# ANALYSIS OF WIENER MEASURE ON PATH AND LOOP GROUPS

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## 1. Introduction

This article gives a unified explanation of many of the results regarding analysis of Wiener and pinned Wiener measure on path and loop groups which have been developed over the last decade. Most of the results discussed below are outgrowths or have been inspired by Leonard Gross' pioneering papers [36, 37]. Although this article is mostly a survey, the result in Theorem 2.5 below is new.

The main theme of this article is to explain how many properties of Wiener and pinned Wiener measure on path and loop groups can be traced back to the fact that the path group of interest is flat in the sense of Riemannian geometry, see Corollary 1.11 below. This point of view is often best understood using certain heuristic type path integral arguments for Wiener measure. As such, it is useful to give heuristic theorems (labelled as Meta-Theorems below) and heuristic proofs (labelled as Meta-Proofs below) of many of the results. Nevertheless we will always give a corresponding honest theorem capturing the "content" of the intuitive Meta-Theorem. Similarly, along with all Meta-Proofs of honest theorems, we will give references to the literature containing a rigorous proof.

### 1.1. **Notation.**

**Notation 1.1.** Let $K$ be a connected compact Lie group, $\mathfrak{k} := T_eK$ be the Lie algebra of $K, T\,\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ be an $Ad_K$-invariant inner product on $\mathfrak{k}$ and let $\langle\cdot,\cdot\rangle$ denote the unique bi-invariant Riemannian metric on $K$ which agrees with $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ on $\mathfrak{k} := T_eK$. To simplify notation later we will assume that $K$ is a matrix group in which case $\mathfrak{k}$ may also be viewed as a matrix Lie algebra. (Since $K$ is compact, this is no restriction, see for example Theorem 4.1 on p. 136 in [11].) Elements $A \in \mathfrak{k}$ will be

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identified with the unique left invariant vector field on $K$ agreeing with $A$ at the identity in $K$, i.e. if $f \in C^\infty(K)$ then

$$Af(x) = \frac{d}{dt}|_0 f(xe^{tA}).$$

**Example 1.2.** As an example, let $K = SO(3)$ be the group of $3\times 3$ real orthogonal matrices with determinant 1. The Lie algebra of $K$ is $\mathfrak{k} = so(3)$, the set of $3\times 3$ real skew symmetric matrices, and the inner product $\langle A, B\rangle_{\mathfrak{k}} := -\mathrm{tr}(AB)$ is an example of an $Ad_K$ – invariant inner product on $\mathfrak{k}$.

Our main interest here is the path and loop spaces built on $K$ which we now define.

**Notation 1.3.** Suppose that $M = \mathfrak{k}$ or $K$ and $o = 0 \in \mathfrak{k}$ or $o = e \in K$ respectively. Let $W(M)$ denote the collection of continuous paths $\sigma : [0,1] \to M$ such that $\sigma(0) = o \in M$. The subset of finite energy paths $H(M)$ consists of those $\sigma \in W(M)$ which are absolutely continuous and satisfy $E_M(\sigma) < \infty$ where

$$E_{\mathfrak{k}}(\sigma) := \int_0^1 |\sigma'(s)|_{\mathfrak{k}}^2\, ds < \infty \text{ and } E_K(\sigma) := \int_0^1 \left|[\sigma(s)]^{-1}\sigma'(s)\right|_{\mathfrak{k}}^2 ds. \tag{1.1}$$

Also let $\mathcal{L}(M)$ and $H_0(M)$ denote the space of loops inside $W(M)$ and $H(M)$ respectively. In particular,

$$W(K) \equiv \{\sigma \in C([0,1] \to K) \,|\sigma(0) = e\} \tag{1.2}$$

and

$$\mathcal{L}(K) \equiv \{\sigma \in W(K) \,|\sigma(1) = e\} \tag{1.3}$$

respectively. Also let $\mathbf{e} \in \mathcal{L}(K)$ denote the constant path at identity $e \in K$, i.e. $\mathbf{e}(s) = e$ for $s \in [0,1]$.

Because $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ is $Ad_K$ – invariant, the formula for the **energy,** $E_K(\sigma)$, in Eq. (1.1) may also be expressed as

$$E_K(\sigma) = \int_0^1 \left|\sigma'(s)\sigma(s)^{-1}\right|_{\mathfrak{k}}^2 ds. \tag{1.4}$$

As usual we will refer to $H(\mathfrak{k})$ equipped with the Hilbertian inner product,

$$(h,k) := \int_0^1 \langle h'(s), k'(s)\rangle\, ds, \tag{1.5}$$

as the Cameron – Martin Hilbert space. Notice that $E_{\mathfrak{k}}(h) = (h,h)$.

*Remark* 1.4. It is well known that $H(K)$ is a Hilbert Lie group under pointwise multiplication and that the map

$$(x,h) \in H(K) \times H(\mathfrak{k}) \to L_{x*}h \in T(H(K))$$

is a trivialization of the tangent bundle of $H(K)$. (We are using $L_x : H(K) \to H(K)$ to denote left multiplication by $x$.) This trivialization induces a left-invariant Riemannian metric $(\cdot,\cdot)$ on $H(K)$ given explicitly by

$$(L_{x*}h, L_{x*}h) = \int_0^1 \langle h'(s), h'(s)\rangle\, ds \quad \forall\, x \in H(K) \text{ and } h \in H(\mathfrak{k}). \tag{1.6}$$

See Appendix A in [20] and the references therein for more details.

**Notation 1.5.** In the sequel we will fix a special type (for technical convergence issues later) of orthonormal bases, $S$ and $S_0$, for $H(\mathfrak{k})$ and $H_0(\mathfrak{k})$ respectively. Namely we assume,

$$S = \{\ell\xi : \ell \in S_{\mathbb{R}} \text{ and } \xi \in \beta\} \text{ and } S_0 = \{\ell\xi : \ell \in (S_{\mathbb{R}})_0 \text{ and } \xi \in \beta\}$$

where $\beta$ is an orthonormal basis for $\mathfrak{k}$ and $S_{\mathbb{R}}$ and $(S_{\mathbb{R}})_0$ are orthonormal bases for $H(\mathbb{R})$ and $H_0(\mathbb{R})$ respectively.

**Definition 1.6** (Differential Operators)**.** For $h \in H(\mathfrak{k})$ $(h \in H_0(\mathfrak{k}))$, let $\tilde{h}$ denote the left invariant vector field on $H(K)$ $(H_0(K))$ such that $\tilde{h}(e) = h$, i.e. if $f \in C^1(H(K))$ $\left(f \in C^1(H_0(K))\right)$ and $x \in H(K)$ $(x \in H_0(K))$ then

$$\tilde{h}f(x) = \frac{d}{dt}\bigg|_0 f\left(xe^{th}\right),$$

where $\left(xe^{th}\right)(s) = x(s)e^{th(s)}$ for all $s \in [0,1]$. Also for $f \in C^2(H(K))$, let

$$\|\mathrm{grad} f\|^2 := \sum_{h\in S}\left(\tilde{h}f\right)^2 \text{ and } \triangle_{H(K)}f := \sum_{h\in S}\tilde{h}^2 f$$

and for $f \in C^2(H_0(K))$ let

$$\|\mathrm{grad}_0 f\|^2 := \sum_{h\in S_0}\left(\tilde{h}f\right)^2 \text{ and } \triangle_{H_0(K)}f := \sum_{h\in S_0}\tilde{h}^2 f.$$

Similarly for $f \in C^2(H(\mathfrak{k}))$ and $h \in H(\mathfrak{k})$ let

$$\partial_h f(x) := \frac{d}{dt}|_0 f(x+th), \;\; \|\mathrm{grad} f\|^2 := \sum_{h\in S}(\partial_h f)^2 \text{ and } \triangle_{H(\mathfrak{k})}f := \sum_{h\in S}\partial_h^2 f$$

and for $f \in C^2(H_0(\mathfrak{k}))$ and $h \in H_0(\mathfrak{k})$ let

$$\partial_h f(x) := \frac{d}{dt}|_0 f(x+th), \;\; \|\mathrm{grad}_0 f\|^2 := \sum_{h\in S_0}(\partial_h f)^2 \text{ and } \triangle_{H_0(\mathfrak{k})}f := \sum_{h\in S_0}\partial_h^2 f.$$

**Notation 1.7.** If $\mu$ is a probability measure on a measure space $(\Omega, \mathcal{F})$ and $f \in L^1(\mu) = L^1(\Omega, \mathcal{F}, \mu)$, we will often write $\mu(f)$ and sometimes $\mathbb{E}f$ for the integral, $\int_\Omega f d\mu$.

### 1.2. **Basis Geometrical Results.**

**Meta-Theorem 1.8.** *The Lie groups $H(K)$ and $H_0(K)$ should be thought of as being unimodular, i.e. the "Riemannian volume form" is formally invariant under both left and right translations. As a result, $\triangle_{H(K)}$ and $\Delta_{H_0(K)}$ should be interpreted as the Laplace Beltrami operators on $H(K)$ and $H_0(K)$ respectively.*

**Proof.** (Meta Proof) One way to "verify" this assertion would be to show that $\mathrm{trace}(ad_h) = 0$ for all $h \in H(\mathfrak{k})$ $(h \in H_0(\mathfrak{k}))$. The problem here however is that $ad_h : H(\mathfrak{k}) \to H(\mathfrak{k})$ is not a trace class operator in general. However, if we compute the "trace" using the type of basis described in Notation 1.5 we do get trace $(ad_h) = 0$ :

$$\begin{aligned}\operatorname{trace}(ad_h) &= \sum_{k\in S}\langle [h,k],k\rangle_{H(\mathfrak{k})} = \sum_{k\in S}\int_0^1 \langle [h,k]'(s),k'(s)\rangle_{\mathfrak{k}} ds\\ &= \sum_{k\in S}\int_0^1 \langle [h',k](s),k'(s)\rangle_{\mathfrak{k}} ds = \sum_{k\in S}\int_0^1 \langle h'(s),[k(s),k'(s)]\rangle_{\mathfrak{k}}\, ds = 0\end{aligned}$$

where in the fourth equality we have used the $Ad_K$ − invariance of $\langle\cdot,\cdot\rangle$ and in the last equality the fact that $[k(s),k'(s)] = 0$ for all $s \in [0,1]$ and $k \in S$. So even though $ad_h$ is not trace class (i.e. the trace is basis dependent) for the type of bases that we consider one should interpret trace $(ad_h)$ to be 0. Similar computations hold in the $H_0(K)$ case as well. ■

**Definition 1.9** (Left and Right Anti-development Motions)**.** The left and right anti-development maps, $b : H(K) \to H(\mathfrak{k})$ and $B : H(K) \to H(\mathfrak{k})$ respectively are defined by

$$b_s(x) := \int_0^s x(s)^{-1}x'(s)ds \text{ and} \tag{1.7}$$

$$B_s(x) := \int_0^s x'(s)x(s)^{-1}ds. \tag{1.8}$$

The following theorem appears (in a disguised form) in Theorem 3.14 and Lemma 3.15 of Gross [36], also see Shigekawa [56], Fang and Franchi [30], Driver and Hall [20].

**Proposition 1.10.** *Let $b, B : H(K) \to H(\mathfrak{k})$ be as in Definition 1.9, then*

$$\left(B_*\tilde{h}(x)\right)_s' = \tilde{h}B_s'(x) = Ad_{x(s)}h'(s) \tag{1.9}$$

*and*

$$\tilde{h}b_s'(x) = h'(s) + [b_s'(x), h(s)]. \tag{1.10}$$

**Proof.** These are routine computations:

$$\begin{aligned}\tilde{h}B_s'(x) &= \frac{d}{dt}|_0\left\{(xe^{th})'(s)(x(s)e^{th(s)})^{-1}\right\}\\ &= \frac{d}{dt}|_0\left\{\left[x'(s)e^{th(s)} + x(s)(e^{th(s)})'\right]e^{-th(s)}x(s)^{-1}\right\}\\ &= \frac{d}{dt}|_0\left\{x'(s)x(s)^{-1} + x(s)(e^{th(s)})'e^{-th(s)}x(s)^{-1}\right\} = Ad_{x(s)}h'(s)\end{aligned}$$

and

$$\begin{aligned}\tilde{h}b_s'(x) &= \frac{d}{dt}|_0\left\{(x(s)e^{th(s)})^{-1}(xe^{th})'(s)\right\}\\ &= \frac{d}{dt}|_0\left\{e^{-th(s)}x(s)^{-1}\left[x'(s)e^{th(s)} + x(s)(e^{th(s)})'\right]\right\}\\ &= \frac{d}{dt}|_0\left\{e^{-th(s)}b_s'(x)e^{th(s)} + e^{-th(s)}(e^{th(s)})'\right\}\\ &= h'(s) + [b_s'(x), h(s)].\end{aligned}$$

■

The next corollary is the most crucial result for the rest of this paper.

