\varepsilon \in (0,1)$ such that, for $0 < s \leq \varepsilon t$ and $f \in \mathcal{H}$,

$$\int_G |f(x)|^2 \rho_s^1(x)dx \leq R(s,t) \int_G |f(x)|^2 \rho_t^2(x)dx.$$

Hence the desired bound (with a constant $C = C(s,t)$ given by Lemma 3.4) follows from the fact that $f \mapsto \hat{f}(e)$ is an isometry from $\mathcal{H}L^2(G, \rho_\tau^j)$ to $J^0_{q^j,\tau}$, $\tau > 0$, $j = 1,2$. See Theorem 2.6. $\square$

Remark 3.7. As an application of Proposition 3.6, one may take q^1 to be a positive definite Hermitian form, i.e., a form inducing a Riemannian metric on G and q^2 to be a nonnegative Hermitian form satisfying Hörmander's condition but not positive definite. Then the proposition gives control of the series

$$\sum_{k=0}^{\infty}(s^k/k!)|\hat{f}(e)|^2_{q^1,k},$$

which involves "all" Taylor coefficients of f in terms of the series

$$\sum_{k=0}^{\infty}(t^k/k!)|\hat{f}(e)|^2_{q^2,k},$$

which only involves "horizontal" Taylor coefficients of f.

Proof. We may assume $\|\alpha\|_{q^2,t} < \infty$, for otherwise there is nothing to prove. Let G be the complex, connected, simply connected Lie group such that $\mathrm{Lie}(G) = \mathfrak{g}$ and let ρ_t^j, $j = 1,2$ be the heat kernels on G associated to q^j for $j = 1,2$. By Theorem 2.8, there exists $f \in \mathcal{H}(G)$ such that $\hat{f}(e) = \alpha$. The result now follows directly form Proposition 3.6. $\square$

Remark 3.8. The proof of Theorem 3.3 given above yields the desired inequality with $C(s,t) = R(s,t)$, $R(s,t)$ being the heat kernel ratio defined in Lemma 3.4. However, the norms $\|\cdot\|_{q^i,\tau}$ are increasing functions of τ. It follows that, in Theorem 3.3, it suffices to treat the case when $s = \epsilon t$. Consequently, one can replace $C(s,t) = R(s,t)$ by $K(s,t) = \min\{R(s,\varepsilon^{-1}s), R(\epsilon t,t)\}$. For general Lie groups, the bounds mentioned in the proof of Lemma 3.4 yield

$$K(s,t) \leq C_1 \min\{(1+1/s)^{c_1}, e^{c_2 t}\},$$

whereas, for nilpotent groups (or groups of polynomial volume growth), they yield

$$K(s,t) \leq C_1(1+1/s)^{c_1}$$

for some $\varepsilon, c_1, c_2, C_1$ depending on G and q^1, q^2. In this latter case, if one assumes in addition that $q_2 \leq \kappa q_1$ for some $\kappa \in (0,\infty)$, then one obtains

$$K(s,t) \leq C_1.$$

Remark 3.9. Let G be a complex Lie group, $\Delta = \sum_1^m \widetilde{X_i}^2 + \widetilde{iX_i}^2$ a sub-Laplacian on G satisfying Hörmander's condition and ρ_t the associated heat kernel. The well-known two-sided heat kernel estimates mentioned earlier easily implies that the spaces

$$\bigcap_{t>0} \mathcal{H}L^2(G,\rho_t) \quad \text{and} \quad \bigcup_{t>0} \mathcal{H}L^2(G,\rho_t)$$

are algebras under pointwise multiplication. Lemma 3.4 shows that these algebras are independent of the choice of the sub-Laplacian Δ, as long as Hörmander's condition is assumed.

As we shall see in the next Section 4, proving Eq. (3.1) by a direct computation involving commutators appears to be a combinatorial challenge, even under very strong assumptions.

4. Combinatorial approach to comparison for the Heisenberg algebra

In this section we will give a direct combinatorial proof of the comparison inequalities (3.2) when $\mathfrak{h}_3^{\mathbb{C}}$ is the three dimensional complex Heisenberg Lie algebra, q^2 is the natural degenerate form and q^1 is a particular nondegenerate form. As pointed out at the end of this section, the same argument applies to the higher dimensional Heisenberg algebras $\mathfrak{h}_{2n+1}^{\mathbb{C}}$. As already noted in Proposition 3.1, the inequalities (3.2) then yield a proof of surjectivity of the Taylor map in our degenerate case quite different from the proof given in [5, Sections 5 and 6.].

Theorem 4.1. *Suppose $\mathfrak{g}$ is the complex 3 dimensional Heisenberg Lie algebra, so that $\mathfrak{g}$ is the span of X, Y and Z with Z in the center of $\mathfrak{g}$ and $[X,Y]=Z$. Given $\alpha \in \mathfrak{g}^*$, define*

$$q^1(\alpha) = |\langle \alpha, X\rangle|^2 + |\langle \alpha, Y\rangle|^2 + |\langle \alpha, Z\rangle|^2$$

and

$$q^2(\alpha) = |\langle \alpha, X\rangle|^2 + |\langle \alpha, Y\rangle|^2.$$

For $\alpha \in T'$, define $\|\alpha\|_{q^j,s}$ as in (3.1). Then, for $\alpha \in J^0$,

$$\|\alpha\|_{q^1,s}^2 \leq C(s,t)\|\alpha\|_{q^2.t}^2$$

where

$$C(s,t)=\sum_{k=0}^{\infty}(es/t)^k\left(\frac{4}{t}+1\right)^k,$$

which is finite provided that $(es/t)\,((4/t)+1)<1$.

Let $\mathfrak{g}$ be a Lie algebra, $\mathcal{U}(\mathfrak{g})$ be its universal enveloping algebra, and for $x\in\mathfrak{g}$, let R_x and L_x denote right and left multiplication by x on $\mathcal{U}=\mathcal{U}(\mathfrak{g})$ and $\mathrm{ad}_x=L_x-R_x$. Hence for $\alpha\in\mathcal{U}$ we have

$$L_x\alpha=x\alpha,\ \ R_x\alpha=\alpha x\text{ and }\mathrm{ad}_x\alpha=x\alpha-\alpha x.$$

Let us recall the following basic result.

Proposition 4.2. *For each* $x\in\mathfrak{g}$, ad_x *acts as a derivation on* $\mathcal{U}$ *and* $e^{t\mathrm{ad}_x}$ *is an automorphism on* $\mathcal{U}$.

Proof. The first assertion is a consequence of the computation

$$\begin{aligned}\mathrm{ad}_x(\alpha\beta)&=x\alpha\beta-\alpha\beta x\\&=(\mathrm{ad}_x\alpha)\,\beta+\alpha x\beta-\alpha\beta x\\&=(\mathrm{ad}_x\alpha)\,\beta+\alpha\mathrm{ad}_x\beta.\end{aligned}$$

For the second assertion, let $\alpha,\beta\in\mathcal{U}(\mathfrak{g})$ and let

$$\sigma(t):=e^{t\mathrm{ad}_x}\alpha\cdot e^{t\mathrm{ad}_x}\beta,$$

Then

$$\begin{aligned}\frac{d}{dt}\sigma(t)&=\mathrm{ad}_xe^{t\mathrm{ad}_x}\alpha\cdot e^{t\mathrm{ad}_x}\beta+e^{t\mathrm{ad}_x}\alpha\cdot\mathrm{ad}_xe^{t\mathrm{ad}_x}\beta\\&=\mathrm{ad}_x\sigma(t)\text{ with }\sigma(0)=\alpha\beta,\end{aligned}$$

and this implies

$$e^{t\mathrm{ad}_x}\alpha\cdot e^{t\mathrm{ad}_x}\beta=\sigma(t)=e^{t\mathrm{ad}_x}(\alpha\beta),$$

i.e., $e^{t\mathrm{ad}_x}$ is an automorphism on $\mathcal{U}$. $\square$

Lemma 4.3. *Let* $\mathfrak{g}$ *be a Lie algebra,* $x,y\in\mathfrak{g}$ *be linearly independent, and suppose*

$$z:=\mathrm{ad}_xy=[x,y]$$

commutes with x *and* y. *Then*

$$z^n=\frac{1}{n!}\mathrm{ad}_x^ny^n=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}x^ky^n(-x)^{n-k}.\tag{4.1}$$

PROOF. Notice that $\mathrm{ad}_x^n y = \mathrm{ad}_x^{n-1} z = 0$ for all $n \geq 2$ so that

$$e^{t\mathrm{ad}_x} y = y + tz,$$

and hence by Proposition 4.2,

$$e^{t\mathrm{ad}_x} y^n = \left(e^{t\mathrm{ad}_x} y\right)^n = (y + tz)^n.$$

Comparing the coefficients of t^n on both sides of this equations proves the first equality in Eq. (4.1). Since L_x and R_x commute,

$$\mathrm{ad}_x^n = (L_x - R_x)^n = \sum_{k=0}^{n} \binom{n}{k} {L_x}^k (-R_x)^{n-k}, \tag{4.2}$$

and hence the second equality in Eq. (4.1) is an immediate consequence of the first and of Eq. (4.2). □

Proof of Theorem 4.1. Since Z is in the center of $\mathfrak{g}$ and $\alpha \in J^0$,

$$|\alpha_k|_{q^1}^2 = \sum_{l=0}^{k} \binom{k}{l} \sum_{A_1, A_2, \ldots, A_{k-l} \in \{X,Y\}} \left|\langle \alpha, Z^{\otimes l} \otimes A_1 \otimes A_2 \otimes \cdots \otimes A_{k-l} \rangle\right|^2. \tag{4.3}$$

Moreover,

$$Z = X \otimes Y - Y \otimes X \bmod J.$$

Thus, dropping the tensor product symbol from the notation, by Lemma 4.3,

$$Z^l = \frac{1}{l!} \mathrm{ad}_X^l Y^l = \frac{1}{l!} \sum_{j=0}^{l} \binom{l}{j} X^j Y^l (-X)^{l-j} \bmod J.$$

Using this equation in Eq. (4.3) implies that

$$\begin{aligned}
|\alpha_k|_{q^1}^2 &= \sum_{l=0}^{k} \binom{k}{l} \sum_{A_1, A_2, \ldots, A_{k-l} \in \{X,Y\}} \left| \frac{1}{l!} \sum_{j=0}^{l} \binom{l}{j} \langle \alpha, X^j Y^l (-X)^{l-j} A_1 A_2 \cdots A_{k-l} \rangle \cdot 1 \right|^2 \\
&\leq \sum_{l=0}^{k} \binom{k}{l} \sum_{A_1, A_2, \ldots, A_{k-l} \in \{X,Y\}} \frac{2^l}{(l!)^2} \sum_{j=0}^{l} \binom{l}{j} |\langle \alpha, X^j Y^l X^{l-j} A_1 A_2 \cdots A_{k-l} \rangle|^2 \\
&\leq \sum_{l=0}^{k} \binom{k}{l} \frac{2^l}{(l!)^2} \sum_{j=0}^{l} \binom{l}{j} |\alpha_{k+l}|_{q^2}^2 = \sum_{l=0}^{k} \binom{k}{l} \frac{4^l}{(l!)^2} |\alpha_{k+l}|_{q^2}^2, \qquad (4.4)
\end{aligned}$$

where, in the second line, we have used the Cauchy-Schwarz inequality for the measure on $\{0, 1, \ldots, l\}$ giving weight $\binom{l}{j}$ to j, along with the binomial formula.

Now suppose that

$$\|\alpha\|_{q^2,t}^2 = \sum_{k=0}^{\infty} (t^k/k!) |\alpha_k|_{q^2}^2 = M < \infty.$$

Then

$$|\alpha_{k+l}|_{q^2}^2 \leq M (k+l)!/t^{(k+l)}. \tag{4.5}$$

Combining Eq. (4.5) with Eq. (4.4) shows that

$$\begin{aligned}|\alpha_k|_{q^1}^2 &\le M\sum_{l=0}^{k}\binom{k}{l}\frac{4^l}{(l!)^2}(k+l)!/t^{(k+l)}\\ &= \frac{M}{t^k}\sum_{l=0}^{k}\binom{k}{l}(4/t)^l\Big\{\frac{(k+l)!}{k!(l!)^2}\Big\}\\ &\le \frac{M}{t^k}\Big(1+\frac{4}{t}\Big)^k C_k,\end{aligned}$$

wherein C_k is given by $C_k = \max\{(k+l)!/(k!(l!)^2); 0 \le l \le k\}$. In order to estimate the constant C_k, we may write $(k+l)!/(k!(l!)^2)$ as a product of factors $(k+j)/(j(l+1-j))$, $j = 1, \ldots, l$. Let $f(x) = (k+x)/(x(l+1-x))$ for $x \in [1, l]$. Then $f'(x) = (x^2 + 2kx - k(l+1))/\{x(l+1-x)\}^2$. The quadratic formula shows that the numerator has only one zero on the positive x axis. Since $f'(l) > 0$, this zero lies to the left of $x = l$. Hence f takes its maximum on $[1, l]$ at one of the two endpoints. Since $f(l) \ge f(1)$ for $l \ge 1$, we see that

$$\frac{(k+l)!}{k!(l!)^2} \le \left(\frac{k+l}{l}\right)^l \le e^k \tag{4.6}$$

for $l \ge 1$. The overall inequality clearly holds for $l = 0$ also. Hence $C_k \le e^k$.

Inserting this estimate for C_k into the previous inequalities, we find that

$$\sum_{k=0}^{\infty}\frac{s^k}{k!}|\alpha_k|_{q^1}^2 \le \sum_{k=0}^{\infty} s^k e^k (4/t+1)^k \|\alpha\|_{q^2,t}^2 = C(s,t)\,\|\alpha\|_{q^2,t}^2$$

where

$$C(s,t) = \sum_{k=0}^{\infty}(es/t)^k\left(\frac{4}{t}+1\right)^k,$$

which is finite provided that $(es/t)\,((4/t)+1) < 1$. □

Remark 4.4. Recall that the (2n+1)-dimensional complex Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ is spanned over $\mathbb{C}$ by $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$, where $[X_j, Y_j] = Z$ and all other commutators are zero. In this case we consider the two Hermitian forms

$$q_{2n+1}^1(\alpha) = \sum_1^n\left(|\langle\alpha, X_i\rangle|^2 + |\langle\alpha, Y_i\rangle|^2\right) + |\langle\alpha, Z\rangle|^2$$

and

$$q_{2n+1}^2(\alpha) = \sum_1^n\left(|\langle\alpha, X_i\rangle|^2 + |\langle\alpha, Y_i\rangle|^2\right).$$

The proof of Theorem 4.1 holds for $\mathfrak{h}_{2n+1}$ with minor modification. One need only use $Z = [X_1, Y_1]$ in Lemma 4.3 instead of $Z = [X, Y]$. The conclusion is that, for $0 < s, t < \infty$ satisfying $(es/t)\,((4/t)+1) < 1$, and for $\alpha \in J^0 \subset T'(\mathfrak{h}_{2n+1})$,

$$\|\alpha\|_{q^1,s}^2 \le C(s,t)\|\alpha\|_{q^2.t}^2$$

with $C(s,t)$ as in Theorem 4.1.

Remark 4.5. In contrast with the previous remark, it is worth pointing out that despite the relative simplicity of the above combinatorial argument, we have not been able to generalize it yet for the $(2n+1)$-dimensional complex Heisenberg Lie algebra. Obvious candidates for a generalization are the so called H-algebras and, more generally, nilpotent step 2 algebras. Difficulties already appear when one tries to treat the simplest possible case, namely, the direct sum of two Heisenberg algebras $\mathfrak{h}_{2n+1} \oplus \mathfrak{h}_{2m+1}$, even for $n = m = 1$. In the next section, we indeed study direct sums and obtain basic functoriality results. Together with Theorem 4.1 (extended to $\mathfrak{h}_{2n+1}$ as in the previous remark), these results yield the obvious generalization of Theorem 4.1 for any direct sum $\mathbb{C}^{n_0} \oplus \mathfrak{h}_{2n_1+1} \oplus \cdots \oplus \mathfrak{h}_{2n_k+1}$ of a finite dimensional abelian algebra and a finite number of Heisenberg algebras $\mathfrak{h}_{2n_i+1}$, $i = 1, \ldots, k$. See Example 5.18.

5. Functoriality under direct sums/products

This section is concerned with the functoriality properties of the Hilbert spaces $J^0_{q,t} = J^0_{q,t}(\mathfrak{g})$ and $\mathcal{H}L^2(G, \rho_t)$ under direct sums (at the Lie algebra level) and direct products (at the Lie group level). These fundamental functoriality properties will be derived independently on both sides from first principles so that they can be used in the study of the Taylor map. Although we have been concerned only with complex Lie algebras because of our focus on holomorphic functions, much of the machinery for dealing with comparison of norms applies to real Lie algebras also. In this section we will consider both real and complex Lie algebras.

5.1. Functoriality of $J^0_{q,t}$ under direct sums.

Notation 5.1. Let $\mathfrak{g}_a$ and $\mathfrak{g}_b$ denote two real (resp. complex) finite dimensional Lie algebras and let q_a and q_b denote non-negative quadratic (resp. Hermitian) forms on the corresponding dual spaces. The direct sum $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_b$ is a Lie algebra and its dual space, which we may identify as $\mathfrak{g}^* = \mathfrak{g}_a^* \oplus \mathfrak{g}_b^*$, supports the quadratic (resp. Hermitian) form, $q := q_a \oplus q_b$. Further, as in Section 2, let $(H_a, (\cdot,\cdot)_a)$, $(H_b, (\cdot,\cdot)_b)$ and $(H, (\cdot,\cdot))$ denote the Hörmander (Hilbertian) subspaces associated to q_a, q_b and q respectively.

We will assume that both q_a and q_b satisfy Hörmander's condition, i.e., $\mathrm{Lie}(H_i) = \mathfrak{g}_i$ for $i \in \{a, b\}$. Under this assumption it is easily shown that $\mathrm{Lie}(H) = \mathfrak{g}$ or, equivalently stated, q also satisfies Hörmander's condition. Let $T_a = T(\mathfrak{g}_a)$, $T_b = T(\mathfrak{g}_b)$ and $T = T(\mathfrak{g})$. Let $J_a \subset T_a$, $J_b \subset T_b$ and $J \subset T$ be the two-sided ideals in each of these tensor algebras as described in Section 2. For each $t > 0$, these structures induce the three Hilbert tensor spaces $(J^0_a)_t$, $(J^0_b)_t$ and $J^0_t(\mathfrak{g})$ with norms denoted by $\|\cdot\|_{q_a,t}$, $\|\cdot\|_{q_b,t}$ and $\|\cdot\|_{q,t}$, respectively (see Eqs. (2.11) and (2.12)). For u and v both in T_a, both in T_b or both in T, we will write uv rather than $u \otimes v$. We will reserve the tensor symbol for the tensor product of two different vector spaces. For example, if $u \in T_a$ and $v \in T_b$, then $u \otimes v \in T_a \otimes T_b$.

It has been long known, and extensively used as a tool in the study of quantum fields that, if $\mathfrak{g}_a$ and $\mathfrak{g}_b$ are commutative and the forms are nondegenerate, then the three tensor Hilbert spaces are just spaces of symmetric tensors over the respective dual spaces and $(J_a^0)_t \otimes (J_b^0)_t$ is naturally isomorphic as a Hilbert space to $J_t^0(\mathfrak{g})$. Indeed, this kind of theorem has even been developed for the sum of continuum many summands [12], [16]. If the Lie algebras are not commutative, then such a functorial relation has already been proved when the forms q_a and q_b are non-degenerate [10, Theorem 4.3]. It has also been proved in the non-degenerate case by probabilistic techniques when the associated groups are compact [9, Corollary 5.8]. Our objective in this section is to extend this theorem to the case where q_a and q_b may be degenerate but satisfy Hörmander's condition.

Notation 5.2. There is a natural embedding of T_a into T. It maps the zero rank tensor $1 \in T_a$ to $1 \in T$. It maps an element $\xi \in \mathfrak{g}_a$ to $\xi \oplus 0 \in \mathfrak{g}$ and is otherwise an algebra isomorphism of the tensor algebra T_a into the tensor algebra T. This isomorphism will be denoted $T_a \ni \beta \mapsto \hat{\beta} \in T(\mathfrak{g})$. The similarly defined isomorphism of T_b into T will also be denoted as $T_b \ni \beta \mapsto \hat{\beta} \in T(\mathfrak{g})$.

The main theorem of this section is the following.

Theorem 5.3. *Assume that q_a and q_b satisfy Hörmander's condition. Then there exists a unique surjective isometry (i.e., an orthogonal or unitary map)*

$$L : (J_a^0)_t \otimes (J_b^0)_t \to (J^0)_t \tag{5.1}$$

such that

$$\left\langle Lw, \hat{\beta}_a\hat{\beta}_b \right\rangle = \langle w, \beta_a \otimes \beta_b\rangle \text{ for all } \beta_a \in T_a,\ \beta_b \in T_b \text{ and } w \in (J_a^0)_t \otimes (J_b^0)_t. \tag{5.2}$$

Here the product $\hat{\beta}_a\hat{\beta}_b$ refers to the product in the algebra T, as already noted.

Remark 5.4. This theorem will be proved by direct algebraic and combinatorial means. In the complex case, one could derive this theorem indirectly from the main theorem of [5] in conjunction with the Section "Functoriality of $\mathcal{H}L^2$" below.

Before starting the proof, let us give the following corollary.

Corollary 5.5. *Suppose that $\mathfrak{g}_a$ and $\mathfrak{g}_b$ are two real (resp. complex) finite dimensional Lie algebras. Let q_a^i and q_b^i, $i = 1, 2$ be non-negative quadratic (resp. Hermitian) forms on $\mathfrak{g}_a^*$ and $\mathfrak{g}_b^*$, respectively. Assume that all four forms satisfy Hörmander's condition. Let $s > 0$ and $t > 0$ and suppose that, for some constants c_a and c_b,*

$$\|\alpha\|_{q_a^1,s} \le c_a \|\alpha\|_{q_a^2,t} \text{ for all } \alpha \in J^0_{q_a^2,t} \tag{5.3}$$

and

$$\|\alpha\|_{q_b^1,s} \le c_b \|\alpha\|_{q_b^2,t} \text{ for all } \alpha \in J^0_{q_b^2,t}. \tag{5.4}$$

Let $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_b$ be the direct sum as Lie algebras. Let $q^1 = q_a^1 \oplus q_b^1$ and $q^2 = q_a^2 \oplus q_b^2$. Then

$$\|\alpha\|_{q^1,s} \le c_a c_b \|\alpha\|_{q^2,t} \text{ for all } \alpha \in J^0_{q^2,t} \subset T'(\mathfrak{g}). \tag{5.5}$$

In particular, if the q_a^2 family controls the q_a^1 family for $\mathfrak{g}_a$ and the q_b^2 family controls the q_b^1 family for $\mathfrak{g}_b^*$, then the q^2 family controls the q^1 family for $\mathfrak{g}$.

Proof. Let

$$L_1 : J^0_{q_a^1,s} \otimes J^0_{q_b^1,s} \to J^0_{q^1,s} \quad \text{and} \quad L_2 : J^0_{q_a^2,t} \otimes J^0_{q_b^2,t} \to J^0_{q^2,t}$$

denote the isometric isomorphisms of Theorem 5.3. By hypotheses (5.3) and (5.4), $J^0_{q_a^2,t} \subset J^0_{q_a^1,s}$, $J^0_{q_b^2,t} \subset J^0_{q_b^1,s}$ and the inclusion operators $\iota_a : J^0_{q_a^2,t} \to J^0_{q_a^1,s}$ and $\iota_b : J^0_{q_b^2,t} \to J^0_{q_b^1,s}$ satisfy the norm bounds

$$\|\iota_a\|_{J^0_{q_a^2,t} \to J^0_{q_a^1,s}} \le c_a \text{ and } \|\iota_b\|_{J^0_{q_b^2,t} \to J^0_{q_b^1,s}} \le c_b. \tag{5.6}$$

Hence it follows that $\iota_a \otimes \iota_b : J^0_{q_a^2,t} \otimes J^0_{q_b^2,t} \to J^0_{q_a^1,s} \otimes J^0_{q_b^1,s}$ is also an inclusion map satisfying the norm bound

$$\|\iota_a \otimes \iota_b\|_{J^0_{q_a^2,t} \otimes J^0_{q_b^2,t} \to J^0_{q_a^1,s} \otimes J^0_{q_b^1,s}} \le c_a c_b. \tag{5.7}$$

Now let $\alpha \in J^0_{q^2,t}$ and then define $\alpha' := L_1(\iota_a \otimes \iota_b) L_2{}^{-1}\alpha$. By virtue of (5.7) and the fact that L_1 and L_2 are unitary, it follows that $\|\alpha'\|_{q^1,s} \le c_a c_b \|\alpha\|_{q^2,t}$. Therefore, to prove the theorem it suffices to prove that α' and α (although possibly lying in different Hilbert spaces) are the same element of $T(\mathfrak{g})'$. To this end, it suffices to show that they have the same value on the fundamental set $\{\hat\beta_a \hat\beta_b ; \beta_a \in T_a,\ \beta_b \in T_b\}$. Let $w = (L_2)^{-1}\alpha \in J^0_{q_a^2,t} \otimes J^0_{q_b^2,t}$. Then

$$\begin{aligned} \left\langle \alpha', \hat\beta_a \hat\beta_b \right\rangle &= \left\langle L_1(\iota_a \otimes \iota_a) w, \hat\beta_a \hat\beta_b \right\rangle \\ &= \langle (\iota_a \otimes \iota_a) w, \beta_a \otimes \beta_b \rangle = \langle w, \beta_a \otimes \beta_b \rangle = \left\langle \alpha, \hat\beta_a \hat\beta_b \right\rangle . \end{aligned} \tag{5.8}$$

□

5.2. **Proof of Theorem 5.3.** Rather than follow the pattern of proof in [10, Theorem 4.3], we are going to give a proof that avoids using completions, as were used in [10], because, in our present context, many of our semi-norms are not norms on convenient subspaces. We will begin with a number of preliminary results needed for the proof of Theorem 5.3, which will be completed at the end of this subsection.

Notation 5.6. We will write $T_a \otimes_{\text{alg}} T_b$ for the algebraic tensor product and $(T_a \otimes_{\text{alg}} T_b)'$ for its algebraic dual space. T' will denote the algebraic dual space of T.

The space $T_a \otimes_{\text{alg}} T_b$ may also be viewed as an algebra over $\mathbb{R}$ or $\mathbb{C}$, respectively, using the multiplication law determined uniquely by

$$(u \otimes v) \cdot (u' \otimes v') = (uu') \otimes (vv') \text{ for all } u, u' \in T_a \text{ and } v, v' \in T_b. \tag{5.9}$$

Definition 5.7. Let

$$\theta : T \to T_a \otimes_{\text{alg}} T_b \tag{5.10}$$

be the unique algebra homomorphism such that

$$\theta(1) = 1 \otimes 1 \quad \text{and} \tag{5.11}$$

$$\theta(u+v) = u \otimes 1 + 1 \otimes v \tag{5.12}$$

for all $u \in \mathfrak{g}_a$ and $v \in \mathfrak{g}_b$. Also, let

$$\kappa : T_a \otimes_{\text{alg}} T_b \to T$$

be the **linear** map uniquely determined by $\kappa(\beta_a \otimes \beta_b) = \hat{\beta}_a \hat{\beta}_b$ for all $\beta_a \in T_a$ and $\beta_b \in T_b$.

Definition 5.8. Let I be the two sided ideal in the algebra T generated by

$$\{\xi \wedge \eta = \xi\eta - \eta\xi; \xi \in \mathfrak{g}_a \text{ and } \eta \in \mathfrak{g}_b\},$$

$\pi : T \to T/I$ be the quotient map, and $I^0 = \{\alpha \in T' : \langle \alpha, I \rangle = 0\}$ be the annihilator of I in T'.

As $[x, y] = 0$ if $x \in \mathfrak{g}_a$ and $y \in \mathfrak{g}_b$, it follows that $I \subset J$. We will make use of this fact in the proof of the next theorem.

Theorem 5.9. *Assuming that θ, κ and I are as in Definitions 5.7 and 5.8, then:*

(1) $\theta \circ \kappa = \mathrm{id}_{T_a \otimes_{\text{alg}} T_b}$.
(2) $\mathrm{Nul}(\theta) = I$ *and if $\hat{\theta} : T/I \to T_a \otimes_{\text{alg}} T_b$ is the factor map associated to θ, then $\hat{\theta}$ is an isomorphism of algebras.*
(3) *The map, $\hat{\kappa} := \pi \circ \kappa : T_a \otimes_{\text{alg}} T_b \to T/I$ is the inverse map to $\hat{\theta}$.*
(4) *The spaces, $(T_a \otimes_{\text{alg}} T_b)'$, $(T/I)'$ and I^0 are all isomorphic. In particular, the map,*

$$\psi : (T_a \otimes_{\text{alg}} T_b)' \to I^0,$$

defined by

$$\psi(\alpha) = \alpha \circ \theta \tag{5.13}$$

is an isomorphism of linear spaces.
(5) *Moreover,*

$$\langle \psi(w), \hat{\beta}_a \hat{\beta}_b \rangle = \langle w, \beta_a \otimes \beta_b \rangle \tag{5.14}$$

for all $w \in (T_a \otimes_{\text{alg}} T_b)'$, $\beta_a \in T_a$ and $\beta_b \in T_b$.

$$\begin{tikzcd} & T \arrow[dr, "\theta"] \arrow[dd, "\pi"] & \\ T_a \otimes_{\mathrm{alg}} T_b \arrow[ur, "\kappa"] \arrow[dr, "\hat{\kappa}"'] & & T_a \otimes_{\mathrm{alg}} T_b \\ & T/I \arrow[ur, "\hat{\theta}"'] & \end{tikzcd}$$

Proof. Suppose that $\{x_i\}_{i=1}^m \subset \mathfrak{g}_a$ and $\{y_j\}_{j=1}^n \subset \mathfrak{g}_b$, then

$$\begin{aligned} \theta \circ \kappa \left((x_1 \cdots x_m) \otimes (y_1 \cdots y_m) \right) &= \theta \left(x_1 \cdots x_m \cdot y_1 \cdots y_m \right) \\ &= (x_1 \otimes 1) \cdots (x_m \otimes 1) \cdot (1 \otimes y_1) \cdots (1 \otimes y_m) \\ &= (x_1 \cdots x_m) \otimes 1 \cdot 1 \otimes (y_1 \cdots y_m) \\ &= (x_1 \cdots x_m) \otimes (y_1 \cdots y_m) , \end{aligned}$$

from which it follows that $\theta \circ \kappa = \mathrm{id}_{T_a \otimes_{\mathrm{alg}} T_b}$. This shows that the map θ is surjective. Since, for $x \in \mathfrak{g}_a$ and $y \in \mathfrak{g}_b$,

$$\begin{aligned} \theta \left(xy - yx \right) &= (x \otimes 1) \cdot (1 \otimes y) - (1 \otimes y) \cdot (x \otimes 1) \\ &= x \otimes y - x \otimes y = 0, \end{aligned}$$

it follows that $I \subset \mathrm{Nul}\,(\theta)$. Therefore, the factor map $\hat{\theta} : T/I \to T_a \otimes_{\mathrm{alg}} T_b$ is well defined. We are going to show that $\hat{\theta}$ is an algebra isomorphism by showing $\hat{\kappa} \circ \hat{\theta} = \mathrm{id}_{T/I}$. Let us begin by computing $\kappa \circ \hat{\theta}$. First observe that the general element of T/I may be written as a linear combination of elements of the form

$$[x_1 \cdots x_m \cdot y_1 \cdots y_m] := x_1 \cdots x_m \cdot y_1 \cdots y_m + I,$$

where $\{x_i\}_{i=1}^m \subset \mathfrak{g}_a$ and $\{y_j\}_{j=1}^n \subset \mathfrak{g}_b$. For such an element, it easily seen that

$$\hat{\theta} \left([x_1 \cdots x_m \cdot y_1 \cdots y_m] \right) = \theta \left(x_1 \cdots x_m \cdot y_1 \cdots y_m \right) = (x_1 \cdots x_m) \otimes (y_1 \cdots y_m)$$

and hence

$$\kappa \circ \hat{\theta} \left([x_1 \cdots x_m \cdot y_1 \cdots y_m] \right) = (x_1 \cdots x_m) \cdot (y_1 \cdots y_m) .$$

Therefore, it follows that $\hat{\kappa} \circ \hat{\theta} = \mathrm{id}_{T/I}$ and hence $\hat{\theta}$ is an algebra isomorphism with inverse $\hat{\kappa}$. This also shows that $I = \mathrm{Nul}\,(\theta)$.

Since $\hat{\theta} : T/I \to T_a \otimes_{\mathrm{alg}} T_b$ is an isomorphism, it follows that $\hat{\theta}^{\mathrm{tr}} : (T_a \otimes_{alg} T_b)' \to (T/I)'$ is also an isomorphism. Clearly $\pi^{\mathrm{tr}} : (T/I)' \to I^0$ is also an isomorphism. By definition of $\hat{\theta}$ we have $\theta = \hat{\theta} \circ \pi$, and therefore $\psi \equiv \theta^{tr} = \pi^{tr} \circ \hat{\theta}^{tr} : (T_a \otimes_{\mathrm{alg}} T_b)' \to I^0$ is also an isomorphism. This proves (4).

Finally, (5.14) follows from the identities

$$\langle \psi(w), \hat{\beta}_a \hat{\beta}_b \rangle = \langle w, \theta(\hat{\beta}_a \hat{\beta}_b) \rangle = \langle w, \beta_a \otimes \beta_b \rangle.$$

□

NOTATION 5.10. Let $\varphi : T_a' \otimes_{\mathrm{alg}} T_b' \to (T_a \otimes_{\mathrm{alg}} T_b)'$ be the natural injection determined by

$$\langle \varphi (u \otimes v), \xi \otimes \eta \rangle = \langle u, \xi \rangle \langle v, \eta \rangle$$

for all $u \in T_a'$, $v \in T_b'$, $\xi \in T_a$ and $\eta \in T_b$.

LEMMA 5.11. *The four ideals, J, J_a, J_b and I are related as follows:*

$$J = \hat{J}_a \cdot \hat{T}_b + \hat{T}_a \cdot \hat{J}_b + I, \tag{5.15}$$

$$\theta(J) = J_a \otimes_{\mathrm{alg}} T_b + T_a \otimes_{\mathrm{alg}} J_b \subset T_a \otimes T_b, \tag{5.16}$$

$$\psi \circ \varphi(J_a^0 \otimes_{\mathrm{alg}} J_b^0) \subset J^0, \tag{5.17}$$

where the map $\beta \mapsto \hat{\beta}$ is given as in Notation 5.2.

PROOF. Let $K = \hat{J}_a \cdot \hat{T}_b + \hat{T}_a \cdot \hat{J}_b + I$. Clearly $\hat{J}_a \cdot \hat{T}_b \subset J$, $\hat{T}_a \cdot \hat{J}_b \subset J$ and $I \subset J$. Therefore $K \subset J$. To prove equality of K and J, it suffices to show that K contains the generators of J and is a two-sided ideal. For $i = 1, 2$, let $x_i \in \mathfrak{g}_a$ and $y_i \in \mathfrak{g}_b$. Define $\gamma_a = x_1 \wedge x_2 - [x_1, x_2]$, $\gamma_b = y_1 \wedge y_2 - [y_1, y_2]$. Then

$$\begin{aligned} \gamma &= (x_1 + y_1) \wedge (x_2 + y_2) - [x_1 + y_1, x_2 + y_2] \\ &= \gamma_a + \gamma_b \mod I. \end{aligned}$$

The generators of J are therefore contained in K. Now if $\xi \in \mathfrak{g}_{\mathfrak{b}}$ and $\beta_a \in T_a$ then $\xi \hat{\beta}_a = \hat{\beta}_a \xi \mod I$. Therefore $\xi(\hat{J}_a \cdot \hat{T}_b) \subset \hat{J}_a \cdot \hat{T}_b + I$. Similarly $\xi(\hat{T}_a \cdot \hat{J}_b) \subset \hat{T}_a \cdot \hat{J}_b + I$ since J_b is a two-sided ideal. A similar argument applies to right multiplication and also to the case where $\xi \in \mathfrak{g}_{\mathfrak{a}}$. Thus $K = J$ and (5.15) holds. Equation (5.16) now follows by applying θ to (5.15).

To prove (5.17) let $\alpha^a \in J_a^0$, $\alpha^b \in J_b^0$, $\beta_a \in J_a$ and $u \in T_b$. Then, by (5.14),

$$\langle \psi \circ \varphi(\alpha_a \otimes \alpha_b), \hat{\beta}_a \cdot \hat{u} \rangle = \langle \varphi(\alpha_a \otimes \alpha_b), \beta_a \otimes u \rangle = \langle \alpha_a, \beta_a \rangle \langle \alpha_b, u \rangle = 0$$

Similarly, $\psi \circ \varphi(\alpha_a \otimes \alpha_b)$ annihilates the second summand on the right in (5.15). It also annihilates I by Theorem 5.9. (5.17) now follows from (5.15). □

The next two lemmas are analytic.