**Corollary 1.11.** *The map $B : H(K) \to H(\mathfrak{k})$ is an isometric isomorphism of infinite-dimensional Riemannian manifolds. In particular, $H(K)$ is flat.*

**Proof.** By Eq. (1.9),

$$\begin{aligned}\langle B_*\tilde{h}(x), B_*\tilde{h}(x)\rangle_{H(\mathfrak{k})} &= \int_0^1 \left|\left(B_*\tilde{h}(x)\right)_s'\right|^2 ds = \int_0^1 \left|Ad_{x(s)}h'(s)\right|^2 ds \\ &= \int_0^1 |h'(s)|^2 ds = \langle h, h\rangle_{H(\mathfrak{k})} = \langle \tilde{h}, \tilde{h}\rangle_{T_x H(K)}.\end{aligned}$$

■

**Theorem 1.12.** *The operators, $\Delta_{H(\mathfrak{k})}$ on $H(\mathfrak{k})$ and $\Delta_{H(K)}$ on $H(K)$, are intertwined by $B$. More precisely if $f : H(K) \to \mathbb{C}$ is a smooth cylinder function, then $\Delta_{H(\mathfrak{k})}(f \circ B^{-1})$ and $\Delta_{H(K)}f$ exist and*

$$\Delta_{H(\mathfrak{k})}(f \circ B^{-1}) = (\Delta_{H(K)}f) \circ B^{-1}. \tag{1.11}$$

**Proof.** (Meta-Proof.) Since $B$ is an isometry and, by Meta-Theorem 1.8, $\Delta_{H(\mathfrak{k})}$ and $\Delta_{H(K)}$ should be thought of as the Laplace Beltrami operators on $H(\mathfrak{k})$ and $H(K)$ respectively, we should expect Eq. (1.11) to hold. This argument would be valid in finite dimensions but because of convergence issues it is not a real proof here. We refer the reader to Appendix $A$ in Driver and Hall [20] for a detailed account. ■

The next proposition points out the well known difficulties when dealing with Laplacians in infinite dimensions.

**Proposition 1.13.** *Let $\mu$ be standard Wiener measure on $\Omega := C([0,1], \mathbb{R})$. For a cylinder function of the form $f(\omega) = F(\omega|_\pi)$, where*

$$\pi := \{0 = s_0 < s_1 < \cdots < s_n = 1\} \tag{1.12}$$

*is a partition of $[0,1]$ and $F : \mathbb{R}^n \to \mathbb{R}$ is a smooth function which along with its derivative has at most polynomial growth, let*

$$\Delta f = \triangle_{H(\mathbb{R})} f := \sum_{h \in S(\mathbb{R})} \partial_h^2 f$$

*be as defined in Definition 1.6 above with $\mathfrak{k} = \mathbb{R}$. Then $\Delta$ is **not** a closable operator on $L^2(\Omega, \mu)$.*

I would like to thank Brian Hall for the natural counter example to the closability of $\Delta$ used in the following proof.

**Proof.** Let

$$f(\omega) := \omega_1^2 - \frac{1}{2} = \int_0^1 \omega_s d\omega_s \text{ (Itô} - \text{integral)}$$

and for a partition $\pi$ as in Eq. (1.12) let

$$f_\pi(\omega) := \sum_{j=0}^{n-1} \omega(t_j)\left(\omega(t_{j+1}) - \omega(t_j)\right).$$

Then by the basic theory of the Itô integral, $f_\pi \to f$ in $L^2(\mu)$ as $|\pi| \to 0$ where $|\pi|$ is the mesh size of $\pi$ defined by

$$|\pi| := \max\left\{|s_{i+1} - s_i| : i = 0, 1, 2, \ldots, n-1\right\}. \tag{1.13}$$

To compute $\Delta f$ and $\Delta f_\pi$ we will make use of the identity

$$\sum_{h\in S(\mathbb{R})} h(s)h(t) = \min(s,t)$$

which follows from the reproducing properties of $H(\mathbb{R})$ and Parseval's inequality, see Lemma 3.3 in [23] for example. By this identity and elementary computations,

$$\begin{aligned}\Delta f_\pi &= \sum_{h\in S(\mathbb{R})} \partial_h^2 f_\pi = 2\sum_{h\in S(\mathbb{R})}\sum_{j=0}^{n-1} h(t_j)\left(h(t_{j+1}) - h(t_j)\right)\\ &= 2\sum_{j=0}^{n-1}\left[\min(t_j,t_{j+1}) - \min(t_j,t_j)\right] = 0\end{aligned}$$

while

$$\Delta f = \sum_{h\in S(\mathbb{R})} \partial_h^2 f = 2\sum_{h\in S(\mathbb{R})} h^2(1) = 2 \neq 0.$$

If $\Delta$ were closable, then $0 = \Delta f_\pi \to \Delta f = 2$ as $|\pi| \to 0$ which clearly does not happen. ■

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## 2. Path Group Results

**Notation 2.1.** If $(\Omega, \mathcal{F}, P)$ is a probability space and $f \in L^1(\Omega,\mu)$, we will often write $\int_\Omega f dP$ as $P(f)$ or sometimes as $\mathbb{E}f$.

Given a Brownian motion $\{W_s : s\in[0,1]\}$ on $\mathfrak{k}$ with variance determined by $\langle\cdot,\cdot\rangle$, the process $k$ defined by Fisk – Stratonovich stochastic differential equation

$$dk_s = k_s \circ \left(\sqrt{t}dW_s\right) \text{ with } k_0 = e. \tag{2.1}$$

is a diffusion on $K$ with generator, $t\Delta/2$, where $\Delta = \sum_{A\in\beta}\tilde{A}^2$ where $\beta$ is an orthonormal basis for $\mathfrak{k}$. Let $dx$ denote Haar measure on $K$ and

$$p_t(x) = e^{t\Delta/2}\delta_e(x) = \frac{d\mathrm{Law}(k_1)}{dx}$$

be the convolution heat kernel on $K$.

**Definition 2.2** (Wiener measure)**.** The measure, $\mu_t = \mathrm{Law}(k)$, is called **Wiener measure** on $W(K)$ with variance $t$.

*Remark* 2.3. Using the $Ad_K$ – invariance of $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ and Levy's criteria of Brownian motion, it is easily seen that $\mu_t = \mathrm{Law}(\tilde{k})$ as well, where $\tilde{k}$ is the solution to the Fisk – Stratonovich stochastic differential equation

$$d\tilde{k}_s = \left(\sqrt{t}dW_s\right)\circ\tilde{k}_s \text{ with } k_0 = e. \tag{2.2}$$

**Meta-Theorem 2.4** (Informal description of Wiener measure)**.** *The measure $\mu_t$ may be described **informally** as*

$$d\mu_t(x) = \rho_t(x)\mathcal{D}x \tag{2.3}$$

*where*

$$\rho_t(x) = \frac{1}{Z_t} e^{-\frac{1}{2t}E_K(x)}, \tag{2.4}$$

*$E_K(x)$ is the energy of $x$ (see Eqs. (1.1) and (1.4)), $\mathcal{D}x$ is the formal Riemann volume measure ("Haar" measure) on $H(K)$ and $Z_t$ is the ill defined "normalization constant,"*

$$\text{“}Z_t := \int_{W(K)} e^{-\frac{1}{2t}E(x)}\mathcal{D}x.\text{”} \tag{2.5}$$

*Moreover the "function" $\rho_t(x)$ "solves" the heat equation*

$$\partial_t \rho_t = \frac{1}{2}\Delta_{H(K)}\rho_t \text{ with } \rho_0 = \delta_e. \tag{2.6}$$

**Proof.** (Meta Proof.) We make use of the well known informal description of Wiener measure, $P_t$, with variance $t$ on $W(\mathfrak{k})$. Namely $dP_t(\omega) = \gamma_t(\omega)\mathcal{D}\omega$ where

$$\gamma_t(\omega)\mathcal{D}\omega = \frac{1}{Z_t}e^{-\frac{1}{2t}E_{\mathfrak{k}}(\omega)}\mathcal{D}\omega,$$

$\mathcal{D}\omega$ is the "Riemann volume measure" on $H(\mathfrak{k})$ and $Z_t$ is a normalization "constant," see Kuo [47] for example. By analogy with the finite dimensional case we should expect the Gaussian "density," $\gamma_t(\omega)$, to be a solution to the heat equation

$$\partial_t \gamma_t = \frac{1}{2}\Delta_{H(\mathfrak{k})}\gamma_t \text{ with } \gamma_0 = \delta_0.$$

Since $B : H(K) \to H(\mathfrak{k})$ is an isometry of Riemannian manifolds, we conclude that $B_*\mathcal{D}x = \mathcal{D}\omega$, i.e. that $B$ takes the Riemann volume element on $H(K)$ to the Riemann volume element on $H(\mathfrak{k})$. Similarly by the definition of $B$, $E_{\mathfrak{k}}$ and $E_K$, $E_K(x) = E_{\mathfrak{k}}(B(x))$ and therefore, $\gamma_t \circ B = \rho_t$ and $\rho_t$ should satisfy the heat Eq. (2.6) because of Theorem 1.12. Furthermore, by Eq. (2.2), $d\mu_t(x) = d\left(P_t \circ \tilde{B}\right)(x)$ where

$$\tilde{B}(x) := \int_0^{\cdot} dx_s \circ x_s^{-1} \text{ (Fisk – Stratonovich integral)},$$

a "stochastic extension" of $B$ to $W(K)$. Therefore we should have

$$d\mu_t(x) = d\left(P_t \circ B\right)(x) = \gamma_t(B(x)) \cdot B_*^{-1}\mathcal{D}\omega = \rho_t(x)\mathcal{D}x$$

which formally proves Eq. (2.3). ■

Theorems 2.5 and 2.6 below give rigorous meaning to the results in the Meta-Theorem 2.4. In the first theorem we follow the ideas in Driver and Andersson [9] and replace $H(K)$ by certain finite dimensional approximations.

**Theorem 2.5.** *To each partition $\pi$ of $[0,1]$ as in Eq. (1.12), let*

$$\begin{aligned} H_\pi(K) &= \left\{ x \in H(K) : \frac{d}{ds}\left[x'(s)x(s)^{-1}\right] = 0 \text{ if } s \notin \pi \right\} \\ &= \left\{ x \in H(K) : \frac{d^2}{ds^2}B_s(x) = 0 \text{ if } s \notin \pi \right\} \end{aligned} \tag{2.7}$$

*– the space of piecewise exponential maps on $K$. Then $H_\pi(K)$ is a submanifold of $H(K)$ and as such inherits a Riemannian metric from $H(K)$. Let $\lambda_\pi^K$ denote the associated volume form on $H_\pi(K)$ and define*

$$d\mu_t^\pi(x) := \frac{1}{Z_\pi} e^{-\frac{1}{2t}E_K(x)} d\lambda_\pi^K(x),$$

*where $Z_\pi$ is chosen so that $\mu_t^\pi$ is a probability measure on $H_\pi(K)$. Then $\lim_{|\pi|\to 0}\mu_t^\pi = \mu_t$ weakly where $|\pi|$ is defined in Eq. (1.13), i.e.*

$$\lim_{|\pi|\to 0}\int_{H_\pi(K)} f d\mu_t^\pi = \int_{W(K)} f d\mu_t$$

*for all bounded continuous functions $f : W(K) \to \mathbb{R}$. (Here we consider $W(K)$ a metric space in the sup – norm topology.)*

**Proof.** This theorem is a fairly direct consequence of Corollary 1.11 and a Wong – Zakai type approximation theorem for solving stochastic differential equations, see Theorem A.4 and Remark A.5 in Appendix A. Here are the details.