LEMMA 5.12. *Let $S_i \subset H_i$ be an orthonormal basis for $(H_i, (\cdot, \cdot)_i)$ for $i = a$ or $i = b$. Then for $\alpha \in J^0$ and $t > 0$ we have*

$$\|\alpha\|_{q,t}^2 = \sum_{k,l=0}^{\infty} \frac{t^{k+l}}{k!l!} \sum_{X_i \in S_a, Y_j \in S_b} |\langle \alpha, X_1 \cdots X_k \cdot Y_1 \cdots Y_l \rangle|^2 . \tag{5.18}$$

PROOF. The set $S = S_a \cup S_b \subset H$ is an orthonormal basis for $(H, (\cdot,\cdot))$, and so, by the definition of $\|\cdot\|_{q,t}$,

$$\|\alpha\|_{q,t}^2 = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\xi_1,\dots,\xi_n \in S} |\langle \alpha, \xi_1 \cdots \xi_n \rangle|^2 . \tag{5.19}$$

For $\xi_1, \cdots, \xi_n \in S$, we may associate a subset $\Lambda \subset \{1, 2, \dots, n\} =: \Gamma_n$ as

$$\Lambda := \{i \in \{1, 2, \dots, n\} ; \xi_i \in S_a\} .$$

From this observation it follows that

$$\sum_{\xi_1,\dots,\xi_n \in S} |\langle \alpha, \xi_1 \cdots \xi_n \rangle|^2 = \sum_{\Lambda \subset \Gamma_n} \sum_{\substack{\xi_i \in S_a \text{ for } i \in \Lambda \\ \xi_j \in S_b \text{ for } j \notin \Lambda}} |\langle \alpha, \xi_1 \cdots \xi_n \rangle|^2 . \tag{5.20}$$

Now suppose that $\Lambda \subset \Gamma_n$ with $\#(\Lambda) = k \in \{0, 1, 2, \dots, n\}$ is given. Because $\xi_i \xi_j - \xi_j \xi_i \in I \subset J$ for each $i \in \Lambda$ and $j \notin \Lambda$, it follows that

$$\sum_{\substack{\xi_i \in S_a \text{ for } i \in \Lambda \\ \xi_j \in S_b \text{ for } j \notin \Lambda}} |\langle \alpha, \xi_1 \cdots \xi_n \rangle|^2 = \sum_{X_i \in S_a} \sum_{Y_j \in S_b} |\langle \alpha, X_1 \cdots X_k \cdot Y_1 \cdots Y_{n-k} \rangle|^2 .$$

As there are $\binom{n}{k}$ such subsets $\Lambda \subset \Gamma_n$ with $\#(\Lambda) = k$, we may rewrite Eq. (5.20) as

$$\sum_{\xi_1,\cdots,\xi_n \in S} |\langle \alpha, \xi_1 \cdots \xi_n \rangle|^2 = \sum_{k=0}^{n} \binom{n}{k} \sum_{X_i \in S_a} \sum_{Y_j \in S_b} |\langle \alpha, X_1 \cdots X_k \cdot Y_1 \cdots Y_{n-k} \rangle|^2 . \tag{5.21}$$

Combining Eqs. (5.19) and (5.21) implies Eq. (5.18). $\square$

LEMMA 5.13. *The restriction of $\psi \circ \varphi$ to $J_t^0(\mathfrak{g}_a) \otimes_{\mathrm{alg}} J_t^0(\mathfrak{g}_b)$ is an isometry into $J_t^0(\mathfrak{g})$.*

PROOF. Suppose that $\alpha^a, \beta^a \in J_t^0(\mathfrak{g}_a)$ and $\alpha^b, \beta^b \in J_t^0(\mathfrak{g}_b)$. Then for $X_i \in S_a$ and $Y_j \in S_b$ we have

$$\begin{aligned}
\left\langle \psi \circ \varphi \left(\alpha^a \otimes \alpha^b\right), X_1 \cdots X_k \cdot Y_1 \cdots Y_l \right\rangle &= \left\langle \varphi \left(\alpha^a \otimes \alpha^b\right), \theta_0 \left(X_1 \cdots X_k \cdot Y_1 \cdots Y_l\right) \right\rangle \\
&= \left\langle \varphi \left(\alpha^a \otimes \alpha^b\right), X_1 \cdots X_k \otimes Y_1 \cdots Y_l \right\rangle \\
&= \langle \alpha^a, X_1 \cdots X_k \rangle \left\langle \alpha^b, Y_1 \cdots Y_l \right\rangle .
\end{aligned}$$

From this identity along with the polarization of Eq. (5.18), we find

$$\begin{aligned}
\left\langle \psi\circ\varphi\left(\alpha^{a}\otimes\alpha^{b}\right),\psi\circ\varphi\left(\beta^{a}\otimes\beta^{b}\right)\right\rangle_{q,t} &= \sum_{k,l=0}^{\infty}\frac{t^{k+l}}{k!\,l!}\times \\
\sum_{X_i\in S_a}\sum_{Y_j\in S_b}\left\langle \alpha^{a},X_1\cdots X_k\right\rangle\left\langle \alpha^{b},Y_1\cdots Y_l\right\rangle &\overline{\left\langle \beta^{a},X_1\cdots X_k\right\rangle\left\langle \beta^{b},Y_1\cdots Y_l\right\rangle} \\
&= \left(\sum_{k=0}^{\infty}\frac{t^{k}}{k!}\sum_{X_i\in S_a}\left\langle \alpha^{a},X_1\cdots X_k\right\rangle\overline{\left\langle \beta^{a},X_1\cdots X_l\right\rangle}\right)\times \\
&\qquad\left(\sum_{l=0}^{\infty}\frac{t^{l}}{l!}\sum_{Y_j\in S_b}\left\langle \alpha^{b},Y_1\cdots Y_l\right\rangle\overline{\left\langle \beta^{b},Y_1\cdots Y_l\right\rangle}\right) \\
&= \left\langle \alpha^{a},\beta^{a}\right\rangle_{q_a,t}\left\langle \alpha^{b},\beta^{b}\right\rangle_{q_b,t}=\left\langle \alpha^{a}\otimes\alpha^{b},\beta^{a}\otimes\beta^{b}\right\rangle_{J_t^0(\mathfrak{g}_a)\otimes J_t^0(\mathfrak{g}_b)},
\end{aligned}$$

which shows that $\psi\circ\varphi$ is isometric on $J_t^0\left(\mathfrak{g}_a\right)\otimes_{alg}J_t^0\left(\mathfrak{g}_b\right)$. □

We are now ready to complete the proof of the main Theorem 5.3.

Completion of proof of Theorem 5.3. By Lemma 5.13, the restriction of $\psi\circ\varphi$ to $J_t^0\left(\mathfrak{g}_a\right)\otimes_{\mathrm{alg}} J_t^0\left(\mathfrak{g}_b\right)$ extends uniquely to an isometry of Hilbert spaces, $L: J_t^0\left(\mathfrak{g}_a\right)\otimes J_t^0\left(\mathfrak{g}_b\right)\to J_t^0\left(\mathfrak{g}\right)$. Moreover, the determining equation, Eq. (5.2), for L follows from Eq. (5.14) by continuity. So to complete the proof of Theorem 5.3, we need only show that $L: J_t^0\left(\mathfrak{g}_a\right)\otimes J_t^0\left(\mathfrak{g}_b\right)\to J_t^0\left(\mathfrak{g}\right)$ is surjective.

Suppose $\alpha\in J_t^0\left(\mathfrak{g}\right)$ and $\left\langle \alpha,L\left(\alpha^{a}\otimes\alpha^{b}\right)\right\rangle=0$ for all $\alpha^{a}\in J_t^0\left(\mathfrak{g}_a\right)$ and $\alpha^{b}\in J_t^0\left(\mathfrak{g}_b\right)$. By polarization of Eq. (5.18) and the identity

$$\begin{aligned}
\left\langle L\left(\alpha^{a}\otimes\alpha^{b}\right),X_1\cdots X_k\cdot Y_1\cdots Y_l\right\rangle &= \left\langle \varphi\left(\alpha^{a}\otimes\alpha^{b}\right),X_1\cdots X_k\otimes Y_1\cdots Y_l\right\rangle \\
&= \left\langle \alpha^{a},X_1\cdots X_k\right\rangle\left\langle \alpha^{b},Y_1\cdots Y_l\right\rangle,
\end{aligned}$$

we have

$$\begin{aligned}
0 &= \left\langle \alpha,L\left(\alpha^{a}\otimes\alpha^{b}\right)\right\rangle_{q,t} \\
&= \sum_{k=0}^{\infty}\frac{t^{k}}{k!}\sum_{X_i\in S_a}\sum_{l=0}^{\infty}\frac{t^{l}}{l!}\sum_{Y_j\in S_b}\left\langle \alpha,X_1\cdots X_k\cdot Y_1\cdots Y_l\right\rangle\overline{\left\langle \alpha^{a},X_1\cdots X_k\right\rangle\left\langle \alpha^{b},Y_1\cdots Y_l\right\rangle}. \qquad (5.22)
\end{aligned}$$

We now are going to show that there is an element $u\in\left(J_a^0\right)_t$ such that

$$\left\langle u,\beta_a\right\rangle=\sum_{l=0}^{\infty}\frac{t^{l}}{l!}\sum_{Y_j\in S_b}\left\langle \alpha,\beta_a\cdot Y_1\cdots Y_l\right\rangle\overline{\left\langle \alpha^{b},Y_1\cdots Y_l\right\rangle}\text{ for all }\beta_a\in T_a, \qquad (5.23)$$

and, in particular, that the sum is convergent. Assuming for the moment that the sum in Eq. (5.23) is convergent, it follows from Eq. (5.15) and Eq. (5.23) that $u\in J_a^0$. Moreover, by the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
|\langle u,\beta_a\rangle|^2 &\le \{\sum_{l=0}^{\infty}\frac{t^l}{l!}\sum_{Y_j\in S_b}\left|\langle\alpha,\beta_a\cdot Y_1\cdots Y_l\rangle\,\overline{\langle\alpha^b,Y_1\cdots Y_l\rangle}\right|\}^2\\
&\le \sum_{l=0}^{\infty}\frac{t^l}{l!}\sum_{Y_j\in S_b}|\langle\alpha,\beta_a\cdot Y_1\cdots Y_l\rangle|^2\left\|\alpha^b\right\|_{q_b,t}^2. \qquad (5.24)
\end{aligned}
$$

So according to Lemma 5.12,

$$
\begin{aligned}
\|u\|_{q_a,t}^2 &= \sum_{k=0}^{\infty}\frac{t^k}{k!}\sum_{X_i\in S_a}|\langle u,X_1\cdots X_k\rangle|^2\\
&\le \left\|\alpha^b\right\|_{q_b,t}^2\sum_{k=0}^{\infty}\frac{t^k}{k!}\sum_{X_i\in S_a}\sum_{l=0}^{\infty}\frac{t^l}{l!}\sum_{Y_j\in S_b}|\langle\alpha,X_1\cdots X_k\cdot Y_1\cdots Y_l\rangle|^2\\
&= \left\|\alpha^b\right\|_{q_b,t}^2\|\alpha\|_{q,t}^2<\infty, \qquad (5.25)
\end{aligned}
$$

and therefore, $u\in(J_a^0)_t$. In order to see that the sum in Eq. (5.23) is convergent, recall from [5, Lemma 2.11] that for any $\beta_a\in T_a$ there exists $\beta_a'\in T(H_a)$ such that $\beta_a-\beta_a'\in J_a$. For each ℓ we have $(\beta_a-\beta_a')\cdot Y_1\cdots Y_\ell\in\hat{J}_a\cdot\hat{T}_b\subset J$. Therefore β_a may be replaced by β_a' in each term of (5.23). Since β_a' may be written as a finite linear combination of elements in $\Gamma_a:=\{1\}\cup_{k=1}^{\infty}\{X_1\cdots X_k;X_j\in S_a\}$, it suffices to verify the convergence in Eq. (5.23) when $\beta_a\in\Gamma_a$. However, the convergence of the sum in this case is now clear from the estimates in Eqs. (5.24) and (5.25).

With u as in Eq. (5.23), Eq. (5.22) may be written as

$$
0=\sum_{k=0}^{\infty}\frac{t^k}{k!}\sum_{X_i\in S_a}\langle u,X_1\cdots X_k\rangle\,\overline{\langle\alpha^a,X_1\cdots X_k\rangle}=(u,\alpha^a)_{q_a,t},
$$

which is assumed to hold for all $\alpha^a\in(J_a^0)_t$. Taking $\alpha^a=u$ in this identity then shows $u=0$, i.e.,

$$
0=\sum_{l=0}^{\infty}\frac{t^l}{l!}\sum_{Y_j\in S_b}\langle\alpha,\beta_a\cdot Y_1\cdots Y_l\rangle\,\overline{\langle\alpha^b,Y_1\cdots Y_l\rangle}\text{ for all }\beta_a\in T_a. \qquad (5.26)
$$

For fixed $\beta_a\in T_a$, we may now define $v\in J^0(\mathfrak{g}_b)$ by

$$
\langle v,\beta_b\rangle:=\langle\alpha,\beta_a\cdot\beta_b\rangle\ \text{ for all }\beta_b\in T_b. \qquad (5.27)
$$

Working as above, v may be written as a linear combination of functions of the form $\{T_b \ni \beta_b \to \langle \alpha, \eta \cdot \beta_b \rangle\}_{\eta \in \Gamma_a}$, and if $\beta_a = \eta \in \Gamma_a$, then

$$\begin{aligned} \|v\|_{q_b,t}^2 &= \sum_{l=0}^{\infty} \frac{t^l}{l!} \sum_{Y_j \in S_b} |\langle v, Y_1 \cdots Y_l \rangle|^2 \\ &= \sum_{l=0}^{\infty} \frac{t^l}{l!} \sum_{Y_j \in S_b} |\langle \alpha, \beta_a \cdot Y_1 \cdots Y_l \rangle|^2 \le \|\alpha\|_{q,t}^2 < \infty, \end{aligned}$$

wherein we have used Lemma 5.12 in the last line. Hence we have shown that $v \in J_t^0(\mathfrak{g}_b)$. Taking $\alpha^b = v$ in Eq. (5.26) shows $0 = (v,v)_{q_b,t}$, i.e., $v = 0$. As $\beta_a \in T_a$ was arbitrary, it follows from Eq. (5.18) that $\langle \alpha, \beta_a \cdot \beta_b \rangle = 0$ for all $\beta_a \in T_a$ and $\beta_b \in T_b$ and this suffices to show $\alpha \equiv 0$. $\square$

5.3. **Functoriality of $\mathcal{H}L^2$ under products.** Suppose that M is a complex manifold equipped with smooth positive measure μ, and let $\mathcal{H}L^2(\mu)$ denote the Hilbert space of complex square integrable functions on M. As is well known (see [4, Lemma 3.4] for example), for any $m \in M$, the evaluation map $e_m(f) = f(m)$ for all $f \in \mathcal{H}L^2(\mu)$, is a bounded linear functional on $\mathcal{H}L^2(\mu)$. So by the Riesz theorem, there exists a unique element $F(\cdot, m) \in \mathcal{H}L^2(\mu)$ such that $f(m) = \langle f, F(\cdot, m) \rangle_{L^2(\mu)}$ for all $f \in \mathcal{H}L^2(\mu)$. (We will often write F_m for $F(\cdot, m)$.) The function $F: M \times M \to \mathbb{C}$ so defined is called the reproducing kernel for $\mathcal{H}L^2(\mu)$. The following proposition summarizes some of the well known properties of this reproducing kernel.

Proposition 5.14. *The reproducing kernel $F: M \times M \to \mathbb{C}$ for $\mathcal{H}L^2(\mu)$ satisfies:*

1. $F(m, m') = \overline{F(m', m)}$ *for all* $m, m' \in M$.
2. *The map,* $M \ni m \to F_m \in \mathcal{H}L^2(\mu)$ *is continuous.*
3. *The reproducing kernel F is continuous.*

Proof. 1. The first assertion is a consequence of the identity

$$(F_{m'}, F_m) = F_{m'}(m) = F(m, m'), \tag{5.28}$$

which follows from the reproducing properties of F.

2. If $K \subset M$ is a compact set and $f \in \mathcal{H}L^2(\mu)$, we have $\sup_{m \in K} |e_m(f)| = \max_{m \in K} |f(m)| < \infty$. Therefore, by the uniform boundedness principle,

$$C(K) := \sup_{m \in K} \|F_m\|_{\mathcal{H}L^2(\mu)} = \sup_{m \in K} \|e_m\|_{\mathcal{H}L^2(\mu)^*} < \infty.$$

Therefore, if $\|f\|_{\mathcal{H}L^2(\mu)} \le 1$ then $\max_{m \in K} |f(m)| \le C(K)$. Hence, using the Cauchy estimates (see [4, Lemma 3.4]), we may easily show that, for any $m \in M$, there exists a chart (x, V) of M with $m \in V$ (V is the domain of x) such that $x(V)$ is a poly disk centered at $x(m) := 0$ in a Euclidean space and

$$C_i := \sup_{\|f\|_{\mathcal{H}L^2(\mu)}\leq 1} \max_{m\in V} \left|\frac{\partial f}{\partial x^i}(m)\right| < \infty \text{ for } i = 1, 2, \ldots, \dim_{\mathbb{R}}(M).$$

Then, for $m' \in V$, we have

$$\begin{aligned}\|F_m - F_{m'}\|_{\mathcal{H}L^2(\mu)} &= \sup_{\|f\|_{\mathcal{H}L^2(\mu)}\leq 1} |(f, F_m - F_{m'})| \\ &= \sup_{\|f\|_{\mathcal{H}L^2(\mu)}\leq 1} |f(m) - f(m')| \leq \sum_{i=1}^{\dim_{\mathbb{R}}(M)} C_i \left|x^i(m')\right|,\end{aligned}$$

which proves $m' \to F_{m'}$ is continuous at $m \in M$.

3. The third assertion now follows from the second and the identity in Eq. (5.28). □

Now suppose that N is another complex manifold equipped with a smooth positive measure ν. Let $G_n(\cdot) = G(\cdot, n)$ be the reproducing kernel for $\mathcal{H}L^2(\nu)$.

Theorem 5.15. *Suppose that M, N, μ, ν, F and G are as defined above. Then the function*

$$(F_m \otimes G_n)(m', n') := F_m(m') \cdot G_n(n') \text{ for } m, m' \in M \text{ and } n, n' \in N$$

is the reproducing kernel for $\mathcal{H}L^2(\mu\otimes\nu)$ where $\mu\otimes\nu$ is the product measure on $M \times N$.

Proof. Since $F_m \otimes G_n \in \mathcal{H}L^2(\mu\otimes\nu)$, we need only show for any $h \in \mathcal{H}L^2(\mu\otimes\nu)$ that $h(m,n) = \tilde{h}(m,n)$, where

$$\tilde{h}(m,n) := (h, F_m \otimes G_n) \text{ for all } m \in M \text{ and } n \in N.$$

We will do this by showing $\tilde{h}$ is continuous and then by showing $h = \tilde{h}$ a.e. with respect to $\mu\otimes\nu$.

The continuity of $\tilde{h}$ follows from the continuities of $M \ni m \to F_m \in \mathcal{H}L^2(\mu)$ and $N \ni n \to F_n \in \mathcal{H}L^2(\nu)$ and the following simple estimate

$$\begin{aligned}\left|\tilde{h}(m,n) - \tilde{h}(m',n')\right| &= |(h, F_m \otimes G_n - F_{m'} \otimes G_{n'})| \\ &\leq \|h\|_{L^2(\mu\otimes\nu)} \|F_m \otimes G_n - F_{m'} \otimes G_{n'}\|_{L^2(\mu\otimes\nu)} \\ &= \|h\|_{L^2(\mu\otimes\nu)} \|(F_m - F_{m'}) \otimes G_n - F_{m'} \otimes (G_n - G_{n'})\|_{L^2(\mu\otimes\nu)} \\ &\leq \|h\|_{L^2(\mu\otimes\nu)} \|F_m - F_{m'}\|_{L^2(\mu)} \|G_n\|_{L^2(\nu)} \\ &\qquad + \|F_{m'}\|_{L^2(\mu)} \|G_n - G_{n'}\|_{L^2(\nu)}.\end{aligned}$$

If

$$M_0 := \left\{m' \in M; \int_N |h(m',n')|^2 \, d\nu(n') < \infty\right\}$$

and

$$N_0 := \left\{n' \in N; \int_M |h(m',n')|^2 \, d\mu(m') < \infty\right\},$$

then by Fubini's theorem we know $\mu(M \setminus M_0) = 0$ and $\nu(N \setminus N_0) = 0$. For $m' \in M_0$, $h(m', \cdot) \in \mathcal{H}L^2(\nu)$ and therefore, for all $n \in N$, we have

$$h(m', n) = (h(m', \cdot), G_n) = \int_N h(m', n') \overline{G_n(n')} d\nu(n'). \tag{5.29}$$

Now take $n \in N_0$ so that $h(\cdot, n) \in \mathcal{H}L^2(\mu)$. Multiply Eq. (5.29) by $\overline{F_m(m')}$ and then integrate with respect to $d\mu(m')$, to find

$$\begin{aligned} h(m, n) &= \int_M h(m', n) \overline{F_m(m')} d\mu(m') \\ &= \int_M \left(\int_N h(m', n') \overline{G_n(n')} d\nu(n') \right) \overline{F_m(m')} d\mu(m') \\ &= (h, F_m \otimes G_n) = \tilde{h}(m, n), \end{aligned}$$

wherein we have used Fubini's theorem for the third equality. Hence we have shown $h(m, n) = \tilde{h}(m, n)$ for all $m \in M$ and $n \in N_0$. As N_0 is dense in N and h and $\tilde{h}$ are continuous, we may now conclude that $h(m, n) = \tilde{h}(m, n) = (h, F_m \otimes G_n)$ for all $m \in M$ and $n \in N$. $\square$

As a corollary we have the following theorem.

THEOREM 5.16. *Suppose M and N are complex manifolds equipped with smooth positive measures μ and ν, respectively. Let*

$$\varphi : L^2(\mu) \otimes L^2(\nu) \to L^2(\mu \otimes \nu)$$

be the natural unitary isomorphism determined uniquely by

$$\varphi(f \otimes g)(m, n) := f(m) g(n) \tag{5.30}$$

for all $f \in L^2(\mu)$ and $g \in L^2(\nu)$. Then

$$\varphi\left(\mathcal{H}L^2(\mu) \otimes \mathcal{H}L^2(\nu)\right) = \mathcal{H}L^2(\mu \otimes \nu)$$

so that φ

$$\varphi|_{\mathcal{H}L^2(\mu)\otimes\mathcal{H}L^2(\nu)} : \mathcal{H}L^2(\mu) \otimes \mathcal{H}L^2(\nu) \to \mathcal{H}L^2(\mu \otimes \nu)$$

is again a unitary isomorphism of Hilbert spaces.

PROOF. Since the projection maps from $M \times N$ to M and N are holomorphic and the product of holomorphic functions is holomorphic, $\varphi(f \otimes g) \in \mathcal{H}L^2(\mu \otimes \nu)$ for all $f \in \mathcal{H}L^2(\mu)$ and $g \in \mathcal{H}L^2(\nu)$. As linear combinations of elements of the form $f \otimes g$ are dense in $\mathcal{H}L^2(\mu) \otimes \mathcal{H}L^2(\nu)$ and $\mathcal{H}L^2(\mu \otimes \nu)$ is a closed subspace of $L^2(\mu \otimes \nu)$, it follows that $\varphi(\mathcal{H}L^2(\mu) \otimes \mathcal{H}L^2(\nu)) \subset \mathcal{H}L^2(\mu \otimes \nu)$. To see that the inclusion is not proper, suppose $h \in \mathcal{H}L^2(\mu \otimes \nu)$ is perpendicular to the closed linear space $\varphi(\mathcal{H}L^2(\mu) \otimes \mathcal{H}L^2(\nu))$. Under this assumption, it follows from Theorem 5.15 that

$$h(m, n) = (h, F_m \otimes G_n) = (h, \varphi(F_m \otimes G_n)) = 0 \text{ for all } (m, n) \in M \times N,$$

i.e., that $h \equiv 0$. $\square$

5.4. **Applications of the sum/product functoriality to the Taylor map.** Let us now suppose that G_a and G_b are two simply connected complex Lie groups with Lie algebras denoted by $\mathfrak{g}_a$ and $\mathfrak{g}_b$. Further assume that q_a and q_b are non-negative Hermitian forms on $\mathfrak{g}_a^*$ and $\mathfrak{g}_b^*$, respectively, which satisfy Hörmander's condition. Then $G = G_a \times G_b$ is again a simply connected Lie group with Lie algebra $\mathfrak{g} := \mathfrak{g}_a \oplus \mathfrak{g}_b$ and $q = q_a \oplus q_b$ being a non-negative Hermitian form on $\mathfrak{g}^*$ satisfying Hörmander's condition. Let us continue to use the assumptions and notation introduced above.

Corollary 5.17. *If we know that*

$$\mathcal{H}L^2\left(G_i, \rho_t^i\right) \ni f \mapsto \hat{f} \in J_t^0\left(\mathfrak{g}_i\right)$$

is an isometry (resp. a unitary isomorphism) for both $i = a$ and $i = b$, then

$$\mathcal{H}L^2\left(G, \rho_t\right) \ni w \to \hat{w} \in J_t^0\left(\mathfrak{g}\right) \tag{5.31}$$

is also an isometry (resp. a unitary isomorphism) of Hilbert spaces, where $\rho_t := \rho_t^a \otimes \rho_t^b$.

Proof. Let $u_a \in \mathcal{H}L^2\left(G_a, \rho_t^a\right)$ and $u_b \in \mathcal{H}L^2\left(G_b, \rho_t^b\right)$ and $u_a \otimes u_b \in \mathcal{H}L^2\left(G_a \times G_b, \rho_t^a \otimes \rho_t^b\right)$ be defined by $\left(u_a \otimes u_b\right)\left(g_a, g_b\right) := u_a\left(g_a\right) u_b\left(g_b\right)$. Then, recalling the defining equation Eq. (5.2) of L, for $\beta_i \in T_i$ for $i = a$ or b, we have

$$\left\langle \widehat{u_a \otimes u_b}, \beta_a \cdot \beta_b \right\rangle = \left\langle \hat{u}_a, \beta_a \right\rangle \left\langle \hat{u}_b, \beta_b \right\rangle = \left\langle L\left(\hat{u}_a \otimes \hat{u}_b\right), \beta_a \cdot \beta_b \right\rangle,$$

from which it follows that $\widehat{u_a \otimes u_b} = L\left(\hat{u}_a \otimes \hat{u}_b\right)$. Since $\mathcal{H}L^2\left(G_i, \rho_t^i\right) \cong J_t^0\left(\mathfrak{g}_i\right)$ and $L : J_t^0\left(\mathfrak{g}_a\right) \otimes J_t^0\left(\mathfrak{g}_b\right) \to J_t^0\left(\mathfrak{g}\right)$ is unitary, it follows that

$$\mathcal{H}L^2\left(G_a, \rho_t^a\right) \otimes_{\mathrm{alg}} \mathcal{H}L^2\left(G_b, \rho_t^b\right) \ni w \mapsto \hat{w} \in J_t^0\left(\mathfrak{g}\right)$$

is an isometry. From Theorem 5.16, we know that $\mathcal{H}L^2\left(G_a, \rho_t^a\right) \otimes_{\mathrm{alg}} \mathcal{H}L^2\left(G_b, \rho_t^b\right)$ is dense in $\mathcal{H}L^2\left(G, \rho_t\right)$, and therefore the map in Eq. (5.31) extends uniquely to an isometry from $\mathcal{H}L^2\left(G, \rho_t\right)$ to $J_t^0\left(\mathfrak{g}\right)$. By the Cauchy estimates (see the proof of Proposition 5.14 below), if $w_n \to w$ in $\mathcal{H}L^2\left(G, \rho_t\right)$ then $\beta w_n \to \beta w$ for all $\beta \in T$. Therefore the isometry from $\mathcal{H}L^2\left(G, \rho_t\right)$ to $J_t^0\left(\mathfrak{g}\right)$ is still given by $w \mapsto \hat{w}$ where $\langle \hat{w}, \beta \rangle = \beta w$.

We will now finish the proof by showing the map in Eq. (5.31) is surjective when the Taylor maps on the factors are surjective. Let

$$D := \left\{ \widehat{u_a \otimes u_b}; u_i \in \mathcal{H}L^2\left(G_i, \rho_t^i\right) \text{ for } i = a \text{ and } b \right\}.$$

Since $\widehat{u_a \otimes u_b} = L\left(\hat{u}_a \otimes \hat{u}_b\right)$ and $\mathcal{H}L^2\left(G_i, \rho_t^i\right) \cong J_t^0\left(\mathfrak{g}_i\right)$, we may write

$$D = \left\{ L\left(\alpha^a \otimes \alpha^b\right); \alpha^i \in J_t^0\left(\mathfrak{g}_i\right) \text{ for } i = a \text{ and } b \right\}.$$

As L is unitary and $\left\{\alpha^a \otimes \alpha^b; \alpha^i \in J_t^0\left(\mathfrak{g}_i\right) \text{ for } i = a \text{ and } b\right\}$ is total in $J_t^0\left(\mathfrak{g}_a\right) \otimes J_t^0\left(\mathfrak{g}_b\right)$, it follows that D is total in $J_t^0\left(\mathfrak{g}\right)$. Hence the range of the map in Eq. (5.31) is dense in $J_t^0\left(\mathfrak{g}\right)$, which suffices to prove that the map is surjective. $\square$

EXAMPLE 5.18. Consider the complex Heisenberg algebras $\mathfrak{h}_{2n+1} = \mathfrak{h}^{\mathbb{C}}_{2n+1}$ each equipped with the two natural Hermitian forms q^i_{2n+1}, $i = 1, 2$, of Remark 4.4 (q^1_{2n+1} is positive definite, whereas q^2_{2n+1} is degenerate but satisfies Hörmander's condition). Let

$$\mathfrak{g} = \mathbb{C}^{n_0} \oplus \mathfrak{h}_{2n_1+1} \oplus \cdots \oplus \mathfrak{h}_{2n_k+1}.$$

Equip this direct sum with the two Hermitian forms

$$q^1 = q^0 \oplus q^1_{2n_1+1} \oplus \cdots \oplus q^1_{2n_k+1}$$

and

$$q^2 = q^0 \oplus q^2_{2n_1+1} \oplus \cdots \oplus q^2_{2n_k+1},$$

where q_0 is the canonical Hermitian form on $\mathbb{C}^{n_0}$. Observe that, of course, q^1 is positive definite whereas q^2 is degenerate but satisfies Hörmander's condition. Now applying the result stated in Remark 4.4 together with Corollary 5.5 yields the inequality

$$\|\alpha\|_{q^1,s} \leq C(s,t)^k \|\alpha\|_{q^2,t} \text{ for all } \alpha \in J^0_{q^2,t} \subset T'(\mathfrak{g}),$$

with $C(s,t)$ as in Theorem 4.1. In particular, $C(s,t)$ is finite for all $s, t \in (0, \infty)$ satisfying $(es/t)((4/t) + 1) < 1$. Hence, the q^2 family controls the q^1 family on $\mathfrak{g}$ (that the q^1 family controls the q^2 family is obvious from the definition). Now, the results of [4] concerning the positive definite case, together with the control of q^1 by q^2 and Proposition 3.1, yields the unitarity of the Taylor map between $J^0_{q^2,t}$ and the corresponding space of holomorphic functions on the associated group G. This proof of the unitarity of the Taylor map for the form q^2 uses functoriality only on the tensor side. However, it also uses Proposition 3.1 which involves knowledge of some properties of the Taylor map in the Hörmander case. The point is that the properties of the Taylor map that are used in Proposition 3.1 are proved in exactly the same way in both the positive definite and the Hörmander case (see [5, Section 4]). Alternatively, one can use Corollary 5.17 which uses functoriality on both the tensor side and the function side. This yields the desired result, i.e., the unitarity of the Taylor map for q^2, using only the properties of the Taylor map in the positive definite case.

6. FUNCTORIALITY UNDER QUOTIENTS

6.1. **The K-invariant Taylor isomorphism.** Let G be a connected, simply connected, complex Lie group, K a connected, closed, complex subgroup of G, $\mathfrak{g} = \mathrm{Lie}(G)$, and $\mathfrak{k} = \mathrm{Lie}(K)$. Recall that $T = T(\mathfrak{g})$ denotes the algebraic tensor algebra over $\mathfrak{g}$ and $J = J(\mathfrak{g})$ is the two sided ideal generated by $\{\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta]; \xi, \eta \in \mathfrak{g}\}$. We let $T\mathfrak{k}$ (resp. $\mathfrak{k}T$) be the left (resp. right) ideal in T generated by $\mathfrak{k}$. Given a subspace V of T, we let V^0 be its annihilator in T' and V^0_t the subspace of those elements $\alpha \in V^0_t$ such that $\|\alpha\|_{q,t} < \infty$.

In this subsection we are going restrict the isomorphism in Theorem 2.8 to the right K-invariant functions in $\mathcal{H}L^2(G)$. The next lemma is key to the main results of this section.

LEMMA 6.1. *Let $f \in \mathcal{H}(G)$. Then*

$$f \text{ is right } K \text{ invariant iff } \langle \hat{f}, T\mathfrak{k} \rangle = 0, \tag{6.1}$$

$$f \text{ is left } K \text{ invariant iff } \langle \hat{f}, \mathfrak{k}T \rangle = 0. \tag{6.2}$$

PROOF. First note that, for any function $f \in C^1(G)$, f is right K-invariant if and only if $\tilde{\eta} f \equiv 0$ on G for all $\eta \in \mathfrak{k}$. Now suppose that $f \in \mathcal{H}(G)$. Let $\xi_1, \ldots, \xi_n \in \mathfrak{g}$ and let $\eta \in \mathfrak{k}$. Since $\{\tilde{\xi}_1 \cdots \tilde{\xi}_n(\tilde{\eta} f)\}(e) = \langle \hat{f}, \xi_1 \cdots \xi_n \eta \rangle$, the holomorphic function $\tilde{\eta} f$ is zero on G if and only if $\langle \hat{f}, T\eta \rangle = 0$. Hence f is right K-invariant if and only if $\langle \hat{f}, T\eta \rangle = 0$ for all $\eta \in \mathfrak{k}$.

To discuss left K invariance, let $\check{\eta}$ denote the right invariant extension of η. Of course, any C^1 function on G is left K invariant if and only if $\check{\eta} f \equiv 0$ on G for all $\eta \in \mathfrak{k}$. Suppose that $f \in \mathcal{H}(G)$. Then $\check{\eta} f$ is holomorphic. So f is left K-invariant if and only if, for all $\eta \in \mathfrak{k}$, $\{\tilde{\xi}_1 \cdots \tilde{\xi}_n(\check{\eta} f)\}(e) = 0$ for all $\xi_j \in \mathfrak{g}$. But

$$\begin{aligned}\{\tilde{\xi}_1 \cdots \tilde{\xi}_n(\check{\eta} f)\}(e) &= \frac{\partial^{n+1}}{\partial s_1 \cdots \partial s_n \partial t}|_{s_1 = \cdots = s_n = t = 0} f(e^{t\eta} e^{s_1 \xi_1} \cdots e^{s_n \xi_n}) \\ &= \{\tilde{\eta} \tilde{\xi}_1 \cdots \tilde{\xi}_n f\}(e) = \langle \hat{f}, \eta \xi_1 \cdots \xi_n \rangle.\end{aligned} \tag{6.3}$$

□

The following theorem is a holomorphic version of the main theorem in [8]. It is based on Lemma 6.1 and Theorem 2.8, which is taken from [5, Theorem 6.1].

THEOREM 6.2. *The Taylor map $\mathcal{H}(G) \ni f \mapsto \hat{f} \in J^0$ restricts to a surjective isometry from the space of right K-invariant functions in $\mathcal{H}L^2(G, \rho_t)$ onto $(J + T\mathfrak{k})_t^0$ and restricts to a surjective isometry from $\mathcal{H}L_K^2(G, \rho_t)$, the space of left K-invariant functions in $\mathcal{H}L^2(G, \rho_t)$, onto $(J + \mathfrak{k}T)_t^0$.*

PROOF. Theorem 2.8 asserts that the map $\mathcal{H}(G) \ni f \mapsto \hat{f}$ is a surjective isometry from $\mathcal{H}L^2(G, \rho_t)$ onto J_t^0. Hence we need only identify the ranges of these two restrictions properly. If f is in $\mathcal{H}(G)$ and is left K-invariant then, by Lemma 6.1, $\hat{f}$ annihilates the right ideal $\mathfrak{k}T$ and is therefore in $(J + \mathfrak{k}T)_t^0$. Conversely, if $\hat{f}$ lies in this space, then $\hat{f}$ annihilates $\mathfrak{k}T$ and by Lemma 6.1 f is left K-invariant. A similar argument applies to right invariant functions. □

6.2. **The quotient theorem.** Let G and K be as in Subsection 6.1, let M be the space of right K cosets $M = K \setminus G$, and let $\pi : G \to M$ be the associated quotient map. In this general situation, the hypothesis that K is connected is equivalent to the simple connectedness of M (for example see [7, I. Chap. 1, Theorem 4.8]). In a standard way,

M admits a smooth right G action defined by $\pi(x)g := \pi(xg)$ for all $x, g \in G$, and the linear map $\pi_{*e} : \mathfrak{g} \to T_{Ke}M$ is surjective with $\ker(\pi_{*e}) = \mathfrak{k}$. Hence if we let $\mathfrak{m} := \mathfrak{g}/\mathfrak{k}$, then

$$\mathfrak{m} \ni A + \mathfrak{k} \to \pi_{*e}A \in T_{Ke}M \tag{6.4}$$

is a linear isomorphism of vector spaces. In the sequel, we will use Eq. (6.4) to identify $\mathfrak{m}$ with $T_{Ke}M$ without further mention.