Let $H_\pi(\mathfrak{k})$ be the subspace of $H(\mathfrak{k})$ given by

$$H_\pi(\mathfrak{k}) = \left\{\omega \in H(\mathfrak{k}) : \frac{d^2}{ds^2}\omega_s = 0 \text{ if } s \notin \pi\right\}.$$

We view $H_\pi(\mathfrak{k})$ as a Riemannian manifold with metric determined by the inner product on $H(\mathfrak{k})$, see Eq. (1.5). By the definition of $H_\pi(K)$ (Eq. (2.7)), $B(H_\pi(K)) = H_\pi(\mathfrak{k})$ and by Corollary 1.11, $B : H_\pi(K) \to H_\pi(\mathfrak{k})$ is an isometry of Riemannian manifolds. Therefore $B_*\lambda_\pi^K = \lambda_\pi^{\mathfrak{k}}$ where $\lambda_\pi^{\mathfrak{k}}$ is the Riemannian volume measure on $H_\pi(\mathfrak{k})$. From these remarks and the identity $E_K = E_{\mathfrak{k}} \circ B$, which follows from Eq. (1.4) and the definition of $B$, we find

$$d\left(B_*\mu_t^\pi\right) = \frac{1}{Z_\pi} e^{-\frac{1}{2t}E_K\circ B} d\left(B_*\lambda_\pi^K\right) = \frac{1}{Z_\pi} e^{-\frac{1}{2t}E_{\mathfrak{k}}} d\lambda_\pi^{\mathfrak{k}} \tag{2.8}$$

where $Z_\pi$ is a normalization constant chosen to make $B_*\mu_t^\pi$ a probability measure.

As in Appendix A, let $\{W_s^\pi : s \in [0,1]\}$ be the piecewise linear process defined by

$$W_s^\pi = W_{s_j} + (s - s_j)\frac{W_{s_{j+1}} - W_{s_j}}{s_{j+1} - s_j} \text{ when } s \in [s_j, s_{j+1}],$$

where $W$ is the $\mathfrak{k}$ – valued Brownian motion used to define $k$ in Eq. (2.1). If $g : H_\pi(\mathfrak{k}) \to \mathbb{R}$ is a bounded measurable function, then using the Gaussian finite dimensional distributions of $W$, we have

$$\mathbb{E}\left[g(\sqrt{t}W^\pi)\right] = \int_{\mathfrak{k}^n} g(\sqrt{t}\omega^\pi_{(\xi_1,\dots,\xi_n)})\left[\prod_{j=1}^n p^{\mathfrak{k}}_{s_{j+1}-s_j}(\xi_{j+1} - \xi_j)\right] d\xi_1 \dots d\xi_n \tag{2.9}$$

where $p_s^{\mathfrak{k}}(\xi) = (2\pi s)^{-\dim\mathfrak{k}/2}\exp\left(-\frac{1}{2s}|\xi|^2\right)$ and $\omega^\pi_{(\xi_1,\dots,\xi_n)} \in H_\pi(\mathfrak{k})$ is determined uniquely by $\omega^\pi_{(\xi_1,\dots,\xi_n)}(s_j) = \xi_j$ for $j = 1, 2, \dots, n$. Using $\sqrt{t}\omega^\pi_{(\xi_1,\dots,\xi_n)} = \omega^\pi_{(\sqrt{t}\xi_1,\dots,\sqrt{t}\xi_n)}$, we may make a change of variables in Eq. (2.9) to find

$$\mathbb{E}\left[g(\sqrt{t}W^\pi)\right] = \int_{\mathfrak{k}^n} g(\omega^\pi_{(\xi_1,\dots,\xi_n)})\left[\prod_{j=1}^n p^{\mathfrak{k}}_{t(s_{j+1}-s_j)}(\xi_{j+1} - \xi_j)\right] d\xi_1 \dots d\xi_n. \tag{2.10}$$

Simple algebra shows

$$\begin{aligned}\prod_{j=1}^{n} p^{\mathfrak{k}}_{t(s_{j+1}-s_j)}(\xi_{j+1}-\xi_j) &= C_\pi \exp\left(-\frac{1}{2t}\sum\nolimits_{j=1}^{n}\frac{\left|\xi_{j+1}-\xi_j\right|^2}{s_{j+1}-s_j}\right)\\ &= C_\pi \exp\left(-\frac{1}{2t}\sum\nolimits_{j=1}^{n}\frac{\left|\xi_{j+1}-\xi_j\right|^2}{\left(s_{j+1}-s_j\right)^2}\left(s_{j+1}-s_j\right)\right)\\ &= C_\pi \exp\left(-\frac{1}{2t}E_{\mathfrak{k}}(\omega^{\pi}_{(\xi_1,\dots,\xi_n)})\right),\end{aligned}$$

where

$$C_\pi := \prod_{j=1}^{n}\left(\frac{1}{2\pi t\left(s_{j+1}-s_j\right)}\right)^{\dim\mathfrak{k}/2}.$$

Since $(\xi_1,\dots,\xi_n)\in\mathfrak{k}^n \xrightarrow{\phi} \omega^{\pi}_{(\xi_1,\dots,\xi_n)}\in H_\pi(\mathfrak{k})$ is linear, $\phi_*\left(d\xi_1\dots d\xi_n\right)$ is a translation invariant measure on $H_\pi(\mathfrak{k})$ as is $\lambda^{\mathfrak{k}}_\pi$, so $\phi_*\left(d\xi_1\dots d\xi_n\right)=c_\pi d\lambda^{\mathfrak{k}}_\pi$ for some constant $c_\pi$. Combining all of these observations shows that Eq. (2.10) may be written as

$$\begin{aligned}\mathbb{E}\left[g(\sqrt{t}W^{\pi})\right] &= \int_{\mathfrak{k}^n} g(\omega^{\pi}_{(\xi_1,\dots,\xi_n)})C_\pi\exp\left(-\frac{1}{2t}E_{\mathfrak{k}}(\omega^{\pi}_{(\xi_1,\dots,\xi_n)})\right)d\xi_1\dots d\xi_n\\ &= \int_{H_\pi(\mathfrak{k})} g(\omega)\frac{1}{Z_\pi}\exp\left(-\frac{1}{2t}E_{\mathfrak{k}}(\omega)\right)d\lambda^{\mathfrak{k}}_\pi(\omega)\end{aligned}\tag{2.11}$$

where $Z_\pi^{-1}=C_\pi c_\pi$ is a normalization constant such that $\frac{1}{Z_\pi}\exp\left(-\frac{1}{2t}E_{\mathfrak{k}}(\omega)\right)d\lambda^{\mathfrak{k}}_\pi(\omega)$ is a probability measure as can be seen by taking $g=1$ in Eq. (2.11). Combining Eq. (2.11) with Eq. (2.8) gives

$$\int_{H_\pi(K)} g(B(x))d\mu^{\pi}_t(x)=\mathbb{E}\left[g(\sqrt{t}W^{\pi})\right]$$

and applying this formula to $g:=f\circ B^{-1}$ shows

$$\int_{H_\pi(K)} f(x)d\mu^{\pi}_t(x)=\mathbb{E}\left[f\circ B^{-1}(\sqrt{t}W^{\pi})\right].\tag{2.12}$$

Using the definition of $B$, $k^\pi := B^{-1}(\sqrt{t}W^{\pi})$ satisfies

$$\sqrt{t}W^{\pi}_s = B_s(k^\pi)=\int_0^s\left(k^\pi(\sigma)\right)^{-1}\frac{dk^\pi(\sigma)}{d\sigma}d\sigma$$

or equivalently

$$\frac{dk^\pi(s)}{ds}=k^\pi(s)\sqrt{t}\frac{dW^{\pi}_s}{ds}\text{ with }k^\pi(0)=e\in K.$$

So by the dominated convergence theorem and a Wong – Zakai type approximation Theorem A.4 and Remark A.5 in Appendix A below,

$$\lim_{|\pi|\to0}\int_{H_\pi(K)} f(x)d\mu^{\pi}_t(x)=\lim_{|\pi|\to0}\mathbb{E}\left[f(k^\pi)\right]=\mathbb{E}\left[f(k)\right]=\int_W f(x)d\mu_t(x)$$

where $k$ is the solution to Eq. (2.1). ■

2.1. **Consequences of Meta – Theorem 2.4.**

**Theorem 2.6.** *If $f : W(K) \to \mathbb{R}$ is a smooth cylinder function then*

$$G(t,x) := \int_{W(K)} f(xy^{-1}) d\mu_t(y)$$

*is a solution to the heat equation,*

$$\left(\partial_t - \frac{1}{2}\Delta_{H(K)}\right) G(t,x) = 0 \text{ with } \lim_{t\downarrow 0} G(t,x) = f(x). \tag{2.13}$$

**Proof.** (Meta-proof) A rigorous proof of this theorem may be found in Driver and Srimurthy [23]. Here we will give a heuristic proof based on Meta-Theorem 2.4 which implies

$$G(t,x) = \int_{W(K)} f(xy^{-1})\rho_t(y)\mathcal{D}y.$$

So (formally)

$$\begin{aligned}
\frac{1}{2}\Delta_{H(K)}G(t,x) &= \frac{1}{2}\sum_{h\in S}\frac{d^2}{d\tau^2}|_0 G(t, xe^{\tau h}) = \frac{1}{2}\int_{W(K)}\sum_{h\in S}\frac{d^2}{d\tau^2}|_0 f(xe^{\tau h}y^{-1})\rho_t(y)\mathcal{D}y \\
&= \frac{1}{2}\int_{W(K)}\sum_{h\in S}\frac{d^2}{d\tau^2}|_0 \left[f(xy^{-1})\rho_t(ye^{\tau h})\right]\mathcal{D}y \\
&= \int_{W(K)} f(xy^{-1})\frac{1}{2}\Delta_{H(K)}\rho_t(y)\mathcal{D}y = \int_{W(K)} f(xy^{-1})\partial_t\rho_t(y)\mathcal{D}y \\
&= \partial_t G(t,x)
\end{aligned}$$

where in the third equality we used the "invariance" of $\mathcal{D}y$ to make the change of variables, $y \to ye^{\tau h}$ and we used Eq. (2.6) in the second to last equality. In analogy with finite dimensional Laplace's method we expect $\lim_{t\downarrow 0} G(t,x) = f(x)$. ■

The following Logarithmic Sobolev inequality appears in Gross [36].

**Theorem 2.7** (Logarithmic Sobolev Inequality on $W(K)$)**.** *For all non-zero smooth cylinder functions, $f : W(K) \to \mathbb{R}$,*

$$\int_{W(K)} f^2 \log\left(\frac{f^2}{\mu_t(f^2)}\right) d\mu_t \le 2t\int_{W(K)} \|gradf\|^2 \, d\mu_t, \tag{2.14}$$

*where*

$$\mu_t(f^2) := \int_{W(K)} f^2 d\mu_t.$$

For analogous results when $K$ is replaced by a compact Riemannian manifold, see Fang [27] (for the weaker Poincaré inequality) and Hsu [42], Aida and Elworthy [5] and Capitaine, Hsu, and Ledoux [12].

**Proof.** (Meta-Proof) Since $B : H(K) \to H(\mathfrak{k})$ is an isometry, it preserves all Riemannian geometric structures. Therefore the original logarithmic Sobolev inequality on $W(\mathfrak{k})$ proved by Gross in [35] transfers directly over to $W(K)$ to give Eq. (2.14), Incidentally, in this case the Meta-proof and the real proof are almost the same, see Gross [36]. ■

2.1.1. *Generalized Segal – Bargmann – Hall Transform.* Another "application" of Meta – Theorem 2.4 and Theorem 1.12 is that they give a simple way to understand the group generalizations of the Segal-Bargmann theorem, see the pioneering work of Hall [39]. Also see Gross and Malliavin [38] and [53, 41, 7, 20].