Notation 6.3. The formula,

$$\dot{A}(m) := \frac{d}{dt}|_0 \left(me^{tA}\right) \text{ for all } m \in M \text{ and } A \in \mathfrak{g} \tag{6.5}$$

defines a linear map $\mathfrak{g} \ni A \mapsto \dot{A} \in \mathrm{Vect}(M)$, where $\mathrm{Vect}(M)$ denotes the linear space of smooth vector fields on M.

Lemma 6.4. *The map $\mathfrak{g} \ni A \mapsto \dot{A} \in \mathrm{Vect}(M)$ has the following properties*:

1. *For all $A \in \mathfrak{g}$, $\dot{A}$ is π – related to $\tilde{A}$, i.e., $\pi_*\tilde{A} = \dot{A} \circ \pi$. (As usual $\tilde{A}$ is the left invariant vector field on G associated to $A \in \mathfrak{g}$.)*
2. *For all $g \in G$ and $A \in \mathfrak{g}$, we have $\dot{A}$ and $(Ad_{g^{-1}}A)^\bullet$ are R_g-related.*
3. *If $A \in \mathfrak{g}$, then $\dot{A}(Ke) = 0$ iff $A \in \mathfrak{k}$.*

Proof. (1) For the first assertion we have, for all $g \in G$, that

$$\pi_*\tilde{A}(g) = \frac{d}{dt}|_0 \pi\left(ge^{tA}\right) = \frac{d}{dt}|_0 \left(\pi(g)e^{tA}\right) = \dot{A}(\pi(g)).$$

(2) Similarly if $g \in G$ and $m \in M$, then

$$\begin{aligned} R_{g*}\dot{A}(m) &= \frac{d}{dt}|_0 me^{tA}g = \frac{d}{dt}|_0 mgg^{-1}e^{tA}g \\ &= \frac{d}{dt}|_0 mge^{tAd_{g^{-1}}A} = (Ad_{g^{-1}}A)^\bullet(mg) \\ &= (Ad_{g^{-1}}A)^\bullet \circ R_g(m). \end{aligned}$$

(3) Observe that $\dot{A}(Ke) = \pi_{*e}(A)$ and therefore the assertion in (3) follows from the comments before Notation 6.3. □

We suppose that q is a given Hermitian form on $\mathfrak{g}^*$ satisfying Hörmander's condition. Let H be the Hörmander subspace of $\mathfrak{g}$ associated with q, equipped with its Hermitian inner product $(\cdot,\cdot)_H$ and an orthonormal basis $(X_i)_1^m$. See (2.8). As in Section 2, let Δ denote the associated hypoelliptic Laplacian on G and let $\rho_t(dx) = \rho_t(x)dx$ denote the associated heat kernel measure (in this section, we will abuse notation and use ρ_t for the heat kernel measure and $\rho_t(x)$ for its density). We may also define a sub-Laplacian Δ_M on M given by

$$\Delta_M = \sum_{j=1}^m \left(\dot{X}_j^2 + \dot{Y}_j^2\right), \text{ where } Y_j := iX_j. \tag{6.6}$$

Using (1) of Lemma 6.4, one sees that Δ_M may also be characterized by the relation

$$\Delta\left(f\circ\pi\right)=\left(\Delta_M f\right)\circ\pi \text{ for all } f\in C^{\infty}\left(M\right), \tag{6.7}$$

and that the family of vector fields $\left\{\dot{X}_j,\dot{Y}_j\right\}_{j=1}^{m}$ satisfies Hörmander's condition, i.e., $\dot{X}_j,\dot{Y}_j$, $j=1,\ldots,m$, together with their brackets of all orders, span the tangent space at any point m of M. Consequently, Δ_M is hypoelliptic.

Definition 6.5. Let $\lambda_t(dm)$ be **heat kernel** measure on M given by

$$\lambda_t(dm)=(\pi_*\rho_t)(dm). \tag{6.8}$$

The interpretation of λ_t as a heat kernel measure will be discussed in Subsection 6.4 below. The following lemma follows readily from the definition of λ_t in (6.8) and the fact that π is a quotient map for the complex differential structures of G and M.

Lemma 6.6. *The pullback map $\pi^*:u\mapsto\pi^*u=u\circ\pi$ is unitary from $L^2(M,\lambda_t)$ onto the subspace $L^2_K(G,\rho_t)\subset L^2(G,\rho_t)$ of left K-invariant functions and from $\mathcal{H}L^2(M,\lambda_t)$ onto $\mathcal{H}L^2_K(G,\rho_t)$, the space of holomorphic left K-invariant functions in $L^2(G,\rho_t)$. In particular, for $u\in\mathcal{H}L^2(M,\lambda_t)$, we have*

$$\int_M|u(m)|^2\lambda_t(dm)=\int_G|\pi^*u(g)|^2\rho_t(dg). \tag{6.9}$$

Definition 6.7 (G-space Taylor map). For $u\in\mathcal{H}(M)$, define $\hat{u}\in T'$ by; $\langle\hat{u},1\rangle=u\left(Ke\right)$ and, for all $n\in\mathbb{N}$,

$$\langle\hat{u},\xi_1\otimes\cdots\otimes\xi_n\rangle=(\dot{\xi}_1\cdots\dot{\xi}_n u)(Ke) \text{ for all } \xi_j\in\mathfrak{g}. \tag{6.10}$$

The map $\mathcal{H}(M)\ni u\mapsto\hat{u}\in T'$ is called the Taylor map on M viewed as a G-space.

The following corollary of Theorem 6.2 can be interpreted in terms of a Taylor map defined for holomorphic functions on the homogeneous space M.

Corollary 6.8 (The quotient theorem). *For all $t>0$, the Taylor map $\mathcal{H}(M)\ni u\mapsto\hat{u}\in T'$ restricts to a unitary map from $\mathcal{H}L^2(M,\lambda_t)$ onto $(J+\mathfrak{k}T)_t^0$.*

Proof. If $f=u\circ\pi$, then $\hat{u}=\hat{f}$ as elements of T'. Moreover, the map $u\to u\circ\pi$ is unitary from $\mathcal{H}L^2(M,\lambda_t)$ onto the space of left invariant functions in $\mathcal{H}L^2(G,\rho_t)$ as already noted in connection with Equation (6.9). The Corollary is therefore a restatement of Theorem 6.2 for the left invariant case. □

6.3. **Normal subgroups.** In this subsection, we are going to specialize the results of subsection 6.2 to the case wherein K is a normal subgroup of G and $\mathfrak{k}$ is an ideal in $\mathfrak{g}$. Recall that a connected subgroup K of G is normal if and only if $\mathfrak{k}=\mathrm{Lie}\left(K\right)$ is an ideal in $\mathfrak{g}$.

Suppose that the closed connected complex subgroup K is normal and $\mathfrak{k} = \mathrm{Lie}\,(K)$ is the associated Lie ideal inside of $\mathfrak{g}$. In this case, $M = K \setminus G$ is itself a Lie group, $\pi : G \to M$ is a surjective Lie homomorphism, and the projection map,

$$\psi : \mathfrak{g} \to \mathfrak{m} := \mathfrak{g}/\mathfrak{k} \cong T_{eK}M = \mathrm{Lie}\,(M) \tag{6.11}$$

is a Lie algebra homomorphism. At the global level, $M = K\backslash G$ is now a simply connected complex Lie group (see, e.g., [7, I. Chap. 1, Theorem 4.8]). Since K is normal, left and right cosets coincide and therefore M admits a smooth left and right action of G.

Lemma 6.9. *To each $A \in \mathfrak{g}$, the vector field $\dot{A} \in \mathrm{Vect}\,(M)$ (see Notation 6.3) is invariant under the left action of G on M. In particular, $\dot{A}$ is a left invariant vector field on M. Moreover, $\dot{A} = 0$ iff $A \in \mathfrak{k}$.*

Proof. For $g \in G$ and $m \in M$, we have

$$\dot{A}\,(gm) = \frac{d}{dt}|_0 gme^{tA} = \frac{d}{dt}|_0 L_g me^{tA} = L_{g*}\dot{A}\,(m)\,.$$

Since $\dot{A}$ is left invariant, it follows that $\dot{A} \equiv 0$ iff $\dot{A}\,(eK) = 0$. Hence the last assertion follows from (3) in Lemma 6.4. $\square$

As we did for $\mathfrak{g}$, we associate to $\mathfrak{m}$ the tensor algebra $T_{\mathfrak{m}}$ over $\mathfrak{m}$ and the two-sided ideal $J_{\mathfrak{m}}$ in $T_{\mathfrak{m}}$ which is generated by the elements

$$\left\{\bar{\xi} \otimes \bar{\eta} - \bar{\eta} \otimes \bar{\xi} - \left[\bar{\xi}, \bar{\eta}\right]_{\mathfrak{m}} ; \bar{\xi}, \bar{\eta} \in \mathfrak{m}\right\}.$$

By universality, the projection map ψ in Eq. (6.11) extends to a surjective homomorphism, $\phi : T \to T_{\mathfrak{m}}$. (We continue to let T and J be the tensor algebra over $\mathfrak{g}$ and the two-sided ideal in T generated by the element in Eq. (2.5).)

Lemma 6.10. *Continuing the notation introduced above, we have*

$$J + T\mathfrak{k}T = J + T\mathfrak{k} \tag{6.12}$$

and

$$\phi^*(J^0_{\mathfrak{m}}) = (J + T\mathfrak{k}T)^0 = (J + T\mathfrak{k})^0. \tag{6.13}$$

Proof. Since $\mathfrak{k}$ is a two-sided ideal in $\mathfrak{g}$, modulo J, we may commute tensors with $\mathfrak{k}$ up to brackets lying in $\mathfrak{k}$. This observation shows that $T\mathfrak{k}T \subset J + T\mathfrak{k}$, which suffices to prove Eq. (6.12). So to complete the proof, we must verify the first equality in Eq. (6.13).

We begin with the claim that

$$\ker \phi = T\mathfrak{k}T. \tag{6.14}$$

Indeed, the kernel of ϕ is a two-sided ideal in T which contains $\mathfrak{k}$ and therefore contains the two-sided ideal $T\mathfrak{k}T$, i.e., $T\mathfrak{k}T \subset \ker \phi$. To prove the opposite inclusion, let $\mathfrak{v}$ be a complementary subspace to $\mathfrak{k}$ in $\mathfrak{g}$. Then ψ restricted to $\mathfrak{v}$ is an isomorphism of $\mathfrak{v}$ onto $\mathfrak{m}$

and therefore $\phi|_{\mathfrak{v}^{\otimes n}} : \mathfrak{v}^{\otimes n} \to \mathfrak{m}^{\otimes n}$ is also an isomorphism. Writing $(T\mathfrak{k}T)_n = (T\mathfrak{k}T) \cap \mathfrak{g}^{\otimes n}$ we have the direct sum decomposition $\mathfrak{g}^{\otimes n} = \mathfrak{v}^{\otimes n} + (T\mathfrak{k}T)_n$. Hence $\ker \phi|\mathfrak{g}^{\otimes \mathfrak{n}} = (T\mathfrak{k}T)_n$. Since ϕ takes n-tensors to n-tensors, (6.14) follows.

Now

$$\phi\left(\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta]\right) = \psi(\xi) \otimes \psi(\eta) - \psi(\eta) \otimes \psi(\xi) - [\psi(\xi), \psi(\eta)],$$

and therefore ϕ takes the generators of J onto the generators of $J_{\mathfrak{m}}$. It follows that

$$\phi(J + T\mathfrak{k}T) = \phi(J) = J_{\mathfrak{m}}. \tag{6.15}$$

Therefore, $\phi^*((J_{\mathfrak{m}})^0) \subset (J + T\mathfrak{k}T)^0$. Furthermore, if $\gamma \in T'$ annihilates $T\mathfrak{k}T$, then we may define $\alpha \in T'_{\mathfrak{m}}$ by $\langle \alpha, \phi(\beta) \rangle = \langle \gamma, \beta \rangle$. This α is well defined because $\phi : T \to T_{\mathfrak{m}}$ is surjective and (6.14) holds. Moreover, in view of (6.15), α annihilates $J_{\mathfrak{m}}$ if γ annihilates J. This establishes the first equality in (6.13). □

Definition 6.11. Given a Hermitian form q on $\mathfrak{g}^*$, let $q_{\mathfrak{m}}$ denote the Hermitian form on $\mathfrak{m}^*$ given by

$$q_{\mathfrak{m}}(\alpha) = \sum_1^m \left|\langle \alpha, \xi_i \rangle\right|^2, \quad \xi_i = \psi(X_i),$$

where $(X_i)_1^m$ is an orthonormal basis (over $\mathbb{C}$) of the Hörmander space $H \subset \mathfrak{g}$ of q. As in Eqs. (2.10) and (2.11), $q_{\mathfrak{m}}$ induces degenerate Hermitian inner products $(\cdot,\cdot)_{(q_{\mathfrak{m}})_k}$ on $(\mathfrak{m}^*)^{\otimes k}$ for all $k \in \mathbb{N}$ and a possibly degenerate semi-norm $\|\cdot\|_{q_{\mathfrak{m}},t}$ on $T'_{\mathfrak{m}}$ for all $t > 0$.

Lemma 6.12. *The map*

$$\phi^* : (J_{\mathfrak{m}})^0_t \to (J + T\mathfrak{k})^0_t$$

is unitary.

Proof. The form $q_{\mathfrak{m}}$ extends as usual to a Hermitian form on the dual $T'_{\mathfrak{m}}$ of the tensor algebra $T_{\mathfrak{m}}$. Although the vectors $\xi_i = \psi(X_i)$, $i = 1, \ldots, m$, are not necessarily orthonormal, for any $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_k \in (\mathfrak{m}^*)^{\otimes k}$, we still have

$$q_{\mathfrak{m}}(\alpha) = \sum_{(i_1,\ldots,i_k)\in\{1,\ldots,m\}^k} \prod_{j=1}^k |\langle \alpha_j, \xi_{i_j} \rangle|^2,$$

and thus

$$\begin{aligned} q(\phi^*(\alpha)) &= \sum_{(i_1,\ldots,i_k)\in\{1,\ldots,m\}^k} \prod_{j=1}^k |\langle \phi^*(\alpha_j), X_{i_j} \rangle|^2 \\ &= \sum_{(i_1,\ldots,i_k)\in\{1,\ldots,m\}^k} \prod_{j=1}^k |\langle \alpha_j, \phi(X_{i_j}) \rangle|^2 \\ &= \sum_{(i_1,\ldots,i_k)\in\{1,\ldots,m\}^k} \prod_{j=1}^k |\langle \alpha_j, \xi_{i_j} \rangle|^2 = q_{\mathfrak{m}}(\alpha). \end{aligned}$$

A similar computation shows that the inner product between two decomposable tensors is preserved by ϕ^*. From this, it follows readily that ϕ^* is an isometry from $(J_{\mathfrak{m}})_t^0$ into $(J+T\mathfrak{k})_t^0$. The surjectivity of ϕ^* follows readily from (6.13). $\square$

The elementary Lemmas 6.6, 6.1 and 6.12 yield the following analogue of Corollary 5.5. We include the real case which can be treated in exactly the same way as the complex case.

PROPOSITION 6.13. *Suppose that $\mathfrak{g}$ is a real (resp. complex) finite dimensional Lie algebra. Let q^i, $i=1,2$, be non-negative quadratic (resp. Hermitian) forms on $\mathfrak{g}^*$. Assume that these two forms satisfy Hörmander's condition. Let $s>0$ and $t>0$ and suppose that, for some constant c,*

$$\|\alpha\|_{q^1,s} \le c\|\alpha\|_{q^2,t} \text{ for all } \alpha \in J^0_{q^2,t} \subset T'. \tag{6.16}$$

Let $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$ which is also an ideal. Let $\mathfrak{m}=\mathfrak{g}/\mathfrak{k}$ be the quotient Lie algebra. Let $q^1_{\mathfrak{m}}$ and $q^2_{\mathfrak{m}}$ be the corresponding forms on $\mathfrak{m}^*$ as in Definition 6.11. Then

$$\|\alpha\|_{q^1_{\mathfrak{m}},s} \le c\|\alpha\|_{q^2_{\mathfrak{m}},t} \text{ for all } \alpha \in J^0_{q^2_{\mathfrak{m}},t} \subset T'_{\mathfrak{m}}. \tag{6.17}$$

In particular, if the q^2 family controls the q^1 family for $\mathfrak{g}$, then the $q^2_{\mathfrak{m}}$ family controls the $q^1_{\mathfrak{m}}$ family for $\mathfrak{m}$.

6.3.1. *Application to quotient groups.* The previous machinery can be used to give an alternative proof of the unitarity of the Taylor map for quotient groups $M=G/K$ when the result is known on G. For instance, in [6], we gave a proof of the unitarity of the Taylor map for stratified nilpotent groups. Since any simply connected nilpotent group is the quotient of a stratified nilpotent group by a connected normal subgroup, this yields the unitarity of the Taylor map on any complex simply connected nilpotent group as an immediate corollary of the stratified case (in [6], this was done by an ad hoc argument).

More generally, let G be a closed complex simply connected Lie group with Lie algebra $\mathfrak{g}$. Let K be a connected, closed, normal Lie subgroup of G with Lie algebra $\mathfrak{k}\subset\mathfrak{g}$. Let $M=G/K$ be the associated quotient group and let $\mathfrak{m}=\mathfrak{g}/\mathfrak{k}$ be its Lie algebra. Let q be a Hermitian form on $\mathfrak{g}^*$ satisfying Hörmander's condition. Let $q_{\mathfrak{m}}$ be the Hermitian form on $\mathfrak{m}^*$ given by Definition 6.11. Let ρ_t and λ_t be the corresponding heat kernel measures on G and M, respectively. If we know that

$$\mathcal{H}L^2(G,\rho_t) \ni f \mapsto \hat{f} \in J_t^0(\mathfrak{g})$$

is an isometry (resp. a unitary isomorphism) then

$$\mathcal{H}L^2(M,\lambda_t) \ni w \mapsto \hat{w} \in J_t^0(\mathfrak{m}) \tag{6.18}$$

is also an isometry (resp. a unitary isomorphism) of Hilbert spaces.

Remark 6.14. Because K is normal, one may easily check that Δ_M in Eq. (6.6) is the Laplacian associated to $q_{\mathfrak{m}}$ and that λ_t in Eq. (6.8) is the associated heat kernel measure on the group M. See the next section for the case where K is not normal.