The classical Segal-Bargmann transform is a unitary isomorphism $S_t^{\mathfrak{k}} : L^2(W(\mathfrak{k}), P_t) \to \mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), P_t^{\mathbb{C}})$ where $\mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), P_t^{\mathbb{C}})$ denotes the "holomorphic" $L^2$ – functions on the path space, $W(\mathfrak{k}_{\mathbb{C}})$, into the complexified Lie algebra $\mathfrak{k}_{\mathbb{C}}$. The measure $P_t^{\mathbb{C}}$ is again the law of a certain $\mathfrak{k}_{\mathbb{C}}$ – valued Brownian motion which we do not bother to describe here. What is important is that, on cylinder functions,

$$S_t^{\mathfrak{k}} f = \left(e^{\frac{t}{2}\Delta_{H(\mathfrak{k})}} f\right)_a,$$

where, for a function $F : W(\mathfrak{k}) \to \mathbb{C}$, $F_a$ refers to the analytic continuation of $F$ to $W(\mathfrak{k}_{\mathbb{C}})$. The $W(K)$ – version of this theorem is the existence of a unitary isomorphism

$$S_t^K : L^2(W(K), \mu_t) \to \mathcal{H}L^2(W(K_{\mathbb{C}}), \mu_t^{\mathbb{C}})$$

where $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu_t^{\mathbb{C}})$ denotes the "holomorphic" $L^2$ – functions on the path space, $W(K_{\mathbb{C}})$, into the complexified Lie group $K_{\mathbb{C}}$. The measure $\mu_t^{\mathbb{C}}$ is again the law of a certain $K_{\mathbb{C}}$ – valued Brownian motion which we do not bother to describe. On cylinder functions, $f : W(K) \to \mathbb{C}$, $S_t^K f : W(K_{\mathbb{C}}) \to \mathbb{C}$ is given analogously to $S_t^{\mathfrak{k}} f$ as

$$S_t^K f = \left(e^{\frac{t}{2}\Delta_{H(K)}} f\right)_a.$$

Moreover, one has the following commutative diagram

$$\begin{array}{ccccc} & L^2(W(\mathfrak{k}), P_t) & \cong & L^2(W(K), \mu_t)) & \\ S_t^{\mathfrak{k}} & \downarrow & & \downarrow & S_t^K \\ & \mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), P_t^{\mathbb{C}}) & \cong & \mathcal{H}L^2(W(K_{\mathbb{C}}), \mu_t^{\mathbb{C}}) & \end{array}$$

where the horizontal equivalences are determined by Itô maps as in (or similar to) Eq. (2.2).

These results are easily understood in terms of Theorem 1.12. Indeed from Eq. (1.11) we expect,

$$S_t^K f = (e^{\frac{t}{2}\Delta_{H(K)}} f)_a = \left(\left[e^{\frac{t}{2}\Delta_{H(\mathfrak{k})}}(f \circ B^{-1})\right] \circ B\right)_a = \left(e^{\frac{t}{2}\Delta_{H(\mathfrak{k})}}(f \circ B^{-1})\right)_a \circ B^{\mathbb{C}},$$

where $B^{\mathbb{C}} : H(K^{\mathbb{C}}) \to H(\mathfrak{k}^{\mathbb{C}})$ is a complex version of $B$. Because of domain issues this is not a rigorous proof, but the spirit is correct. We refer the reader to Driver and Hall [20] for the full details and further generalizations.

## 3. Pinned Wiener Measure Results

We now wish to consider analogous theorems for loop groups where it no longer true that $H_0(K) \subset H(K)$ is a flat Riemannian manifold. However, we can still exploit the fact that $H_0(K)$ is an embedded submanifold of $H(K)$.

### 3.1. **Pinned Wiener Measure.**

**Definition 3.1** (Pinned Wiener Measure)**.** Pinned Wiener measure $\mu_t^e$ is the measured $\mu_t$ "conditioned" to live on $\mathcal{L}(K)$. Informally, $\mu_t^e$ is given by

$$\mu_t^e(f) = \mathbb{E}(f(k)|k_1 = e) = \frac{\int_{W(K)} f(k)\delta_e(k_1)d\mu_t(k)}{\int_{W(K)} \delta_e(k_1)d\mu_t(k)} = \frac{\int_{W(K)} f(k)\delta_e(k_1)d\mu_t(k)}{p_t(e)}.$$

A rigorous definition of $\mu_t^e$ may be given using Doob's construction, namely, $\mu_t^e$ is the unique measure on $\mathcal{L}(K)$ such that if $f$ is a bounded measurable function depending on $\sigma|_{[0,\alpha]}$ for some $\alpha \in (0,1)$, then

$$\mu_t^e(f) \equiv \frac{1}{p_t^K(e)} \mu_t(f p_{t(1-\alpha)}^K(\pi_\alpha)).$$

For more details, see [8], [50], [37], and [19] for example.

We now want to develop the pinned analogues of Meta-Theorem 2.4. For this we will make formal use of the co-area formula described in the next theorem.

**Theorem 3.2** (Co-area Formula)**.** *Suppose that $\pi : M \to N$ is a $C^\infty$ – map of smooth finite dimensional Riemannian manifolds and that $n \in N$ is a regular value of $\pi$. (Recall that the implicit function theorem then implies $M_n := \pi^{-1}(\{n\})$ is a smooth submanifold of $M$.) Let $\lambda_M$ and $\lambda_{M_n}$ denote the Riemann volume measure on $M$ and $M_n$ respectively. Then*

$$\int_M f(x)\delta_n(\pi(x))d\lambda_M(x) = \int_{M_n} f(x) \frac{1}{\sqrt{\det(\pi_{*x}\pi_{*x}^{\mathrm{tr}})}} d\lambda_{M_n}(x)$$

*where $\pi_{*x} : T_xM \to T_{\pi(x)}N$ is the differential of $\pi$ and $\pi_{*x}^{\mathrm{tr}}$ is the adjoint of $\pi_{*x}$ relative to the Riemannian structures on $T_xM$ and $T_{\pi(x)}N$.*

**Proof.** I will not attempt to give a general proof of this fact here but only indicate the main ideas. First off, it suffices to prove the result for functions which have support in a small neighborhood of a point $m \in M_n$. By choosing appropriate coordinates on $M$ and $N$, we may assume that both $M$ and $N$ are vector spaces and $m = 0$, and $n = 0$. We may also assume that $\pi$ is linear. Let $X = Nul(\pi) = \pi^{-1}(\{0\}) = M_n$ and $Y = Nul(\pi)^\perp = Ran(\pi^*)$. Then

$$\begin{aligned}\int_M f\delta_n(\pi)d\lambda_M &= \int_{X\times Y} f(x+y)\delta(\pi(y))dxdy \\ &= \int_{X\times Y} f(x)\delta(\pi(y))dxdy \\ &= \int_X f(x)dx \int_Y \delta(\pi(y))dy.\end{aligned}$$

Now letting $z = \pi(y)$, we find that $dz = |\det(\pi)|\, dy$, so that

$$\int_Y \delta(\pi(y))dy = \frac{1}{|\det(\pi)|}\int_N \delta(z)dy = \frac{1}{|\det(\pi)|}.$$

In order to compute the Jacobian factor $\det(\pi)$, choose an orthogonal transformation $U : N \to Y$, then

$$|\det(\pi)| = |\det(\pi U)| = \det(\pi U\,(\pi U)^{\mathrm{tr}})^{1/2} = \det(\pi\pi^{\mathrm{tr}})^{1/2}.$$

■

We will now formally apply Theorem 3.2 with $M = H(K)$, $N = K$ and $\pi : H(K) \to K$ the projection map $\pi(x) := x_1$.

**Meta-Theorem 3.3** (Informal Description of Pinned Wiener)**.** *Pinned Wiener measure, $\mu_t^e$, on $\mathcal{L}(K)$ is given informally by the expression*

$$d\mu_t^e(x) = \frac{1}{p_t^K(e)}\rho_t(x)\mathcal{D}_0 x. \tag{3.1}$$

*where $\mathcal{D}_0 x$ is the Riemann volume "measure" on $H_0(K)$.*

**Proof.** (Meta-Proof) By Theorem 3.2 we have (formally),

$$d\mu_t^e(x) = \frac{1}{p_t^K(e)}\rho_t(x)\delta_e(x_1)\mathcal{D}x = \frac{1}{p_t^K(e)}\frac{1}{\sqrt{\det(\pi_{*x}\pi_{*x}^{\mathrm{tr}})}}\rho_t(x)\mathcal{D}_0 x.$$

So to finish the proof it suffices to show, $\det(\pi_{*x}\pi_{*x}^{\mathrm{tr}}) = 1$.

Let $h \in H(\mathfrak{k})$, then $L_{x*}h \in T_x H(K)$ and

$$\pi_* L_{x*}h = \frac{d}{dt}|_0 \pi(xe^{th}) = \frac{d}{dt}|_0 (x(1)e^{th(1)}) = L_{x(1)*}h(1).$$

So if $\xi \in \mathfrak{k}$,

$$\langle \pi_* L_{x*}h, L_{x(1)*}\xi\rangle = \langle h(1), \xi\rangle = (h, s \to s\xi)_{H(\mathfrak{k})}$$

which implies $\pi_{*x}^{\mathrm{tr}} L_{x(1)*}\xi = L_{x*}(s \to s\xi)$. Therefore

$$\pi_{*x}\pi_{*x}^{\mathrm{tr}} L_{x(1)*}\xi = \pi_{*x}L_{x*}(s \to s\xi) = L_{x(1)*}\xi,$$

i.e. $\pi_{*x}\pi_{*x}^{\mathrm{tr}} = id_{T_{x(1)}K}$ and thus $\det(\pi_{*x}\pi_{*x}^{\mathrm{tr}}) = 1$. ■

This Meta-Theorem suggests the following quasi – invariance theorem.

**Theorem 3.4** (Malliavin & Malliavin [50])**.** *Suppose that $k \in H_0(K)$, then*

$$d\mu_t^e(xk) = J_k(x)d\mu_t^e(x)$$

*where*

$$\begin{aligned} J_k &= \exp\left(-\frac{1}{t}\int_0^1 \langle k'(s)k^{-1}(s), db_s\rangle - \frac{1}{2t}\int_0^1 \left|k'(s)k^{-1}(s)\right|^2 ds\right) \\ &= \exp\left(-\frac{1}{t}\int_0^1 \langle k'(s)k^{-1}(s), db_s\rangle - \frac{1}{2t}E(k)\right) \end{aligned}$$

*and $b_s(x) := \int_0^s x^{-1}(r)\circ dx(r)$. Here $\circ dx(r)$ is used to denote the Fisk – Stratonovich differential of $x$.*

**Proof.** (Meta-Proof) Formally using Eq. (3.1) and the "invariance" of $\mathcal{D}_0 x$,

$$d\mu_t^e(xk) = \frac{1}{p_t^K(e)}\rho_t(xk)\mathcal{D}_0 x = \frac{\rho_t(xk)}{\rho_t(x)}d\mu_t^e(x) =: J_k(x)d\mu_t^e(x)$$

where

$$\begin{aligned} J_k(x) &= \frac{\rho_t(xk)}{\rho_t(x)} = \frac{e^{-\frac{1}{2t}E_K(xk)}}{e^{-\frac{1}{2t}E_K(x)}} = \exp\left(-\frac{1}{2t}\left[E_K(xk) - E_K(x)\right]\right) \\ &= \exp\left(-\frac{1}{2t}\left[E_{\mathfrak{k}}(b(xk)) - E_{\mathfrak{k}}(b(x))\right]\right). \end{aligned}$$

Using the $Ad_K$ invariance of $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$,

$$\begin{aligned}
E_K(xk) &= \int_0^1 \left|(x(s)k(s))^{-1}\frac{d}{ds}(x(s)k(s))\right|_{\mathfrak{k}}^2 ds \\
&= \int_0^1 \left|k(s)^{-1}x(s)^{-1}\left[x'(s)k(s)+x(s)k'(s)\right]\right|_{\mathfrak{k}}^2 ds \\
&= \int_0^1 \left|Ad_{k(s)^{-1}}\left[x(s)^{-1}x'(s)\right]+k(s)^{-1}k'(s)\right|_{\mathfrak{k}}^2 ds \\
&= E_K(x) + 2\int_0^1 \langle Ad_{k(s)^{-1}}\left[x(s)^{-1}x'(s)\right], k(s)^{-1}k'(s)\rangle_{\mathfrak{k}} ds + E(k) \\
&= E_K(x) + 2\int_0^1 \langle b_s'(x), Ad_{k(s)}\left[k(s)^{-1}k'(s)\right]\rangle_{\mathfrak{k}}\, ds + E(k).
\end{aligned}$$

Therefore,

$$\begin{aligned}
J_k(x) &= \exp\left(-\frac{1}{2t}\left[2\int_0^1 \langle b_s'(x), k'(s)k(s)^{-1}\rangle_{\mathfrak{k}}\, ds + E(k)\right]\right) \\
&= \exp\left(-\frac{1}{t}\int_0^1 \langle k'(s)k^{-1}(s), db_s(x)\rangle - \frac{1}{2t}E(k)\right).
\end{aligned}$$

For a rigorous proof see Malliavin [50], Shigekawa [55, 54], Driver [15, 16], Hsu [43], Enchev and Stroock [25, 26] and Lyons and Qian[49]. Also see Albeverio and Hoegh-Krohn [8] for the path group analogue. ■

We now wish to find the "heat equation" solved by $\mu_t^e$.