6.4. **Intrinsic interpretation of λ_t.** Observe that although the "heat kernel measure" λ_t in Definition 6.5 is well defined by Eq. (6.8), its interpretation in terms of a heat semigroup requires some care because it is not clear, in general, what reference measure on M is the appropriate one in discussing the heat kernel density. However, on G, we may regard the heat semigroup $e^{t\Delta/4}$ as a semigroup acting on $L^\infty(G)$ (obviously, this is not a strongly continuous semigroup). Namely, (with some abuse of notation),

$$e^{t\Delta/4}f(x) = \int_G f(xy)\rho_t(dy) = \int_G f(y)h_t(x,y)dy, \quad f \in L^\infty(G).$$

The first equality can be taken as the definition of $e^{t\Delta/4}$ on $L^\infty(G)$. It reflects the fact that $\Delta/4$ is the generator of a symmetric Markov semigroup acting originally on $L^2(G)$ and which commutes with left translation, i.e., $[e^{t\Delta/4}f](zx) = [e^{t\Delta/4}f_z](x)$ where $f_z : x \mapsto f(zx)$. This property is of course inherited from Δ. As ρ_t admits a continuous density with respect to Haar measure, it implies that $e^{t\Delta/4}L^\infty(G) \subset \mathcal{C}_b(G)$, where $\mathcal{C}_b(G)$ is the space of bounded continuous function on G. Because Δ is subelliptic, the heat kernel $h_t(x,y)$ is a positive smooth symmetric function of (x,y) and is related to the density $z \mapsto \rho_t(z)$ of the measure ρ_t by $h_t(x,y) = \rho_t(x^{-1}y)m(x)$, where m is the modular function on G (see, e.g., [5]).

Obviously, the semigroup $e^{t\Delta/4}$ leaves invariant the subspace $L^\infty_K(G)$ of bounded measurable K-left invariant functions on G. Hence, $e^{t\Delta/4}$ induces a semigroup on $L^\infty(M)$ (again, not a strongly continuous semigroup) defined by the formula

$$(H_t\phi) \circ \pi = e^{t\Delta/4}(\phi \circ \pi) \text{ for all } \phi \in L^\infty(M). \tag{6.19}$$

In particular, the measure λ_t in Definition 6.5 is given for any Borel set $A \subset M$, by

$$\lambda_t(A) = \rho_t(\pi^{-1}(A)) = H_t\mathbf{1}_A(o), \quad o = Ke. \tag{6.20}$$

Further, making use of Eq. (6.7) and of the Gaussian bounds satisfied by $x \mapsto \rho_t(x)$ and its derivatives on G (see, e.g., [5, 18]), one checks that

$$\frac{d}{dt}H_t\phi = \frac{1}{4}H_t(\Delta_M\phi), \quad \phi \in \mathcal{C}_c^\infty(M).$$

Write $\mu(\phi)$ for $\int \phi d\mu$ when μ is a probability measure and observe that Eq. (6.20) implies $\lambda_t(\phi) = (H_t\varphi)(o)$ for all bounded and measurable $\varphi : M \to \mathbb{R}$. Hence it follows that, for any $\phi \in \mathcal{C}_c^\infty(M)$, the function $t \mapsto \lambda_t(\phi) = (H_t\varphi)(o)$ is continuously differentiable and satisfies

$$\frac{d}{dt}\lambda_t(\phi) = \frac{1}{4}\lambda_t(\Delta_M\phi), \quad \lim_{t\to 0}\lambda_t(\phi) = \phi(o).$$

The following theorem shows that these properties yield an alternative intrinsic definition of the measure λ_t. See [3, Theorem 2.6] for a similar theorem in the setting of Riemannian geometry.

Theorem 6.15. *The family of probability measures* $\{\lambda_t; t \in (0,\infty)\}$ *introduced in* Definition 6.5 *is the unique family of probability measures on* M *such that, for all* $\phi \in C_c^\infty(M)$, *the function* $t \to \lambda_t(\phi) := \int_M \phi d\lambda_t$ *is continuously differentiable and satisfies*

$$\frac{d}{dt}\lambda_t(\phi) = \frac{1}{4}\lambda_t(\Delta_M \phi) \quad \text{and} \quad \lim_{t\downarrow 0}\lambda_t(\phi) = \phi(o). \tag{6.21}$$

Proof. As noted above, $\{\lambda_t\}$ in Definition 6.5 satisfies Eq. (6.21). Hence we are left to prove the uniqueness assertion of the theorem.

We need to introduce some notation. For $f \in \mathcal{C}_c(G)$, set

$$f_K(g) = \int_K f(kg)dk,$$

where dk denotes the right invariant measure on K. As $f_K \in \mathcal{C}(G)$ is invariant under the left action of K on G,

$$Kg \mapsto f_K^\#(Kg) := f_K(g)$$

is a well defined function on M. Observe that $f_K^\# \in \mathcal{C}_c(M)$ and is in $\mathcal{C}_c^\infty(M)$ whenever f is in $\mathcal{C}_c^\infty(G)$. Moreover, the image of $\mathcal{C}_c(G)$ by the map $f \mapsto f_K^\#$ is $\mathcal{C}_c(M)$. See [2, chap VII,§2, Prop. 2]. For any probability measure ν on M, define the measure $\widetilde{\nu}$ on G (using the Riesz theorem) by requiring

$$\tilde{\nu}(f) = \nu(f_K^\#) \text{ for all } f \in \mathcal{C}_c(G). \tag{6.22}$$

Note that the map $\nu \mapsto \widetilde{\nu}$ is injective since $\mathcal{C}_c(G) \ni f \mapsto f_K^\# \in \mathcal{C}_c(M)$ is surjective.

Remark 6.16. Given a measurable section $s : M \to G$, the map $\nu \mapsto \tilde{\nu}$ defined above can be described as follows. Observe that $\Theta : K \times M \to G$, $\Theta(k,m) := ks(m)$, is a measure theoretic isomorphism, and set $\tilde{\nu} := \Theta_*[\alpha \otimes \nu]$, where α is the right invariant Haar measure on K. This map is easily seen to be injective and one can check that the measures ν, $\tilde{\nu}$ are related by (6.22).

If $f \in \mathcal{C}_c^\infty(G)$ and $g \in G$, then

$$\begin{aligned}
(\Delta f)_K^\#(Kg) &= \int_K (\Delta f)(kg)\,dk = \int_K \Delta_g[f(kg)]\,dk = \Delta_g \int_K f(kg)\,dk \\
&= (\Delta f_K)(g) = \Delta\left(f_K^\# \circ \pi\right)(g) = \left(\Delta_M f_K^\# \circ \pi\right)(g) \\
&= \left(\Delta_M f_K^\#\right)(Kg).
\end{aligned}$$

This computation gives the key identities

$$\Delta_M f_K^\# = (\Delta f)_K^\#, \quad \text{for all } f \in \mathcal{C}_c^\infty(G). \tag{6.23}$$

Let $\{\nu_t; t \in (0,\infty)\}$ be any family of probability measures satisfying (6.21) (in particular, this includes $\widetilde{\lambda}_t$). Using the definitions and Eq. (6.23), we have

$$\frac{d}{dt}\tilde{\nu}_t(f) = \frac{1}{4}\tilde{\nu}_t(\Delta f), \text{ and } \lim_{t\downarrow 0}\tilde{\nu}_t(f) = \int_K f dk \quad \text{for all } f \in \mathcal{C}_c^\infty(G). \tag{6.24}$$

It follows that the measure-valued map $t \mapsto \widetilde{\nu}_t$ can be viewed as a solution of the heat equation on $(0,\infty)\times G$ in the sense of distributions. That is, for any $\eta \in \mathcal{C}_c^\infty((0,\infty)\times G)$,

$$\int_0^\infty \int_G \left(\frac{d}{dt} - \Delta\right)\eta(t,g)\, \widetilde{\nu}_t(dg)dt = 0.$$

As $d/dt - \Delta$ is hypoelliptic, it follows that $\tilde{\nu}_t(dg) = w(t,g)dg$, where w is a non-negative classical solution of heat equation on $(0,\infty)\times G$. However, it is known (see [1]) that the positive solutions u of the heat equation on $(0,T)\times G$ are exactly the functions of the form

$$(t,x) \mapsto u(t,x) = \int h_t(x,y)\omega(dy), \tag{6.25}$$

where ω is a Radon measure on G such that $\int_G e^{-\alpha d_G(e,g)^2}\omega(dg) < \infty$ for some $\alpha > 0$ large enough (for any $\alpha > 0$ if $T = \infty$). Here $d_G(e,x)$ is the sub-Riemannian distance on G. Further, for any such solution u and any $f \in \mathcal{C}_c(G)$, we have $\lim_{t\to 0}\int_G u(t,x)f(x)dx = \omega(f)$.

This yields a very strong uniqueness result for the positive Cauchy problem on G. In particular, it implies that $\widetilde{\nu}_t = \widetilde{\lambda}_t$ since both families of measures can be identified with the unique positive solution with initial data ω given by $\omega(f) = \int_K f(k)dk$, $f \in \mathcal{C}_c(G)$. Thus $\nu_t = \lambda_t$ as desired since the map $\nu \mapsto \widetilde{\nu}$ is injective. $\square$

Remark 6.17. The description (6.25) of the positive solutions of the heat equation on G used above follows from the known uniform local Harnack inequality and the Gaussian heat kernel bounds on G. See [1, Theorem 4.2 and Remark 2] where this representation is proved in a very general context. The relevant heat kernel bounds on G are given in [5, 18]. The harmonic function h appearing in [1, Theorem 4.2] vanishes because of the validity of the uniform local Harnack inequality.

It is worth noting that, by Fubini's theorem, for any non-negative $f \in \mathcal{C}_c(G)$,

$$\begin{aligned}\widetilde{\lambda}_t(f) &= \int_G\int_K f(kg)h_t(e,g)dkdg = \int_G\int_K f(g)h_t(k,g)dkdg \\ &= \int_G f(g)\left\{\int_K h_t(g,k)dk\right\}dg.\end{aligned} \tag{6.26}$$

Hence, the proof of Theorem 6.15 yields the interesting fact that, for any closed subgroup K of G and $(t,g) \in (0,\infty)\times G$,

$$u(t,g) = \int_K h_t(g,k)dk < \infty$$

and $(t,g) \mapsto u(t,g)$ is in $\mathcal{C}^\infty((0,\infty)\times G)$. It seems rather difficult to prove this by direct inspection, even if one uses the known Gaussian bounds on the heat kernel. In terms of $x \mapsto \rho_t(x)$, the function u is given by

$$u(t,g) = \int_K \rho_t(k^{-1}g) m_G(k) dk,$$

where $m = m_G$ is the modular function of G.

REMARK 6.18. Since Δ_M satisfies Hörmander's condition, the measure $\lambda_t(dm)$ admits a smooth positive density with respect to any natural reference measure $\nu(dm)$ associated with the smooth structure of the manifold M, but it is not clear, in general, whether or not there is such a reference measure ν with the property that $\nu(H_t f) = \nu(f)$ for all $t > 0$ and $f \in \mathcal{C}_c(M)$. If M admits a G-invariant measure, call it dm, then H_t is actually self-adjoint on $L^2(M, dm)$ and $\int_M H_t f dm = \int_M f dm$ for all $t > 0$ and $f \in \mathcal{C}_c(M)$.

A well-known necessary and sufficient condition for the existence of a G invariant measure dm on M is that the modular function of G and that of K coincide on K. Under this condition, the decomposition formula

$$\int_G f(g) d_G g = \int_M \int_K f(k g_m) d_K k dm$$

holds for any continuous compactly supported f on G. Here, $g_m \in G$ is any representative of $m \in M$, $\int_K f(kg_m) d_K k$ is understood as a function on M, and $d_G g$ (resp. $d_K k$) denotes an appropriately fixed right invariant measure on G (resp. K). The abuse of notation used in this formula is standard. Using this formula, it is possible to show that the vector fields $\pi_*(\widetilde{X}_i)$, $i = 1, \dots, m$, are skew-adjoint on $L^2(M, dm)$. Thus H_t is self-adjoint on $L^2(M, dm)$ as stated above. Moreover, writing $\lambda_t(dm) = \lambda_t(m) dm$ on M for $m = Kg$, we have

$$\lambda_t(m) = \int_K \rho_t(kg) d_K k.$$

7. THE TAYLOR MAP ON HOMOGENOUS SPACES: TWO EXAMPLES

The aim of this section is to illustrate Corollary 6.8 by two very concrete examples. As far as we know, Corollary 6.8 is the first result that provides the unitarity of the Taylor map on complex manifolds that do not carry a group structure.

7.1. **The Grushin complex 2-space.**

The Grushin plane is $\mathbb{R}^2$ equipped with the sub-Riemannian geometry associated with the sub-Laplacian $\partial_x^2 + x^2 \partial_y^2$. In some sense, it is the simplest sub-Riemannian object although it is best understood by observing that it can be viewed as the quotient of the Heisenberg group by a non central one dimensional subgroup. In this section, we consider the complex version of this object.

NOTATION 7.1. The complex Heisenberg group is $H_3^{\mathbb{C}} = \mathbb{C}^3$ with the group law

$$(z_1, z_2, z_3) \cdot (z_1', z_2', z_3') = (z_1 + z_1', z_2 + z_2', z_3 + z_3' + (1/2)(z_1 z_2' - z_2 z_1'))\,.$$

Let us observe here that if $z_j = x_j + iy_j$ then

$$z_1 z_2' - z_2 z_1' = [x_1 x_2' - x_2 x_1' - (y_1 y_2' - y_2 y_1')] + i[x_1 y_2' - y_2 x_1' + y_1 x_2' - x_2 y_1'].$$

Consider the elements X_i, Y_i, $i = 1, 2$, of the Lie algebra $\mathfrak{h}_3^{\mathbb{C}}$ given by

$$X_1 = (1, 0, 0), \quad X_2 = (0, 1, 0), \quad Y_1 = (i, 0, 0), \quad Y_2 = (0, i, 0).$$

(In Section 4, we considered the complex Heisenberg Lie algebra $\mathfrak{h}_3^{\mathbb{C}}$ with generators X, Y and $[X, Y] = Z$. To match this with the present notation, set $X = X_1, Y = X_2, Z = [X_1, X_2]$).

Let q be the Hermitian form on $(\mathfrak{h}_3^{\mathbb{C}})^*$ given by

$$q(\alpha) = |\langle \alpha, X_1 \rangle|^2 + |\langle \alpha, X_2 \rangle|^2.$$

The left-invariant vector fields associated to X_1, X_2, Y_1, Y_2 are

$$\begin{aligned}
\widetilde{X}_1 &= \partial/\partial x_1 - (x_2/2)\partial/\partial x_3 - (y_2/2)\partial/\partial y_3 \\
\widetilde{X}_2 &= \partial/\partial x_2 + (x_1/2)\partial/\partial x_3 + (y_1/2)\partial/\partial y_3 \\
\widetilde{Y}_1 &= \partial/\partial y_1 + (y_2/2)\partial/\partial x_3 - (x_2/2)\partial/\partial y_3 \\
\widetilde{Y}_2 &= \partial/\partial y_2 - (y_1/2)\partial/\partial x_3 + (x_1/2)\partial/\partial y_3.
\end{aligned}$$

The sub-Laplacian associated with q is

$$\Delta = \widetilde{X}_1^2 + \widetilde{X}_2^2 + \widetilde{Y}_1^2 + \widetilde{Y}_2^2. \tag{7.1}$$

Define the kernel ρ_t on $H_3^{\mathbb{C}}$ by the identity $e^{t\Delta/4} = *\rho_t$.

The complex line $K = \mathbb{C} = \{(z_1, z_2, z_3); z_2 = z_3 = 0\} \subset H_3^{\mathbb{C}}$ is a complex closed Lie subgroup of $H_3^{\mathbb{C}}$ with quotient space $M = K \backslash H_3^{\mathbb{C}} = \mathbb{C}^2$. Let $S := \{(0, z_2, z_3)\,; z_2, z_3 \in \mathbb{C}\} \cong \mathbb{C}^2$. For $g = (z_1, z_2, z_3) \in G$, we have

$$Kg = \{(z, 0, 0)\,(z_1, z_2, z_3)\,; z \in \mathbb{C}\} = \{(z_1 + z, z_2, z_3 + zz_2/2)\,; z \in \mathbb{C}\},$$

and

$$Kg \cap S = \{(z_1 + z, z_2, z_3 + zz_2/2)\,; z = -z_1\} = \{(0, z_2, z_3 - z_1 z_2/2)\}.$$

This shows that S is a global section for the left K-action on G. Therefore the maps,

$$\begin{aligned}
\mathbb{C}^2 \ni (z_2, z_3) &\mapsto (0, z_2, z_3) \in S \text{ and} \\
S \ni (0, z_2, z_3) &\mapsto K(0, z_2, z_3) \in K \setminus G,
\end{aligned}$$

are holomorphic diffeomorphisms. Hence we may now view the natural projection map $\pi : G \to K \setminus G$ as a map from $G \to \mathbb{C}^2$ given by

$$\pi(g) = \left(z_2, z_3 - \frac{1}{2} z_1 z_2 \right) \text{ for all } g = (z_1, z_2, z_3) \in G.$$

Since G acts on the right on $K \setminus G$, there is an induced right action of G on $M = \mathbb{C}^2$ which we now compute. If $(w_2, w_3) = \pi(0, w_2, w_3)$ and $g = (z_1, z_2, z_3) \in G$, then

$$\begin{aligned}(w_2, w_3) \cdot g &:= \pi((0, w_2, w_3) g) = \pi((0, w_2, w_3)(z_1, z_2, z_3)) \\ &= \pi\left(z_1, z_2 + w_2, z_3 + w_3 - \frac{1}{2} w_2 z_1\right) \\ &= \left(z_2 + w_2, z_3 + w_3 - \frac{1}{2} w_2 z_1 - \frac{1}{2} z_1 (z_2 + w_2)\right) \\ &= \left(z_2 + w_2, z_3 + w_3 - w_2 z_1 - \frac{1}{2} z_1 z_2\right).\end{aligned}$$

Hence, if $A = (a_1, a_2, a_3) \in \mathfrak{g} = \mathbb{C}^3$, then, since the exponential map is the identity in our coordinates, we have

$$\begin{aligned}\dot{A}(w_2, w_3) &= \frac{d}{dt}|_0 (w_2, w_3) \cdot e^{tA} = \frac{d}{dt}|_0 (w_2, w_3) \cdot (tA) \\ &= \frac{d}{dt}|_0 \left(ta_2 + w_2, ta_3 + w_3 - w_2 t a_1 - \frac{1}{2} t^2 a_1 a_2\right) \\ &= (a_2, a_3 - w_2 a_1). \qquad (7.2)\end{aligned}$$

To simplify notation, let us now write

$$w_2 = w = u + iv, \quad w_3 = z = x + iy.$$

Using Eq. (7.2), we find

$$\dot{X}_1(w, z) = (0, -w), \ \dot{X}_2(w, z) = (1, 0), \qquad (7.3)$$

$$\dot{Y}_1(w, z) = (0, -iw) \text{ and } \dot{Y}_2(w, z) = (i, 0). \qquad (7.4)$$

In particular, the action of these vector fields on any holomorphic function f may be written as

$$\dot{X}_1 f = -w \partial_z f, \ \dot{X}_2 f = \partial_w f, \qquad (7.5)$$

$$\dot{Y}_1 f = -iw \partial_z f, \ \dot{Y}_2 f = i \partial_w f, \qquad (7.6)$$

where

$$\partial_w := \frac{1}{2}(\partial/\partial u - i\partial/\partial v), \quad \partial_z := \frac{1}{2}(\partial/\partial x - i\partial/\partial y). \qquad (7.7)$$