**Meta-Theorem 3.5.** *The "function" $\tilde{\mu}_t^e(x) := \frac{1}{p_t^K(e)}\rho_t(x)$ on $H_0(K)$ solves the heat equation with potential,*

$$\partial_t\tilde{\mu}_t^e(x) = \left(\frac{1}{2}\triangle_{\mathcal{L}(K)} + V_t\right)\tilde{\mu}_t^e \text{ with } \tilde{\mu}_t^e \to \delta_{\mathbf{e}} \text{ as } t\downarrow 0 \tag{3.2}$$

*where*

$$V_t := \frac{1}{2t^2}\left|b_1\right|_{\mathfrak{k}}^2 - \left(\frac{\dim\mathfrak{k}}{2t} + \partial_t \log p_t^K(e)\right) \tag{3.3}$$

*and $b$ is defined in Eq. (1.7).*

**Proof.** (Meta-Proof.) Let $\beta = \{\xi_i : i = 1,\ldots,\dim\mathfrak{k}\}$ be an orthonormal basis for $\mathfrak{k}$ and let $h_i(s) = s\xi_i$. It is easily seen that $S_0 \cup \{h_i : i = 1,\ldots,\dim\mathfrak{k}\}$ is an orthonormal basis for $H(\mathfrak{k})$ and therefore,

$$\Delta_{H(K)} = \Delta_{H_0(K)} + \sum_{i=1}^{\dim(\mathfrak{k})}\tilde{h}_i^2.$$

Using these remarks and Eq. (2.6),

$$\begin{aligned}\left(\partial_t - \Delta_{H_0(K)}/2\right)\tilde{\mu}_t^e &= \left(\partial_t - \Delta_{H(K)}/2\right)\tilde{\mu}_t^e + \frac{1}{2}\sum_{i=1}^{\dim(\mathfrak{k})} \tilde{h}_i^2 \tilde{\mu}_t^e \\ &= \left(\partial_t \frac{1}{p_t^K(e)}\right)\rho_t + \frac{1}{p_t^K(e)}\left(\partial_t - \Delta_{H(K)}/2\right)\rho_t + \frac{1}{p_t^K(e)}\frac{1}{2}\sum_{i=1}^{\dim(\mathfrak{k})} \tilde{h}_i^2 \rho_t \\ &= -\partial_t \ln(p_t^K(e)) \cdot \tilde{\mu}_t^e + \frac{1}{p_t^K(e)}\frac{1}{2}\sum_{i=1}^{\dim(\mathfrak{k})} \tilde{h}_i^2 \rho_t.\end{aligned}$$

Moreover,

$$\begin{aligned}\tilde{h}_i^2 \rho_t &= \tilde{h}_i\left(-\rho_t \frac{1}{2t}\tilde{h}_i E_K\right) \\ &= \left[\frac{1}{4t^2}\left(\tilde{h}_i E_K\right)^2 - \frac{1}{2t}\tilde{h}_i^2 E_K\right]\rho_t,\end{aligned}$$

and therefore

$$\left(\partial_t - \Delta_{H_0(K)}/2\right)\tilde{\mu}_t^e = V_t \tilde{\mu}_t^e,$$

where

$$V_t = \frac{1}{2}\sum_{i=1}^{\dim(\mathfrak{k})}\left[\frac{1}{4t^2}\left(\tilde{h}_i E_K\right)^2 - \frac{1}{2t}\tilde{h}_i^2 E_K\right] - \partial_t \ln(p_t^K(e)). \tag{3.4}$$

Let us now proceed to work on Eq. (3.4). Making use of Eqs. (1.1), (1.7) and (1.10) we find

$$\begin{aligned}\tilde{h}_i E_K(x) &= \tilde{h}_i E_{\mathfrak{k}}(b(x)) = \tilde{h}_i \int_0^1 \left|b_s'(x)\right|^2 ds \\ &= 2\int_0^1 \langle b_s'(x), \tilde{h}_i b_s'(x)\rangle ds = 2\int_0^1 \langle b_s'(x), h_i'(s) + [b_s'(x), h_i(s)]\rangle ds \\ &= 2\int_0^1 \langle b_s'(x), h_i'(s)\rangle ds = 2\langle b_1(x), \xi_i\rangle \qquad (3.5)\end{aligned}$$

and

$$\tilde{h}_i^2 E_K(x) = 2\int_0^1 \langle h_i'(s) + [b_s'(x), h_i(s)], h_i'(s)\rangle ds = 2\left|\xi_i\right|^2 = 2 \tag{3.6}$$

wherein we have used the $Ad_K$ – invariance of $\langle \cdot, \cdot \rangle$ to conclude

$$\langle [b_s'(x), h_i(s)], h_i'(s)\rangle = s\langle [b_s'(x), \xi_i], \xi_i\rangle = -s\langle b_s'(x), [\xi_i, \xi_i]\rangle = 0.$$

Combining Eqs. (3.4), (3.5) and (3.6) shows $V_t$ may be written as in Eq. (3.3). ■

The following theorem and corollary is a rigorous version of Meta – Theorem 3.5.

**Theorem 3.6** (Airault & Malliavin, [6])**.** *For any smooth cylindrical function $f : \mathcal{L}(K) \to \mathbb{R}$,*

$$\partial_t \mu_t^e(f) = \mu_t^e\left[\left(\frac{1}{2}\triangle_{\mathcal{L}(K)} + V_t\right) f\right], \tag{3.7}$$

*where* $\triangle_{\mathcal{L}(K)} = \sum_{h \in S_0} \tilde{h}_i^2$,

$$V_t := \frac{1}{2t^2} |b_1|_{\mathfrak{k}}^2 - \left( \frac{\dim \mathfrak{k}}{2t} + \partial_t \log p_t^K(e) \right) \text{ with} \tag{3.8}$$

$$b_1(x) := \int_0^1 x(s)^{-1} \circ dx(s). \tag{3.9}$$

**Proof.** See Airault and Malliavin[6] and also Driver and Srimurthy [23] for a simplified proof. ■

**Corollary 3.7.** *Let*

$$F(t,x) := \int_{\mathcal{L}(K)} f(xy^{-1}) d\mu_t^e(y), \tag{3.10}$$

*then*

$$\partial_t F(t,x) = \frac{1}{2} \triangle_{\mathcal{L}(K)} F(t,x) + \int_{\mathcal{L}(K)} V_t(y) f(xy^{-1}) d\mu_t^e(y). \tag{3.11}$$

*where* $V_t$ *is defined in Eq. (3.8).*

**Proof.** (Meta-proof. A rigorous proof of this theorem may be found in Driver and Srimurthy [23].) Writing (formally)

$$F(t,x) = \int_{H_0(K)} f(xy^{-1}) \tilde{\mu}_t^e(y) \mathcal{D}_0 y,$$

we find, as in the proof of Theorem 2.6, that

$$\begin{aligned} \left( \partial_t - \frac{1}{2} \Delta_{H_0(K)} \right) F(t,x) &= \int_{H_0(K)} f(xy^{-1}) \left( \partial_t - \frac{1}{2} \Delta_{H_0(K)} \right) \tilde{\mu}_t^e(y) \mathcal{D}_0 y \\ &= \int_{H_0(K)} f(xy^{-1}) V_t(y) \tilde{\mu}_t^e(y) \mathcal{D}_0 y \\ &= \int_{\mathcal{L}(K)} V_t(y) f(xy^{-1}) d\mu_t^e(y) \end{aligned}$$

where in the second equality we have used Eq. (3.2). ■

3.2. **Gross' Pioneering Results.** Much of the work described above was an outgrowth of the following two fundamental results of Leonard Gross.

**Theorem 3.8** (Gross' Ergodicity Theorem, [37])**.** *Suppose* $\int_{\mathcal{L}(K)} \|gradf\|^2 d\mu_t^e = 0$ *then* $f$ *is constant* $\mu_t^e$ *– a.e. on each homotopy class of* $\mathcal{L}(K)$.

See Gross [37], Aida [1, 3], Leandre [48] and the article by Brian Hall [40] in this volume for more on Gross' ergodicity theorem.

**Theorem 3.9** (Gross' Logarithmic Sobolev Inequality, [36])**.** *For all* $\lambda > 0$ *there are constant* $C$ *and* $B$ *such that*

$$\int_{\mathcal{L}(K)} f^2 \log \left( \frac{f^2}{\mu_t^e(f^2)} \right) d\mu_t^e \leq C \int_{\mathcal{L}(K)} \left\{ \|grad_0 f\|^2 + \left( \lambda |b_1|^2 + B \right) f^2 \right\} d\mu_t^e, \tag{3.12}$$

*for all non-zero smooth cylinder functions,* $f : \mathcal{L}(K) \to \mathbb{R}$.

**Proof.** The reader is referred to [36] for Gross' original proof. His proof is very geometrical and fits nicely into the picture presented in this article. The reader should also consult Getzler [31, 32], Aida [2], and Gong and Ma [33]. ■

**Open Question:** Is is possible to take $\lambda = 0$ in Eq. (3.12)?

Recent work of Eberle [24] has given examples of compact simply connected Riemannian manifolds where the analogue of Eq. (3.12) does **not** hold with $\lambda = 0$. These manifolds have a "dumb bell" shape unlike the very symmetric Lie group case. It is still not known if there exists any compact Riemannian manifolds so that Eq. (3.12) holds with $\lambda = 0$. However, it is known from [22] that Eq. (3.12) does hold when $d\mu_t^e$ is replaced by the heat kernel measure $\nu_t^e$ on $\mathcal{L}(K)$ (see Section 4 below). It also known by the work of Gong, Röckner, and Wu [34] that by modifying pinned Wiener measure by certain positive densities, the weaker Poincaré inequality can be made to hold on the loop space of any compact Riemannian manifold. These results make the question of whether Eq. (3.12) holds with $\lambda = 0$ all the more intriguing.

## 4. Comparing Heat Kernel and Pinned Wiener Measures

**Definition 4.1** (Heat Kernel Measure)**.** The heat kernel measures on $\mathcal{L}(K)$ are formed by the one parameter family of probability measures $\{\nu_t^e : t \geq 0\}$ satisfying $\nu_0^e = \lim_{t\downarrow 0} \nu_t^e = \delta_{\mathbf{e}}$ and

$$\partial_t \nu_t^e(f) = \frac{1}{2}\nu_t^e\left(\triangle_{H_0(K)} f\right)$$

for all smooth cylinder functions $f : \mathcal{L}(K) \to \mathbb{R}$.

The reader is referred to Malliavin [51], Driver and Lohrenz [22], and Driver and [17] for the existence of $\nu_t^e$. Our description of the results in this section will be rather brief compared to the previous sections. This is because to understand the heat kernel measure, $\nu_t^e$, one must go to the path space of $\mathcal{L}(K)$ which is beyond the scope of this article. Nevertheless, I would like to include some basic properties of $\nu_t^e$ and its relationship to $\mu_t^e$.

**Theorem 4.2** (Heat Kernel Logarithmic Sobolev Theorem, [22])**.** *There is a constant $C < \infty$ such that*

$$\int_{\mathcal{L}(K)} f^2 \log \frac{f^2}{\nu_t^e(f^2)} d\nu_t^e \leq C \int_{\mathcal{L}(K)} \|grad_0 f\|^2 \, d\nu_t^e \tag{4.1}$$

*for all smooth cylinder functions $f : \mathcal{L}(K) \to \mathbb{R}$.*

**Proof.** See Driver and Lohrenz [22], Carson [13, 14] and Fang [29] ■

**Corollary 4.3** (Quasi-invariance for heat kernel measure)**.** *For each $k \in H_0(G)$ which is null homotopic, $\nu_t^e$ quasi-invariant under the right and left translations by $k$.*

**Proof.** See Driver [17, 18] and Fang [28, 29]. The free loop space version of these results was carried out by Trevor Carson in [13, 14]. ■

**Theorem 4.4** (V. Srimurthy,[57])**.** *Let $\mathcal{F}_s$ denote the $\sigma$ – algebra on $W(K)$ generated by the coordinate functions $x \in W(K) \to x(r) \in K$ for $r \leq s$. Then for any $s < 1$, the measures $\nu_t^e$ and $\mu_t^e$ are mutually absolutely continuous on $\mathcal{F}_s$.*

**Proof.** This is proved in Srimurthy [57] using two parameter stochastic calculus along the lines developed by J. Norris [52]. ■

More generally we have the following two theorems.

**Theorem 4.5** (Driver & Srimurthy [23]). *Heat kernel measure $\nu_t^e$ is absolutely continuous relative to pinned Wiener measure $\mu_t^e$ for all $t>0$. Moreover, $d\nu_t^e/d\mu_t^e$ is a bounded function on $\mathcal{L}(K)$.*

**Proof.** (Sketch of the proof.) By standard heat kernel asymptotics, one shows $c_t := \frac{\dim \mathfrak{k}}{2t} + \partial_t \log p_t^K(e) = O(|\ln t|)$ as $t \downarrow 0$ so that

$$C_t := \int_0^t c_\tau d\tau = \int_0^t \left( \frac{\dim \mathfrak{k}}{2\tau} + \partial_\tau \log p_\tau^K(e) \right) d\tau \tag{4.2}$$

is a continuous function for $t \geq 0$. Let $f : \mathcal{L}(K) \to [0,\infty)$ be a cylinder function and define

$$F(t,x) := e^{C_t} \int_{\mathcal{L}(K)} f(xy^{-1}) d\mu_t^e(y)$$

and

$$G(t,x) := \int_{\mathcal{L}(K)} f(xy^{-1}) d\nu_t^e(y).$$

Then, by Corollary 3.7 and the definition of $\nu_t^e$, $F$ and $G$ satisfy

$$\begin{aligned} \left(\partial_t - \frac{1}{2}\Delta_{H_0(K)}\right) F(t,x) &= \int_{\mathcal{L}(K)} \left[V_t(y) + \dot{C}_t\right] f(xy^{-1}) d\mu_t^e(y) \\ &= \int_{\mathcal{L}(K)} \frac{1}{2t^2} |b_1(y)|^2 f(xy^{-1}) d\mu_t^e(y) \geq 0. \end{aligned}$$

and

$$\left(\partial_t - \frac{1}{2}\Delta_{H_0(K)}\right) G(t,x) = 0.$$

Since both $G(0,x) = F(0,x)$, we may apply the Maximum principle (or Du Hamel's principle) to conclude that

$$F(t,x) \geq G(t,x)$$

for all $(t,x)$. Setting $x = \mathbf{e}$ gives

$$e^{C_t} \int_{\mathcal{L}(K)} f(y) d\mu_t^e(y) \geq \int_{\mathcal{L}(K)} f(y) d\nu_t^e(y) \tag{4.3}$$

for all positive cylinder functions on $\mathcal{L}(K)$. By standard measure theoretic arguments, it then follows that Eq. (4.3) holds for all bounded measurable $f$ from which the result easily follows. ■

**Theorem 4.6** (Aida & Driver [4]). *For each $h \in H_0(K)$, let $\nu_t^e(h,A) := \nu_t^e(h^{-1}A)$ – heat kernel measure on $\mathcal{L}(K)$ starting at $h$. Let $\Pi \subset H_0(K)$ be chosen so that to each homotopy class in $\mathcal{L}(K)$, there is a unique representative in $\Pi$. Then $\mu_t^e$ is absolutely continuous relative to $\sum_{h\in\Pi} \nu_t^e(h,\cdot)$. In particular, if $K$ is simply connected, then $\mu_t^e$ and $\nu_t^e$ are mutually absolutely continuous.*

**Proof.** I will only give a sketch of the proof when $K$ is simply connected. By Theorem 4.5, $Z_t(x) = \frac{d\nu_t^e}{d\mu_t^e}(x)$ exists. Let $N := \{x \in \mathcal{L}(K) : Z_t(x) = 0\}$. Using the quasi-invariance of the pinned measure $\mu_t^e$ (Theorem 3.4) and of $\nu_t^e$ (Corollary 4.3) under left translation by $H_0(K)$, one shows that $h \cdot N = N$ modulo sets of $\mu_t^e$ –

measure zero for all $h \in H_0(K)$. Using Gross' ergodicity Theorem 3.8, it follows that either $\mu_t^e(N) = 0$ or that $\mu_t^e(\mathcal{L}(K) \setminus N) = 0$. Since $\nu_t^e$ is not the zero measure, we conclude that $\mu_t^e(N) = 0$, i.e. that $Z(x) > 0$ on $\mathcal{L}(K)$ for $\mu_t^e$ - a.e. $x$ and hence $\mu_t^e$ is absolutely continuous relative to $\nu_t^e$. ■

4.1. **Feynman-Kac Formula for Pinned Wiener Measure.** At this point one should expect when $K$ is simply connected that $d\mu_t^e/d\nu_t^e$ can be expressed in terms of a Feynman – Kac type formula. This is technically problematic to prove owing to the singular behavior of the potential $V_t$ as $t \downarrow 0$. Xiang - Dong Li and the author have been able to find partial results in this direction. The statement of these formula require the notion of Brownian motion on the $\mathcal{L}(K)$ and hence will be omitted. Let me mention though that these formula do begin to give more quantitative information about $d\mu_t^e/d\nu_t^e$ and can be used to give another proof of Theorem 4.5.

## Appendix A. A Wong - Zakai Approximation Theorem

A.1. **Basic notation and the Theorem.** Let $n \in \mathbb{N}$ and $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$ and $T \in (0, \infty)$. Let $gl_n$ denote the $n \times n$ matrices with entries from $\mathbb{F}$ and $GL_n$ – denote the invertible matrices in $gl_n$. For $A, B \in gl_n$ let $(A, B) = tr(A^*B)$ and $|A| := \sqrt{(A, A)}$. We will also use $\|A\|$ to denote the operator norm of $A$ of any bounded operator on an inner product space.

*Remark* A.1. We will use, without further comment, the following basic properties of the Hilbert Schmidt norm $|A|$,

(1) If $I \in gl_n$ is the identity matrix, then $|I| = \sqrt{n}$.
(2) $\|A\| \leq |A|$ and $|A^*| = |A|$.
(3) If $A, B \in gl_n$ then $|AB| \leq \|A\| |B| \leq |A| |B|$ and $|AB| \leq |A| \|B\| \leq |A| |B|$ and for all $p \geq 0$,

$$\left|Ae^B\right|^p \leq |A|^p \left\|e^B\right\|^p \leq |A|^p e^{p\|B\|} \leq |A|^p e^{p|B|}.$$

Now suppose that $W_t$ is a $gl_n$ – valued Brownian motion, i.e. $W_t = \sum_{A \in \beta} W_t^A A$ where $\beta \subset gl_n$ is an $\mathbb{R}$ – linearly independent subset of $gl_n$ and $\left\{W_t^A\right\}_{A \in \beta}$ is a collection of independent $\mathbb{R}$ – valued Brownian motions.

A partition, $\pi$, of $[0, T]$ is a set of the form

$$\pi = \{0 = t_0 < t_1 < t_2 < \cdots < t_N = T\}.$$

For $t \in J_n := [t_n, t_{n+1})$ we will write $t_-$ for $t_n$, $t_+$ for $t_{n+1}$, $\Delta_n := t_{n+1} - t_n$, $\Delta_n W := W(t_{n+1}) - W(t_n)$ and

$$W^\pi(t) := W(t_-) + (t - t_-)\left(\frac{W(t_+) - W(t_-)}{t_+ - t_-}\right).$$

We will also write $\Delta_t W := W(t_+) - W(t_-)$ and $\Delta_t = t_+ - t_-$. Notice that $\dot{W}^\pi(t) = \Delta_t W/\Delta_t$ for $t \notin \pi$. As usual, define

$$|\pi| := \max\{\Delta_n : n = 0, 1, \ldots, N-1\}.$$

**Definition A.2** (SDE)**.** Let $g_t$ be the $GL_n$ – valued process solving the stochastic differential equation,

$$\begin{aligned} dg &= g \circ dW \text{ with } g_0 = I \\ &= gdW + \frac{1}{2}g(dW)^2 = gdW + \frac{1}{2}g\eta dt \end{aligned} \tag{A.1}$$

where $\eta := \sum_{A\in\beta} A^2$.

**Definition A.3** (Wong – Zakai Approximation)**.** The Wong – Zakai approximation associated to $g$ and a partition $\pi$ is $g_\pi(t) = y(t)$ where $y$ solves the ordinary differential equation,

$$\dot{y}(t) = y(t)\dot{W}^\pi(t) \text{ with } y(0) = I.$$

Explicitly if $t \in J_n$ then

$$g_\pi(t) = e^{\Delta_0 W} e^{\Delta_1 W} \dots e^{\Delta_{n-1} W} e^{\frac{(t-t_n)}{\Delta_n}\Delta_n W}.$$

**Theorem A.4** (Wong – Zakai Approximation Theorem)**.** *For any $p \geq 2$ and there is a constant $C < \infty$ such that*

$$\sup_{t\in\pi} \mathbb{E}\left|g_\pi(t) - g_t\right|^p \leq \mathbb{E}\sup_{t\in\pi}\left|g_\pi(t) - g_t\right|^p \leq C\left|\pi\right|^{p/2} \tag{A.2}$$

*and*

$$\mathbb{E}\sup_{t\leq T}\left|g_\pi(t) - g_t\right|^p \leq C\left|\pi\right|^{(p/2)-1}. \tag{A.3}$$

*Remark* A.5. A similar proof as below shows Theorem A.4 also holds if we assume $g$ and $g_\pi$ solve:

$$dg = dW \circ g \text{ with } g_0 = I$$

and $g_\pi = y$ where $y$ solves

$$\dot{y}(t) = \dot{W}^\pi(t)y(t) \text{ with } y(0) = I.$$

In this case

$$g_\pi(t) = e^{\Delta_{n-1} W} e^{\frac{(t-t_n)}{\Delta_n}\Delta_n W} \dots e^{\Delta_1 W} e^{\Delta_0 W}.$$

Theorems of this type have a long history starting with Wong and Zakai [59], also see Bismut [10], Ikeda and Watanabe [45] , Kloeden and Platen [46] and Hu [44] to name a few references. Despite all of these references, it seems worthwhile to sketch a proof here in the Lie group case for two reasons. Firstly, the main ideas are more easily seen in this setting and secondly, it is still hard to find results of this nature when the coefficients of the stochastic differential equations are not bounded. This later case is needed when working on the complexified Lie groups as in Brian Hall's article. In a forthcoming paper [21], Hu and the author will give a more general version of the Theorem A.4 which will include a better convergence estimate than the one described in Eq. (A.3). The rest of this appendix will be devoted to the proof of Theorem A.4. Throughout the proof, the letter $C$ will be used to denote a constant **not** depending on $\pi$. This constant may vary from line to line.