7.2. **The heat kernel on the Grushin complex 2-space.** Identifying $\mathbb{C}^2$ with $\mathbb{R}^4$ by $(w, z) = (u, v, x, y)$, the vector fields in (7.2) through (7.4) act on $\mathcal{C}^\infty(\mathbb{R}^4)$ by

$$\dot{X}_1 = -(u\partial/\partial x + v\partial/\partial y), \quad \dot{X}_2 = \partial/\partial u$$

and

$$\dot{Y}_1 = v\partial/\partial x - u\partial/\partial y, \quad \dot{Y}_2 = \partial/\partial v.$$

They form a Hörmander system $\{\dot{X}_1, \dot{X}_2, \dot{Y}_1, \dot{Y}_2\}$ on M and the associated sub-Laplacian

$$\Delta_M = (\partial/\partial u)^2 + (\partial/\partial v)^2 + (u^2+v^2)((\partial/\partial x)^2 + (\partial/\partial y)^2)$$

is in fact elliptic at each point of $\mathbb{C}^2$ except along the complex line $\{w=0\}$ where it is (step two) subelliptic. It is the prototype of a class of subelliptic operators introduced in [11] and often called Grushin operators. The fields $\dot{X}_i, \dot{Y}_i$, $j=1,2$, are divergence free on $M=\mathbb{C}^2$ (equipped with Lebesgue measure) and $e^{t\Delta_M/4}$ is a semigroup of self-adjoint operators on $L^2(\mathbb{C}^2)$. The associated heat kernel measure λ_t admits a density, the "heat kernel" on M based at $(0,0)=Ke$, and, abusing notation, we write $\lambda_t(d\xi)=\lambda_t(\xi)d\xi$, $\xi=(w,z)\in M$. This heat kernel is studied in [15, 14, 17]. It satisfies the two-sided heat kernel estimates

$$\frac{c_1}{V(\sqrt{t})}\exp\left(-C_1\frac{\delta(\xi)^2}{t}\right) \le \lambda_t(\xi) \le \frac{C_2}{V(\sqrt{t})}\exp\left(-c_2\frac{\delta(\xi)^2}{t}\right), \tag{7.8}$$

where $\delta(\xi)$ is the subelliptic distance between the origin and $\xi=(w,z)$ associated with the Hörmander system of vector fields $\{\dot{X}_1, \dot{X}_2, \dot{Y}_1, \dot{Y}_2\}$. The volume function $V(r)$ is the Lebesgue volume of the subelliptic ball $\{\xi\in M; \delta(\xi)<r\}$ around the origin. Furthermore, it is not hard to estimate $\delta(\xi)$ and $V(r)$. Namely, there are constants $c, C\in(0,\infty)$ such that, for $\xi=(w,z)=(u+iv, x+iy)\in M$, we have

$$c(|u|+|v|+\sqrt{|x|}+\sqrt{y|}) \le \delta(\xi) \le C(|u|+|v|+\sqrt{|x|}+\sqrt{|y|}).$$

This, in turn, implies that

$$V(r)\simeq r^6, \quad r>0.$$

This means that the condition that a holomorphic function f be in $L^2(M,\lambda_t)$ imposes quite different growth conditions in the w direction and in the z direction.

7.3. **Taylor coefficients and the unitary Taylor map.** We now translate our main result concerning the Taylor map on homogeneous spaces to the present context.

NOTATION 7.2. For $k\in\mathbb{N}$, $1\le j\le k$, and $m,n\in\mathbb{N}_0^k$, let

$$m! := \prod_{j=1}^k m_j!, \quad |m|_j := \sum_{l=1}^j m_l, \quad |m| = |m|_k \quad \text{and}$$

$$\Omega(m,n) = m!\cdot\binom{|n|_1}{m_1}\cdot\binom{|n|_2-|m|_1}{m_2}\cdot\binom{|n|_3-|m|_2}{m_3}\cdots\binom{|n|_k-|m|_{k-1}}{m_k},$$

where we use the usual convention that $0!=1$ and $\binom{p}{q}=0$ if $p<q$. We may also write

$$\Omega(m,n) := \prod_{j=1}^k P\left(m_j, |n|_j - |m|_{j-1}\right),$$

where $|m|_0 := 0$ and

$$P(m,n) := 1_{\{m\le n\}}\frac{n!}{(n-m)!},$$

where $1_{\{m\le n\}}$ is the function which is one if $m \le n$ and zero otherwise.

EXAMPLE 7.3. For $m=(3,2,0)$ and $n=(4,1,1)$, we have

$$\Omega(m,n)=(3!)(2!)\binom{4}{3}\binom{2}{2}\binom{1}{0}=48.$$

NOTATION 7.4. For $k\in\mathbb{N}$ and $m,n\in\mathbb{N}_0^k$, define $m\preceq n$ if and only if $|m|_j\le|n|_j$ for $1\le j\le k$. (Then $m\preceq n$ if and only if $\Omega(m,n)\ne 0$.)

PROPOSITION 7.5. *Let $f(w,z)$ be a holomorphic function on $\mathbb{C}^2$. For $k\in\mathbb{N}$ and $m,n\in\mathbb{N}_0^k$, we have*

$$\left(\dot{X}_2^{n_1}\left(-\dot{X}_1\right)^{m_1}\cdots\dot{X}_2^{n_k}\left(-\dot{X}_1\right)^{m_k}f\right)(0,0)=\Omega(m,n)\left(\partial_w^{|n|-|m|}\partial_z^{|m|}f\right)(0,0). \tag{7.9}$$

PROOF. The proof is by induction on k. We start with the identity

$$\begin{aligned}\dot{X}_2^{n}\left(-\dot{X}_1\right)^{m}f&=\partial_w^n(w\partial_z)^m f=\partial_w^n(w^m\partial_z^m)f\\&=\sum_{k=0}^n\binom{n}{k}\partial_w^k w^m\cdot\partial_w^{n-k}\partial_z^m f\\&=\sum_{k=0}^n\binom{n}{k}1_{\{k\le m\}}\frac{m!}{(m-k)!}w^{m-k}\cdot\partial_w^{n-k}\partial_z^m f.\end{aligned}$$

When evaluating this expression at $w=z=0$, we must have $k=m\le n$. Therefore,

$$\left(\dot{X}_2^{n}\left(-\dot{X}_1\right)^{m}f\right)(0,0)=1_{\{m\le n\}}\binom{n}{m}m!\cdot\left(\partial_w^{n-m}\partial_z^m f\right)(0,0), \tag{7.10}$$

which proves Eq. (7.9) for $k=1$. For the induction step, replace f in Eq. (7.9) by

$$\dot{X}_2^{n_{k+1}}\left(-\dot{X}_1\right)^{m_{k+1}}f=\partial_w^{n_{k+1}}\left(w^{m_{k+1}}\partial_z^{m_{k+1}}f\right)$$

to find

$$\begin{aligned}&\left(\dot{X}_2^{n_1}\left(-\dot{X}_1\right)^{m_1}\cdots\dot{X}_2^{n_{k+1}}\left(-\dot{X}_1\right)^{m_{k+1}}f\right)(0,0)\\&\quad=\Omega(m,n)\cdot\left(\partial_w^{|n|-|m|}\partial_z^{|m|}\partial_w^{n_{k+1}}\left(w^{m_{k+1}}\partial_z^{m_{k+1}}f\right)\right)(0,0)\\&\quad=\Omega(m,n)\cdot\left(\partial_w^{|n|+n_{k+1}-|m|}\left(w^{m_{k+1}}\partial_z^{|m|+m_{k+1}}f\right)\right)(0,0)\\&\quad=\Omega(m,n)\cdot m_{k+1}!\binom{|n|+n_{k+1}-|m|}{m_{k+1}}\left(\partial_w^{|n|+n_{k+1}-|m|-m_{k+1}}\partial_z^{|m|+m_{k+1}}f\right)(0,0).\end{aligned}$$

which is the desired result. $\square$

Now given $\xi_1,\dots,\xi_N\in\left\{\dot{X}_1,\dot{X}_2\right\}$, in order for $(\xi_1\cdots\xi_N f)(0,0)\ne 0$, we must have $\xi_1=\dot{X}_2$. When $\xi_1=\dot{X}_2$, there is unique $k\in\mathbb{N}$, $n\in\mathbb{N}^k$ and $m\in\mathbb{N}^{k-1}\times\mathbb{N}_0$ such that $|m|+|n|=N$ and

$$\xi_1\cdots\xi_N f(0,0)=\dot{X}_2^{n_1}\dot{X}_1^{m_1}\cdots\dot{X}_2^{n_k}\dot{X}_1^{m_k}f(0,0).$$

Making use of Proposition 7.5 allows us to conclude that

$$\xi_1 \cdots \xi_N f(0,0) = (-1)^{|m|}\, \Omega(m,n) \cdot \left(\partial_w^{|n|-|m|} \partial_z^{|m|} f\right)(0,0). \tag{7.11}$$

Notation 7.6. Set $I(0) = \{((0),(0))\}$ and, for $N \in \mathbb{N}$,

$$I(N) = \{(m,n) \in (\mathbb{N}^{k-1} \times \mathbb{N}_0) \times \mathbb{N}^k \text{ for some } k \in \mathbb{N} \text{ and } |m| + |n| = N,\ m \preceq n\}.$$

According to Eq. (7.11) and the definition of $\Omega(m,n)$ (see Notations 7.2 and 7.4), only those pairs of integer tuples which belong to $I(N)$ for some N will play a role in what follows.

Example 7.7. Here is a listing of $I(N)$ for $1 \le N \le 4$;

$I(1) = \{((0),(1))\}$,

$I(2) = \{((0),(2)),((1),(1))\}$,

$I(3) = \{((0),(3)),((1),(2)),((1,0),(1,1))\}$ and

$I(4) = \{((0),(4)),((1),(3)),((2),(2)),((1,0),(1,2)),((1,0),(2,1)),((1,1),(1,1))\}$.

Corollary 7.8. *Suppose that f is a holomorphic function on $\mathbb{C}^2$ and $\Omega(m,n)$ and $I(N)$ are as in Notations 7.2 and 7.6. Then*

$$\|f\|_t^2 = \|\hat{f}\|_t^2 = \sum_{N=0}^{\infty} \frac{t^N}{N!} \sum_{(m,n)\in I(N)} \Omega^2(m,n) \left|\left(\partial_w^{|n|-|m|} \partial_z^{|m|} f\right)(0,0)\right|^2. \tag{7.12}$$

Proof. Recall from Corollary 6.8 that $\|f\|_t^2 = \|\hat{f}\|_t^2$ where

$$\|\hat{f}\|_t^2 = \sum_{N=0}^{\infty} \frac{t^N}{N!} \sum_{\xi_1,\dots,\xi_N \in \left\{\dot{X}_1, \dot{X}_2\right\}} \left|(\xi_1 \cdots \xi_N f)(0,0)\right|^2. \tag{7.13}$$

However, from Eq. (7.11), we know that

$$\sum_{\xi_1,\dots,\xi_N \in \left\{\dot{X}_1, \dot{X}_2\right\}} \left|(\xi_1 \cdots \xi_N f)(0,0)\right|^2 = \sum_{(m,n)\in I(N)} \Omega^2(m,n) \cdot \left|\left(\partial_w^{|n|-|m|} \partial_z^{|m|} f\right)(0,0)\right|^2. \tag{7.14}$$

Eq. (7.12) now follows from Eqs. (7.12) and (7.14). ☐

We now illustrate Corollary 7.8 by a number of special cases and explicit examples. If $f(w,z) = g(w)$ (the simplest case) then Eq. (7.12) reduces to

$$\|f\|_t^2 = \|\hat{f}\|_t^2 = \sum_{N=0}^{\infty} \frac{t^N}{N!} \left|f^{(N,0)}(0,0)\right|^2 = \sum_{N=0}^{\infty} \frac{t^N}{N!} \left|g^{(N)}(0)\right|^2.$$

Indeed, $f^{(|n|-|m|,|m|)}(0,0) = 0$ unless $|m| = 0$. Further, if $(m,n) \in I(N)$, $|m| = 0$, and $N = |n| + |m| \ge 1$, then $k = 1$, $m = (0)$ and $n = (N)$, and in this case, $\Omega((0),(N)) = 1$. In particular, if $g(w) = e^w$ it follows that $\|f\|_t^2 = e^t < \infty$ for all $t > 0$.

On the other hand, if $f(w,z)=g(z)$, we obtain

$$\|f\|_t^2=\|\hat{f}\|_t^2=\sum_{N=0}^{\infty}\frac{t^{2N}}{(2N)!}\sum_{\substack{(m,n)\in I(2N)\\|m|=|n|}}\Omega^2(m,n)\left|\left(\partial_z^{|m|}g\right)(0)\right|^2. \tag{7.15}$$

Observe that the combinatorial factor $\sum_{\substack{(m,n)\in I(2N)\\|m|=|n|}}\Omega^2(m,n)$ always contains the term corresponding to $m=n=(N)$ for which $\Omega(m,m)=N!$. Thus

$$\|\hat{f}\|_t^2\geq\sum_{N=0}^{\infty}\frac{(N!)^2}{(2N)!}t^{2N}\left|g^{(N)}(0)\right|^2. \tag{7.16}$$

Example 7.9. If we pick $f(w,z)=g(z)=e^z$, then Eq. (7.16) implies that

$$\|\hat{f}\|_t^2\geq\sum_{N=0}^{\infty}\frac{(N!)^2t^{2N}}{(2N)!}$$

will certainly be infinite if $t\geq 2$. In other words, e^z is not $L^2(M,\lambda_t)$ if $t\geq 2$.

The final two examples give exact values for the integrals $\int_M|z|^6\lambda_t(d\xi)$ and $\int_M|w|^2|z|^6\lambda_t(d\xi)$, where $\xi:=(w,z)$.

Example 7.10. When $f(w,z)=g(z)=z^3$,the only non-zero derivative at $(0,0)$ is $(\partial^3 f/\partial^3 z)(0,0)=6$. So according to Eq. (7.15),

$$\int_M(x^2+y^2)^3\lambda_t(d\xi)=\|\hat{f}\|_t^2=\frac{t^6}{6!}\sum_{\substack{(m,n)\in I(6)\\|m|=|n|=3}}\Omega^2(m,n)\cdot 6^2. \tag{7.17}$$

The pairs (m,n), which contribute to the above sum, are

$(k=1):m=n=(3)$,
$(k=2):m=n=(1,2)$, $m=(1,2)$, $n=(2,1)$ and $m=n=(2,1)$ and
$(k=3):m=n=(1,1,1)$.

The corresponding Ω-values are given by

$$\Omega((3),(3))=6,\ \Omega((1,2),(1,2))=\Omega((2,1),(2,1))=2,$$

$$\Omega((1,2),(2,1))=4\text{ and }\Omega((1,1,1),(1,1,1))=1,$$

which combined with Eq. (7.17) gives,

$$\int_M(x^2+y^2)^3\lambda_t(d\xi)=(36+4+4+16+1)\frac{t^6}{6!}\cdot 6^2=\frac{61}{20}t^6.$$

Example 7.11. For $f(w,z)=wz^3$, we have

$$\|\hat{f}\|_t^2=6^2\frac{t^7}{7!}\left(\sum_{\substack{(n,m)\in I(7)\\|m|=3,|n|=4}}\Omega(m,n)^2\right). \tag{7.18}$$

The possible pairs (m,n) and values of $\Omega(m,n)$ are given in Table 1.

TABLE 1. When necessary, an additional 0 should be appended to the end of m so that m and n have the same number of components.

$m \setminus n$	$(1,1,1,1)$	$(1,1,2)$	$(1,2,1)$	$(2,1,1)$	$(1,3)$	$(2,2)$	$(3,1)$	(4)
$(1,1,1)$	1	2	4	8	-	-	-	-
$(1,2)$	-	-	2	4	6	12	18	-
$(2,1)$	-	-	-	2	-	4	12	-
(3)	-	-	-	-	-	-	6	24

Hence

$$\sum_{\substack{(n,m)\in I(7)\\ |m|=3,|n|=4}} \Omega(m,n)^2 = 1385$$

and therefore from Eq. (7.18) we have,

$$\int_M |w|^2 |z|^6 \lambda_t(d\xi) = \|\hat{f}\|_t^2 = \frac{277}{28} t^7.$$

7.4. **A one-dimensional complex G-space.**

NOTATION 7.12. We let $G := \mathbb{C}^\times \ltimes \mathbb{C}$, where we identify G with the affine transformation group on $\mathbb{C}$ by

$$G \ni (a,b) \mapsto (z \mapsto az + b)\,. \tag{7.19}$$

The map in Eq. (7.19) is an isomorphism of groups provided we define multiplication on G by

$$(a,b)\,(a',b') = (aa', ab' + b)\,.$$

The identity in G is $e = (1,0)$ and the inverse to $(a,b) \in G$ is $(a,b)^{-1} = (a^{-1}, -a^{-1}b)$. Let $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2)$ denote the standard linear coordinates on $\mathbb{C}^2$ restricted to G.

For $A = (\alpha, \beta) \in \mathfrak{g} := \mathrm{Lie}\,(G) = \mathbb{C}^2$, the left invariant vector field $\tilde{A}$ on G associated to A is given by

$$\tilde{A}\,(a,b) = \frac{d}{dt}|_0\,(a,b)\,(1 + t\alpha, t\beta) = \frac{d}{dt}|_0\,(a\,(1+t\alpha)\,, at\beta + b) = (a\alpha, a\beta)\,.$$

We may express $\tilde{A}$ in coordinates as

$$\begin{aligned}\tilde{A} &= z_1\,(\alpha,\beta) = \mathrm{Re}\,(\alpha z_1)\,\frac{\partial}{\partial x_1} + \mathrm{Im}\,(\alpha z_1)\,\frac{\partial}{\partial y_1} + \mathrm{Re}\,(\beta z_1)\,\frac{\partial}{\partial x_2} + \mathrm{Im}\,(\beta z_1)\,\frac{\partial}{\partial y_2} \\ &= \alpha z_1 \partial_1 + \overline{\alpha z_1}\bar{\partial}_1 + \beta z_1 \partial_2 + \overline{\beta z_1}\bar{\partial}_2,\end{aligned}$$

where $\partial_i = \partial_{z_i}$, $i = 1,2$ (see (7.7)). In particular, it follows that $\tilde{A} = \alpha z_1 \partial_1 + \beta z_1 \partial_2$ when acting on holomorphic functions.

Let

$$X_1 = (1,0), \quad X_2 = (0,1), \quad Y_1 = (i,0), \quad Y_2 = (0,i) \in \mathfrak{g}, \tag{7.20}$$

and q be the Hermitian form on $\mathfrak{g}^*$ given by

$$q(\alpha) = |\langle \alpha, X_1 \rangle|^2 + |\langle \alpha, X_2 \rangle|^2.$$

The left-invariant vector fields associated to $X_i, Y_i \in \mathfrak{g}$, $i = 1, 2$, are given by

$$\begin{aligned}
\widetilde{X}_1 &= x_1 \partial/\partial x_1 + y_1 \partial/\partial y_1, \\
\widetilde{X}_2 &= x_1 \partial/\partial x_2 + y_1 \partial/\partial y_2, \\
\widetilde{Y}_1 &= -y_1 \partial/\partial x_1 + x_1 \partial/\partial y_1, \\
\widetilde{Y}_2 &= -y_1 \partial/\partial x_2 + x_1 \partial/\partial y_2.
\end{aligned}$$

The sub-Laplacian associated with q is

$$\Delta = \widetilde{X}_1^2 + \widetilde{X}_2^2 + \widetilde{Y}_1^2 + \widetilde{Y}_2^2 \tag{7.21}$$

$$= r_1^2 \left(\partial^2/\partial x_1^2 + \partial^2/\partial y_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial y_2^2 \right), \tag{7.22}$$

where $r_1^2 := x_1^2 + y_1^2$. Define the heat kernel ρ_t on G by the identity $e^{t\Delta/4} = *\rho_t$ and observe that Δ is actually an elliptic operator in this example.

We now let $K = \mathbb{C}^\times \times \{0\}$, a complex closed Lie subgroup of G, and $M := K \backslash G$ be the associated quotient space of right cosets. Using

$$(a,b)(c,d)^{-1} = (a,b)\left(c^{-1}, -c^{-1}d\right) = \left(ac^{-1}, b - ac^{-1}d\right),$$

we have

$$K(a,b) = K(c,d) \iff (a,b)(c,d)^{-1} \in K \iff b - ac^{-1}d = 0 \iff \frac{b}{a} = \frac{d}{c}.$$

In particular, this shows that $K(a,b) = K(1, b/a)$ and the map, $\pi : M \to \mathbb{C}$ defined by $\pi(K(a,b)) = b/a$ is one to one and onto. Thus, using π, we may and shall identify M with $\mathbb{C}$. Using this identification, the right action of G on M induces a right action on $\mathbb{C}$ given by

$$b \cdot (c,d) = c^{-1}(b+d). \tag{7.23}$$

To each $A = (\alpha, \beta)_e \in \mathfrak{g}$, the right action of G on $\mathbb{C}$ induces a vector field $\dot{A}$ on $\mathbb{C}$ via

$$\dot{A}(b) := \frac{d}{dt}|_0 bg(t), \tag{7.24}$$

where $g(t)$ is any smooth curve in G such that $g(0) = e = (1,0)$ and $\dot{g}(0) = A$. Explicitly, we may take $g(t) = (1 + t\alpha, t\beta)$ in Eq. (7.24) to find

$$\dot{A}(b) = \frac{d}{dt}|_0 (b \cdot (1 + t\alpha, t\beta)) = \frac{d}{dt}|_0 \left((1+t\alpha)^{-1}(b + t\beta)\right) = \beta - \alpha b. \tag{7.25}$$

In particular, it follows that

$$\dot{X}_1(b) = -b,\ \dot{X}_2(b) = 1,\ \dot{Y}_1(b) = -ib,\ \text{and } \dot{Y}_2(b) = i,$$

where $X_1, X_2, Y_1,\ Y_2$ are as in Eq. (7.20). If $z = x + iy$ is the standard holomorphic coordinate on $\mathbb{C}$, ∂_z is as in (7.7) and $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function, then $\dot{A}f = (\beta - \alpha z)\,\partial_z f$ and, in particular,

$$\dot{X}_1 f = -z\partial_z f,\ \dot{X}_2 f = \partial_z f, \tag{7.26}$$

$$\dot{Y}_1 f = -iz\partial_z f,\ \dot{Y}_2 f = i\partial_z f. \tag{7.27}$$

7.5. **The heat kernel on the G-space.** We may rewrite Eq. (7.25) as

$$\dot{A} = \operatorname{Re}(\beta - \alpha z)\frac{\partial}{\partial x} + \operatorname{Im}(\beta - \alpha z)\frac{\partial}{\partial y}. \tag{7.28}$$

Taking $A = X_1, X_2, Y_1$ and Y_2 (see Eq. (7.20)) in Eq. (7.28) shows

$$\dot{X}_1 = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},\ \dot{X}_2 = \frac{\partial}{\partial x},\ \dot{Y}_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\ \text{and } \dot{Y}_2 = \frac{\partial}{\partial y}.$$

A straightforward computation then gives

$$\Delta_M := \dot{X}_1^2 + \dot{X}_2^2 + \dot{Y}_1^2 + \dot{Y}_2^2 = \left[1 + x^2 + y^2\right]\left(\partial_x^2 + \partial_y^2\right). \tag{7.29}$$

Remark 7.13. Notice that Δ_M is elliptic and is in fact the Laplace Beltrami operator on $\mathbb{C}$ when equipped with the Riemannian metric determined by

$$g(\partial_x, \partial_x) = g(\partial_y, \partial_y) = \frac{1}{\rho} \text{ and } g(\partial_x, \partial_y) = 0,$$

where $\rho := 1 + x^2 + y^2$. Indeed, $\sqrt{g} = 1/\rho$ and therefore,

$$\Delta_g f = \frac{1}{\sqrt{g}}\partial_i\left(g^{ij}\sqrt{g}\partial_j f\right) = \rho\partial_i\left(\rho\delta_{ij}\frac{1}{\rho}\partial_j f\right) = \rho\left(\partial_x^2 + \partial_y^2\right)f.$$

The geodesics starting at $0 \in \mathbb{C}$ are radial curves, say $\sigma_w(t) = tw$, for $w \in \mathbb{C}$. Therefore, the distance, in this metric, from 0 to $w \in \mathbb{C}$ is given by

$$d(0, w) = \int_0^1 \sqrt{\frac{|w|^2}{1 + t^2|w|^2}}dt = \sinh^{-1}(|w|) \cong \ln(2|w|) \ \text{ for } \ |w| \gg 1, \tag{7.30}$$

where $|w|$ is the Euclidean norm of w. The volume $V(s)$ of the ball of radius s centered at $0 \in \mathbb{C}$ is given by

$$\begin{aligned} V(s) &= 2\pi\int_0^{\sinh(s)} \frac{1}{1 + r^2} r\,dr \\ &= \pi\ln\left(1 + \sinh^2(s)\right) = \pi\ln\left(\cosh^2(s)\right). \end{aligned} \tag{7.31}$$

The metric g on $\mathbb{C}$ is complete (see Eq. (7.30)) and therefore $\Delta_M|_{C_c^\infty(\mathbb{R}^2\cong\mathbb{C})}$ is essentially self-adjoint on $L^2(\rho^{-1}\, dxdy)$. The associated heat kernel measure λ_t admits a density, the "heat kernel" on M based at $0 = Ke$, and, abusing notation, we write $\lambda_t(d\xi) = \lambda_t(\xi)\rho(\xi)^{-1}d\xi$, $\xi \in \mathbb{C} \cong M$. A simple computation shows that the metric g has non-negative Ricci curvature. Hence the Li-Yau estimates gives

$$\frac{c_\epsilon}{\ln(\cosh^2\sqrt{t})}e^{-(1+\epsilon)(\sinh^{-1}|\xi|)^2/t} \le \lambda_t(\xi) \le \frac{C_\epsilon}{\ln(\cosh^2\sqrt{t})}e^{-(1-\epsilon)(\sinh^{-1}|\xi|)^2/t}$$

for any ϵ small enough.

7.6. **Taylor coefficients and the unitary Taylor map.** We now express Corollary 6.8 in the context of the one dimensional G-space described in the previous two subsections.

$$\|f\|_t^2 = \int_{\mathbb{C}} |f|^2 d\lambda_t.$$

LEMMA 7.14. *If $a, b \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ with $a \ge 1$ and $g : \mathbb{C} \to \mathbb{C}$ a holomorphic function, then*

$$\left(\partial_z^a (z\partial_z)^b g\right)(0) = a^b g^{(a)}(0).$$

PROOF. First observe that, for any holomorphic function $u : \mathbb{C} \to \mathbb{C}$, we have

$$\partial_z^a (z\partial_z u(z)) = \sum_{k=0}^{a} \binom{a}{k} \partial_z^k z \cdot \partial_z^{a-k}\partial_z u(z) = z\partial_z^{a+1}u(z) + a\partial_z^a u(z)$$

and hence

$$\partial_z^a (z\partial_z u(z))|_{z=0} = a(\partial_z^a u)(0).$$

The result now follows by induction on b. $\square$

NOTATION 7.15. Let $\Gamma := \bigcup_{\ell\in\mathbb{N}} \Gamma_\ell$ where Γ_ℓ contains pairs of sequences $(a, b) \in \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty$ such that $a_i \ge 1$ if $i \le \ell$, $b_i \ge 1$ if $i < \ell$, and $a_i = b_i = 0$ if $i > \ell$. (Notice that b_ℓ may be zero in this definition.) Furthermore, for $(a, b) \in \Gamma_\ell$, let

$$|a| := \sum_{i=1}^{\ell} a_i = \sum_{i=1}^{\infty} a_i, \ |b| := \sum_{i=1}^{\ell} b_i = \sum_{i=1}^{\infty} b_i, \text{ and}$$

$$\gamma(a, b) := \prod_{i=1}^{\ell} (a_1 + \cdots + a_i)^{2b_i} = \prod_{i=1}^{\infty} (a_1 + \cdots + a_i)^{2b_i}.$$

COROLLARY 7.16. *Suppose that f is a holomorphic function on $\mathbb{C}$, then*

$$\|f\|_t^2 = \|\hat{f}\|_t^2 = |f(0)|^2 + \sum_{1\le m\le n<\infty} A(m, n)\frac{t^n}{n!}\left|f^{(m)}(0)\right|^2 \tag{7.32}$$

$$= \sum_{m=0}^{\infty} c_m(t)\left|f^{(m)}(0)\right|^2, \tag{7.33}$$

where $c_0(t) \equiv 1$,

$$c_m(t) := \sum_{n=m}^{\infty} A(m,n) \frac{t^n}{n!} \text{ for all } m \geq 1, \text{ and} \tag{7.34}$$

$$A(m,n) := \sum_{(a,b)\in\Gamma} 1_{\{|a|=m\}} 1_{\{|a|+|b|=n\}} \gamma(a,b) \quad \text{for all } n \geq m \geq 1. \tag{7.35}$$

Proof. If $W_1, \ldots, W_n \in \{\partial_z, z\partial_z\}$, then either $W_1 \cdots W_n f(0)$ is zero or

$$W_1 \cdots W_n f(0) = \left(\partial_z^{a_1} (z\partial_z)^{b_1} \cdots \partial_z^{a_\ell} (z\partial_z)^{b_\ell} f\right)(0)$$

for some $\ell \in \mathbb{N}$ and $(a,b) \in \Gamma_\ell$ with $|a| + |b| = n$. Moreover, making repeated use of Lemma 7.14 shows

$$\left(\partial_z^{a_1} (z\partial_z)^{b_1} \cdots \partial_z^{a_\ell} (z\partial_z)^{b_\ell} f\right)(0) = \gamma(a,b)^{1/2} f^{(|a|)}(0).$$

Therefore, it follows that

$$\|\hat{f}\|_t^2 = |f(0)|^2 + \sum_{(a,b)\in\Gamma} \frac{t^{|a|+|b|}}{(|a|+|b|)!} \gamma(a,b) \left|f^{(|a|)}(0)\right|^2,$$

from which Eqs. (7.32) and (7.33) easily follow. □

Example 7.17. If $m = 1$ and $n \geq 1$ is given, then $a = (1,0,0,\ldots)$, $b = (n-1,0,0,\ldots)$, and $A(1,n) = 1$. Therefore,

$$c_1(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} = e^t - 1.$$

Example 7.18. If $m = 2$ and $n \geq 2$ is given, then $a = (2,0,\ldots)$ and $b = (n-2,0,\ldots)$ or $a = (1,1,0,\ldots)$ and $b = (b_1, b_2, 0, \ldots)$ with $b_1 \geq 1$ and $b_1 + b_2 = n-2$. Therefore,

$$\begin{aligned} A(2,n) &= 2^{2(n-2)} + \sum_{b_1=1}^{n-2} 2^{2(n-2-b_1)} = \sum_{b_1=0}^{n-2} 2^{2(n-2-b_1)} \\ &= \sum_{l=0}^{n-2} 2^{2l} = \frac{4^{n-1}-1}{4-1} = \frac{1}{3}\left(4^{n-1}-1\right) \end{aligned}$$

and

$$\begin{aligned} c_2(t) &= \sum_{n=2}^{\infty} A(2,n) \frac{t^n}{n!} = \frac{1}{3} \sum_{n=2}^{\infty} \left(4^{n-1}-1\right) \frac{t^n}{n!} \\ &= \frac{1}{12} e^{4t} - \frac{1}{3} e^t + \frac{1}{4}. \end{aligned}$$

Combining Examples 7.17 and 7.18 with Corollary 7.16 shows

$$\|f\|_t^2 = \|\hat{f}\|_t^2 = |f(0)|^2 + \left(e^t - 1\right) |f'(0)|^2 + \frac{1}{2} \left(\frac{1}{12} e^{4t} - \frac{1}{3} e^t + \frac{1}{4}\right) |f''(0)|^2 + \cdots.$$

REMARK 7.19 (Asymptotic estimate of $c_m(t)$). In order to estimate $c_m(t)$ from below, first observe that if we choose $(a,b) \in \Gamma_1$ with $a_1 = m$ and $b_1 = n - m$, then $A(m,n) \geq \gamma(a,b) = m^{2(n-m)}$. Thus we may conclude

$$c_m(t) = \sum_{n=m}^{\infty} A(m,n) \frac{t^n}{n!} \geq m^{-2m} \sum_{n=m}^{\infty} m^{2n} \frac{t^n}{n!} \geq m^{-2m} \cdot \max_{n \geq m} \left(m^{2n} \frac{t^n}{n!} \right).$$

By Stirling's formula,

$$m^{2n} \frac{t^n}{n!} \asymp m^{2n} \frac{t^n}{\sqrt{2\pi n}\,(n/e)^n} = \frac{1}{\sqrt{2\pi n}} \left(\frac{m^2 te}{n} \right)^n.$$

The latter expression has a maximum at approximately $n = m^2 t$, which allows us to conclude

$$\max_{n \geq m} \left(m^{2n} \frac{t^n}{n!} \right) \gtrapprox \frac{1}{m\sqrt{2\pi t}} e^{m^2 t}.$$

Thus we get the following rough approximation to a lower bound for $c_m(t)$:

$$c_m(t) \gtrapprox \frac{1}{\sqrt{2\pi t}} \frac{e^{m^2 t}}{m^{2m+1}} = \frac{1}{m\sqrt{2\pi t}} \left(\frac{e^{mt}}{m^2} \right)^m.$$

REFERENCES

[1] A. ANCONA AND J. C. TAYLOR, Some remarks on Widder's theorem and uniqueness of isolated singularities for parabolic equations, Partial differential equations with minimal smoothness and applications (Chicago, IL, 1990), 15–23, IMA Vol. Math. Appl. 42, Springer, New York, 1992.

[2] N. BOURBAKI, Éléments de mathématique. Fascicule XXIX. Livre VI: Intégration. Chapitre 7: Mesure de Haar. Chapitre 8: Convolution et représentations, Actualités Scientifiques et Industrielles, No. 1306, Hermann, Paris, 1963.

[3] B. K. DRIVER, Heat kernels measures and infinite dimensional analysis, Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 101–141, Contemp. Math. 338, Amer. Math. Soc., Providence, RI, 2003.

[4] B. K. DRIVER AND L. GROSS, Hilbert spaces of holomorphic functions on complex Lie groups, New trends in stochastic analysis (Charingworth, 1994), 76–106, World Sci. Publ., River Edge, NJ, 1997.

[5] B. K. DRIVER, L. GROSS, AND L. SALOFF-COSTE, Holomorphic functions and subelliptic heat kernels over Lie groups, J. Eur. Math. Soc. (JEMS)11 (2009), 941–978.

[6] B. K. DRIVER, L. GROSS AND L. SALOFF-COSTE, Surjectivity of the Taylor map for complex nilpotent Lie groups, Math. Proc. Cambridge Philos. Soc. 146 (2009), 177–195.

[7] V. V. GORBATSEVICH, A. L. ONISHCHIK AND E. B. VINBERG, Foundations of Lie theory and Lie transformation groups, Translated from the Russian by A. Kozlowski, Reprint of the 1993 translation [*Lie groups and Lie algebras. I*, Encyclopaedia Math. Sci., 20, Springer, Berlin, 1993], Springer-Verlag, Berlin, 1997.

[8] L. GROSS, Harmonic analysis for the heat kernel measure on compact homogeneous spaces, Stochastic analysis on infinite-dimensional spaces (Baton Rouge, LA, 1994), 99–110, Pitman Res. Notes Math. 310, Longman Sci. Tech., Harlow, 1994.

[9] L. GROSS, Uniqueness of ground states for Schrödinger operators over loop groups, J. Funct. Anal. 112 (1993), 373–441.

[10] L. GROSS, Some norms on universal enveloping algebras, Canad. J. Math. 50 (1998), 356–377.

[11] V. V. Grušin, A certain class of elliptic pseudodifferential operators that are degenerate on a submanifold, Mat. Sb. (N.S.) 84 (126) (1971), 163–195.

[12] A. Guichardet, Symmetric Hilbert spaces and related topics, Springer-Verlag, Berlin, 1972. Infinitely divisible positive definite functions, Continuous products and tensor products, Gaussian and Poissonian stochastic processes, Lecture Notes in Mathematics, Vol. 261.

[13] L. Hörmander, Hypoelliptic second order differential equations, Acta Math. 119 (1967), 147–171.

[14] P. Maheux, Analyse et géométrie sur les espaces homogènes, C. R. Acad. Sci. Paris Sér. I Math. 314 (1992), 441–444.

[15] P. Maheux, Estimations du noyau de la chaleur sur les espaces homogènes, J. Geom. Anal. 8 (1998), 65–96.

[16] K. R. Parthasarathy and K. Schmidt, Positive definite kernels, continuous tensor products, and central limit theorems of probability theory, Springer-Verlag, Berlin, 1972, Lecture Notes in Mathematics, Vol. 272.

[17] L. Saloff-Coste, Parabolic Harnack inequality for divergence-form second-order differential operators, Potential theory and degenerate partial differential operators (Parma), Potential Anal. 4 (1995), 429–467.

[18] N. T. Varopoulos, L. Saloff-Coste and T. Coulhon, Analysis and geometry on groups, Cambridge University Press, Cambridge, 1992.

Department of Mathematics, 0112
University of California, San Diego
La Jolla, CA 92093-0112
U.S.A.
E-mail address: bdriver@math.ucsd.edu

Department of Mathematics
Cornell University
Ithaca, NY 14853-4201
U.S.A.
E-mail address: gross@math.cornell.edu

Department of Mathematics
Cornell University
Ithaca, NY 14853-4201
U.S.A.
E-mail address: lsc@math.cornell.edu