A.2. **Bounds on the approximate solution.**

**Proposition A.6.** *Let $y = g_\pi$ as above. There exists $C < \infty$ such that*

$$\sup_{t\in\pi} \mathbb{E}\,|y(t)|^p \le n^{p/2} e^{Cp^2t/2}\ \forall\, 2 \le p \le |\pi|^{-1/2}. \tag{A.4}$$

*Moreover, for each $p \in [2,\infty)$ there is a constant $C_p < \infty$ such that*

$$\mathbb{E}\left[\sup_{t\in J_n} |y(t) - y(t_n)|^p\right] \le C_p \Delta_n^{p/2} \le C_p\,|\pi|^{p/2}\ \ \forall\, n = 0, 1, \ldots, N-1. \tag{A.5}$$

**Proof.** Taylor theorem with integral remainder states,

$$f(t_+) - f(t_-) = \dot f(t_-)(t_+ - t_-) + \frac{1}{2}(t_+ - t_-)^2 \int_{t_-}^{t_+} \ddot f(t)d\nu(t) \tag{A.6}$$

where $d\nu(t) = 2(t_+ - t)/(t_+ - t_-)^2 dt$ – a probability measure on $[t_-, t_+]$. Let $f(t) = |y(t)|^p$ , then elementary computations and estimates show,

$$\dot f(t) = p\,|y(t)|^{p-2}\operatorname{Re}(y(t), y(t)\dot W^\pi(t)),$$

$$\begin{aligned}\ddot f(t) = p(p-2)\,|y(t)|^{p-4}&\left[\operatorname{Re}(y(t), y(t)\dot W^\pi(t))\right]^2\\ &+ p\,|y(t)|^{p-2}\operatorname{Re}\left[(y(t)\dot W^\pi(t), y(t)\dot W^\pi(t)) + (y(t), y(t)\dot W^\pi(t)\dot W^\pi(t))\right]\end{aligned}$$

and

$$\left|\ddot f(t)\right| \le (p(p-2) + 2p)\,|y(t)|^p \left|\dot W^\pi(t))\right|^2 = p^2\,|y(t)|^p \left|\dot W^\pi(t))\right|^2. \tag{A.7}$$

Therefore by Eq. (A.6)),

$$|y(t_+)|^p - |y(t_-)|^p = p\,|y(t_-)|^{p-2}\operatorname{Re}(y(t_-), y(t_-)\dot W^\pi(t_-)) + R \tag{A.8}$$

where the remainder,

$$R = \frac{1}{2}(t_+ - t_-)^2 \int_{t_-}^{t_+} \frac{d^2}{dt^2}\,|y(t)|^p\,d\nu(t),$$

satisfies (by Eq. (A.7),

$$|R| \le \frac{1}{2}(t_+ - t_-)^2 \int_{t_-}^{t_+} p^2\,|y(t)|^p \left|\dot W^\pi(t))\right|^2 d\nu(t) = \frac{p^2}{2}\,|\Delta_t W|^2 \int_{t_-}^{t_+} |y(t)|^p\,d\nu(t).$$

Taking expectations of Eq. (A.8) gives,

$$|\mathbb{E}\,|y(t_+)|^p - \mathbb{E}\,|y(t_-)|^p| = |\mathbb{E}R| \le \frac{p^2}{2}\int_{t_-}^{t_+} \mathbb{E}\left[|y(t)|^p\,|\Delta_t W|^2\right] d\nu(t) \tag{A.9}$$

Since

$$y(t) = y(t_-)\exp\Big((t - t_-)\,\dot W^\pi(t))\Big) = y(t_-)\exp\left(\frac{t - t_-}{t_+ - t_-}\Delta_t W\right), \tag{A.10}$$

Remark A.1 implies

$$|y(t)|^p = |y(t_-)|^p \exp\left(p\frac{t - t_-}{t_+ - t_-}\,|\Delta_t W|\right) \le |y(t_-)|^p\, e^{p|\Delta_t W|}$$

and combining this equation with Eq. (A.9) gives

$$\begin{aligned}\left|\mathbb{E}\left|y(t_+)\right|^p - \mathbb{E}\left|y(t_-)\right|^p\right| &\le \frac{p^2}{2}\int_{t_-}^{t_+} \mathbb{E}\left[\left|y(t_-)\right|^p e^{p\left|\Delta_t W\right|}\left|\Delta_t W\right|^2\right] d\nu(t) \\ &= \frac{p^2}{2}\mathbb{E}\left[\left|y(t_-)\right|^p\right]\mathbb{E}\left[e^{p\left|\Delta_t W\right|}\left|\Delta_t W\right|^2\right].\end{aligned}$$

Using Taylor's theorem with remainder and the fact that $s \to \mathbb{E}\left[e^{s|W_1|}\left|W_1\right|^2\right]$ is a smooth function,

$$\mathbb{E}\left[e^{p\left|\Delta_t W\right|}\left|\Delta_t W\right|^2\right] = \Delta_t \cdot \mathbb{E}\left[e^{p\sqrt{\Delta_t}|W_1|}\left|W_1\right|^2\right] = \Delta_t\left(C_0 + O\left(p\sqrt{\Delta_t}\right)\right) \le C\Delta_t$$

where the last inequality holds provided $0 \le p\sqrt{\Delta_t} \le 1$. Putting this all together implies,

$$\begin{aligned}\mathbb{E}\left|y(t_+)\right|^p &\le \mathbb{E}\left|y(t_-)\right|^p + C\frac{p^2}{2}\mathbb{E}\left[\left|y(t_-)\right|^p\right](t_+ - t_-) \\ &= \mathbb{E}\left|y(t_-)\right|^p\left[1 + C\frac{p^2}{2}(t_+ - t_-)\right] \le \mathbb{E}\left|y(t_-)\right|^p e^{Cp^2(t_+ - t_-)/2}\end{aligned}$$

which upon iteration (using $y(0) = y(t_0) = I$) proves Eq. (A.4).

For Eq. (A.5), the fundamental theorem of calculus shows

$$\begin{aligned}\left|y(t) - y(t_-)\right| &= \left|\int_{t_-}^t y(\tau)\dot{W}_\pi(\tau)d\tau\right| = \left|\int_{t_-}^t y(t_-)\exp\left(\left(\tau - t_-\right)\dot{W}^\pi(\tau)\right)\right)\dot{W}_\pi(\tau)d\tau\right| \\ &\le \int_{t_-}^t \left|y(t_-)\right|\left|\dot{W}_\pi(\tau)\right|\exp\left(\left(\tau - t_-\right)\left|\dot{W}^\pi(\tau)\right)\right|\right)d\tau \\ &\le \left|y(t_-)\right|\left|\Delta_t W\right| e^{\left|\Delta_t W\right|}.\end{aligned}$$

and therefore

$$\mathbb{E}\left[\sup_{t\in J_n}\left|y(t) - y(t_n)\right|^p\right] \le \mathbb{E}\left|y(t_n)\right|^p \cdot \mathbb{E}\left(\left|\Delta_n W\right|^p e^{p\left|\Delta_n W\right|}\right) \le C_p\Delta_n^{p/2} \le C\left|\pi\right|^{p/2}.$$

■

A.3. **Bounds on the solution.**

**Proposition A.7.** *If $p \in [2, \infty)$, there exists a constant $C_p < \infty$ such that*

$$\mathbb{E}\left|g_t\right|^p \le \mathbb{E}\sup_{t\le T}\left|g_t\right|^p \le C_p e^{C_p t} \le C_p e^{C_p T} \text{ for all } t \in [0, T] \tag{A.11}$$

*and*

$$\mathbb{E}\sup_{t\in J_n}\left|g_t - g_{t_n}\right|^p \le C_p\Delta_n^{p/2} \le C_p\left|\pi\right|^{p/2}\ \forall\ n = 0, 1, \ldots, N-1. \tag{A.12}$$

*for all $n$.*

**Proof.** The estimate in Eq. (A.11) is very standard and follows from the Picard iteration method for solving Eq. (A.1). Alternatively one may prove this using Burkholder's and Gronwall's inequalities, see [58] for example. Let $q = \frac{p/2}{p/2-1} = \frac{p}{p-2}$ and $g_T^* := \sup_{t\le T}\left|g_t\right|$. By the definition of $g$ in Eq. (A.1),

$$g_t - g_{t_-} = \int_{t_-}^t g dW + \frac{1}{2}\int_{t_-}^t g\eta dt$$

and so by Burkholder's inequality

$$
\begin{aligned}
\mathbb{E}\sup_{t\in J_n}|g_t-g_{t_n}|^p &\le C_p\left(\mathbb{E}\left[\int_{t_n}^{t_{n+1}}\sum_{A\in\beta}|g_tA|^2\,dt\right]^{p/2}+\mathbb{E}\left[\int_{t_n}^{t_{n+1}}|g_t\eta|\,dt\right]^p\right)\\
&\le C_p\left(\mathbb{E}\left(g_T^*\right)^p|t_{n+1}-t_n|^{p/2}+\mathbb{E}\left(g_T^*\right)^p|t_{n+1}-t_n|^p\right)\\
&\le C_p\Delta_n^{p/2}\le C_p|\pi|^{p/2}
\end{aligned}
$$

which proves Eq. (A.12). ∎

A.4. **Matrix exponential properties and estimates.**

**Notation A.8.** For $A,B\in gl_n$ let

$$\partial_Ae^B:=\frac{d}{ds}|_0e^{B+sA}\text{ and }\Lambda_B(A):=e^{-B}\partial_Ae^B$$

and

$$\partial_A^2e^B:=\frac{d^2}{ds^2}|_0e^{B+sA}\text{ and }\Gamma_B(A,A):=e^{-B}\partial_A^2e^B.$$

**Proposition A.9.** *Using the notation above,*

$$\Lambda_B(A)=\int_0^1e^{-\tau ad_B}Ad\tau \tag{A.13}$$

*and*

$$\Gamma_B(A,A)=[\Lambda_B(A)]^2-\int_0^1\int_0^1\tau\left([e^{-s\tau ad_B}A,e^{-\tau ad_B}A]\right)d\tau ds \tag{A.14}$$

*Moreover there exists $C\in(0,\infty)$ such that*

$$\|\Lambda_B-I\|:=\sup_{|A|=1}|\Lambda_B(A)-A|\le C|B|e^{C|B|} \tag{A.15}$$

*and*

$$\left|\Gamma_B(A,A)-A^2\right|\le C\left(|B|+|B|^2\right)e^{C|B|}|A|^2. \tag{A.16}$$

**Proof.** Integrating the identity,

$$
\begin{aligned}
\frac{d}{dt}\left[\left(\partial_Ae^{tB}\right)e^{-tB}\right]&=\left[\partial_A\left(e^{tB}B\right)e^{-tB}-\left(\partial_Ae^{tB}\right)Be^{-tB}\right]\\
&=\left[\left(\partial_Ae^{tB}\right)Be^{-tB}+e^{tB}Ae^{-tB}-\left(\partial_Ae^{tB}\right)Be^{-tB}\right]\\
&=e^{tB}Ae^{-tB},
\end{aligned}
$$

shows

$$
\begin{aligned}
\partial_Ae^B&=\left[\int_0^1e^{\tau B}Ae^{-\tau B}d\tau\right]e^B=\int_0^1e^{\tau B}Ae^{(1-\tau)B}d\tau=\int_0^1e^{(1-\tau)B}Ae^{\tau B}d\tau\\
&=e^B\int_0^1e^{-\tau B}Ae^{\tau B}d\tau=e^B\Lambda_B(A)
\end{aligned} \tag{A.17}
$$

which proves Eq. (A.13). Differentiating Eq. (A.17) gives

$$\partial_A^2e^B=e^B[\Lambda_B(A)]^2+e^B\int_0^1\left(\partial_Ae^{-\tau ad_B}\right)Ad\tau, \tag{A.18}$$

where $ad_A B := [A, B]$. Applying Eq. (A.13) to $e^{-\tau ad_B}$ and using $\partial_A(-\tau ad_B) = -\tau ad_A$ we learn

$$\partial_A e^{-\tau ad_B} = -\tau \int_0^1 e^{(s-1)\tau ad_B} ad_A e^{-s\tau ad_B} ds. \tag{A.19}$$

Eq. (A.14) follows from Eqs. (A.18) and (A.19) after a bit of algebra.

Now for the estimates in Eqs. (A.15) and (A.16). By the fundamental theorem of calculus

$$\begin{aligned}\left\|e^{-ad_B} - I\right\| &= \left\|\int_0^1 -ad_B e^{-sad_B} ds\right\| \le \int_0^1 ds \left\|ad_B e^{-sad_B}\right\| \\ &\le \int_0^1 ds \left\|ad_B\right\| \cdot \left\|e^{-sad_B}\right\| \le \left\|ad_B\right\| \cdot e^{\left\|ad_B\right\|} \le C\left|B\right| e^{C|B|}. \end{aligned}\tag{A.20}$$

and hence

$$\left\|\Lambda_B - I\right\| = \left\|\int_0^1 \left(e^{-\tau ad_B} - I\right) d\tau\right\| \le \int_0^1 \left\|e^{-\tau ad_B} - I\right\| d\tau \le C\left|B\right| e^{C|B|}$$

which proves Eq. (A.15).

By Eq. (A.20),

$$R_B(A) := A - e^{-ad_B} A$$

satisfies $\left\|R_B\right\| \le C\left|B\right| e^{C|B|}$. Because $[A, A] = 0$,

$$[e^{-s\tau ad_B} A, e^{-\tau ad_B} A] = [R_{-s\tau B}(A), A] + [A, R_{-\tau B}(A)] + [R_{-s\tau B}(A), R_{-\tau B}(A)]$$

and thus

$$\left|\int_0^1 \int_0^1 \tau \left([e^{-s\tau ad_B} A, e^{-\tau ad_B} A]\right) d\tau ds\right| \le C\left(\left|B\right| + \left|B\right|^2\right) e^{C|B|} \left|A\right|^2.$$

Similarly,

$$\begin{aligned}\left|\left[\Lambda_B(A)\right]^2 - A^2\right| &= \left|\left[A + R_B(A)\right]^2 - A^2\right| \\ &= \left|AR_B(A) + R_B(A)A + \left[R_B(A)\right]^2\right| \\ &\le C\left(\left|B\right| + \left|B\right|^2\right) e^{C|B|} \left|A\right|^2.\end{aligned}$$

Combining these estimates with the definition of $\Gamma_B$ in Eq. (A.14) proves Eq. (A.16). ■

A.5. **Proof of Theorem A.4. Proof.** We are now ready to complete the proof of Theorem A.4. Let us begin by introducing a third process:

$$z_t = z(t) = e^{\Delta_0 W} e^{\Delta_1 W} \dots e^{\Delta_{n-1} W} e^{W_t - W_{t_n}} \text{ for } t \in J_n.$$

Notice that $z$ is adapted whereas $y = g_\pi$ is not. However $z = y$ on $\pi$ and $z_t = g_{\pi \cup \{t\}}(t)$.

Itô's formula (for $t \ge t_n$) implies

$$de^{W_t - W_{t_n}} = e^{W_t - W_{t_n}} \Lambda_{W_t - W_{t_n}}(dW_t) + \frac{1}{2} e^{W_t - W_{t_n}} \Gamma_{W_t - W_{t_n}}(dW_t, dW_t)$$

from which it follows that

$$dz_t = z_t \left[\Lambda_{W_t - W_{t_-}}(dW_t) + \frac{1}{2}\Gamma_{W_t - W_{t_-}}(dW_t, dW_t)\right]. \tag{A.21}$$

We are now going to compare $z$ and $g$. In order to do this we will need the SDE solved by $g^{-1}$ :

$$dg_t^{-1} = -dW_t \circ g_t^{-1} = -dW_t g_t^{-1} + \frac{1}{2}(dW_t)^2 g_t^{-1} \text{ with } g_0 = I. \tag{A.22}$$

Letting $Q := zg^{-1}$ and using Eqs. (A.21) and (A.22) we find

$$\begin{aligned}
dQ_t &= z_t \left[\Lambda_{W_t - W_{t_}}(dW_t) + \frac{1}{2}\Gamma_{W_t - W_{t_}}(dW_t, dW_t)\right] g_t^{-1} \\
&\quad + z_t \left[-dW_t + \frac{1}{2}(dW_t)^2\right] g_t^{-1} - z_t \left[\Lambda_{W_t - W_{t_}}(dW_t)\right] dW_t g_t^{-1} \\
&= z_t \left[\Lambda_{W_t - W_{t_}}(dW_t) - dW_t\right] g_t^{-1} \\
&\quad + z_t \left[\frac{1}{2}\left\{\Gamma_{W_t - W_{t_}}(dW_t, dW_t) + (dW_t)^2\right\} - \left[\Lambda_{W_t - W_{t_}}(dW_t)\right] dW_t\right] g_t^{-1} \\
&= z_t \left[\Lambda_{W_t - W_{t_}}(dW_t) - dW_t\right] g_t^{-1} \\
&\quad + \frac{1}{2}\sum_{A \in \beta} z_t \left[\frac{1}{2}\left\{\Gamma_{W_t - W_{t_}}(A, A) + A^2\right\} - \left[\Lambda_{W_t - W_{t_}}(A)\right] A\right] g_t^{-1} dt.
\end{aligned}$$

From this equation and Burkholder's inequality,

$$\begin{aligned}
\mathbb{E} \sup_{0 \le t \le \tau} |Q_t - I|^p &\le C\mathbb{E}\left[\int_0^\tau \sum_A \left| z_t \left[\Lambda_{W_t - W_{t_}}(A) - A\right] g_t^{-1}\right|^2 dt\right]^{p/2} \\
&\quad + C\mathbb{E}\left[\int_0^\tau \left|\sum_A z_t \left[\frac{1}{2}\left(\Gamma_{W_t - W_{t_}}(A, A) + A^2\right) - \left[\Lambda_{W_t - W_{t_}}(A)\right] A\right] g_t^{-1}\right| dt\right]^p \\
&\le C\tau^{p/2}\mathbb{E}\int_0^\tau \sum_A |z_t|^p \left|g_t^{-1}\right|^p \left|\Lambda_{W_t - W_{t_}}(A) - A^2\right|^p \frac{dt}{\tau} \\
&\quad + C\tau^p \mathbb{E}\int_0^\tau \sum_A |z_t|^p \left|g_t^{-1}\right|^p \left|\frac{1}{2}\left(\Gamma_{W_t - W_{t_}}(A, A) + A^2\right) - \left[\Lambda_{W_t - W_{t_}}(A)\right] A\right|^p \frac{dt}{\tau}.
\end{aligned} \tag{A.23}$$

From our estimates in Eq. (A.15) and (A.16),

$$\left|\Lambda_{W_t - W_{t\_}}(A) - A\right| \le C \left|W_t - W_{t\_}\right| e^{C\left|W_t - W_{t\_}\right|} \text{ and}$$

$$\begin{aligned}
&\left|\frac{1}{2}\left(\Gamma_{W_t - W_{t_}}(A, A) + A^2\right) - \left[\Lambda_{W_t - W_{t_}}(A)\right] A\right| \\
&\qquad \le C\left(\left|W_t - W_{t_}\right| + \left|W_t - W_{t_}\right|^2\right) e^{C\left|W_t - W_{t_}\right|}
\end{aligned}$$

which combined with Eq. (A.23) leads to

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le\tau}|Q_t-I|^p &\le C\tau^{p/2-1}\mathbb{E}\int_0^\tau |z_t|^p\left|g_t^{-1}\right|^p\left|W_t-W_{t_-}\right|^p e^{pC\left|W_t-W_{t_-}\right|}dt\\ &\quad+C\tau^{p-1}\mathbb{E}\int_0^\tau |z_t|^p\left|g_t^{-1}\right|^p\left|C\left(\left|W_t-W_{t_-}\right|+\left|W_t-W_{t_-}\right|^2\right)e^{C\left|W_t-W_{t_-}\right|}\right|^p dt\\ &\le CT^{p/2-1}\mathbb{E}\int_0^T |z_t|^p\left|g_t^{-1}\right|^p\left|W_t-W_{t_-}\right|^p e^{pC\left|W_t-W_{t_-}\right|}dt\\ &\quad+CT^{p-1}\mathbb{E}\int_0^T |z_t|^p\left|g_t^{-1}\right|^p\left(\left|W_t-W_{t_-}\right|+\left|W_t-W_{t_-}\right|^2\right)^p e^{pC\left|W_t-W_{t_-}\right|}dt.\end{aligned}\tag{A.24}$$

Let $\delta := t-t_-\le\Delta_t$, then

$$\begin{aligned}\mathbb{E}\left[\left|W_t-W_{t_-}\right|^\rho e^{C\left|W_t-W_{t_-}\right|}\right] &= \mathbb{E}\left[\left|\sqrt{\delta}W_1\right|^\rho e^{C\left|\sqrt{\delta}W_1\right|}\right]\\ &= \delta^{\rho/2}\mathbb{E}\left[|W_1|^\rho e^{C\sqrt{\delta}|W_1|}\right]\le C\Delta_t^{\rho/2}.\end{aligned}$$

So using this estimate, Propositions A.6 and A.7 along with Hölder's inequality implies there exists a constant $C$ such that

$$\begin{aligned}&\mathbb{E}[|z_t|^p\left|g_t^{-1}\right|^p\left|W_t-W_{t_-}\right|^p e^{pC\left|W_t-W_{t_-}\right|}]\\ &\qquad\le\left(\mathbb{E}|z_t|^{4p}\right)^{1/4}\left(\mathbb{E}\left|g_t^{-1}\right|^{4p}\right)^{1/4}\left(\mathbb{E}\left[\left|W_t-W_{t_-}\right|^{2p}e^{2pC\left|W_t-W_{t_-}\right|}\right]\right)^{1/2}\\ &\qquad\le C\Delta_t^{p/2}\le C|\pi|^{p/2}.\end{aligned}$$

Using this estimate in Eq. (A.24) shows there exists a constant $C<\infty$ such that

$$\sup_{t\in[0,T]}\mathbb{E}|Q_t-I|^p\le\mathbb{E}\sup_{t\in[0,T]}|Q_t-I|^p\le C|\pi|^{p/2}.$$

One more application of Proposition A.7 and Hölder's inequality shows

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}|z_t-g_t|^p &= \mathbb{E}\sup_{t\in[0,T]}\left|\left(z_tg_t^{-1}-I\right)g_t\right|^p\\ &\le\left(\mathbb{E}\sup_{t\in[0,T]}|Q_t-I|^{2p}\,\mathbb{E}\sup_{t\in[0,T]}|g_t|^{2p}\right)^{1/2}\le C|\pi|^{p/2}\end{aligned}$$

which proves Eq. (A.2).

For $t\in J_n$,

$$\begin{aligned}|y_t-g_t| &\le |y_t-y_{t_n}|+|y_{t_n}-g_{t_n}|+|g_t-g_{t_n}|\\ &\le\sup_{\tau\in J_n}|y_\tau-y_{\tau_n}|+\sup_{\tau\in J_n}|g_\tau-g_{\tau_n}|+\sup_{\tau\in\pi}|y_\tau-g_\tau|\end{aligned}$$

and therefore,

$$\begin{aligned}
&\mathbb{E}\sup_{t\le T}|y_t-g_t|^p \\
&\quad\le 3^{1/q}\mathbb{E}\left[\max_n \sup_{\tau\in J_n}|y_\tau-y_{\tau_n}|^p+\max_n\sup_{\tau\in J_n}|g_\tau-g_{\tau_n}|^p+\sup_{\tau\in\pi}|y_\tau-g_\tau|^p\right]\\
&\quad\le 3^{1/q}\mathbb{E}\left[\sum_{n=0}^{N-1}\sup_{\tau\in J_n}|y_\tau-y_{\tau_n}|^p+\sum_{n=0}^{N-1}\sup_{\tau\in J_n}|g_\tau-g_{\tau_n}|^p+\sup_{\tau\in\pi}|y_\tau-g_\tau|^p\right]\\
&\quad\le C_p\left(\sum_{n=0}^{N-1}|t_{n+1}-t_n|^{p/2}+|\pi|^{p/2}\right)\le C_p\left(|\pi|^{(p/2)-1}+|\pi|^{p/2}\right).
\end{aligned}$$

This proves (A.3). ■

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