Path Integrals over a Manifold

with Lars Andersson and Adrian Lim

Bruce Driver

Visiting Miller Professor

(Permanent Address) Department of Mathematics, 0112

University of California at San Diego, USA

http://math.ucsd.edu/∼driver

Topology Seminar

University of California, Berkeley, August 29, 2007

Canonical Quantization

CONCEPT	CLASSICAL	QUANTUM
CONFIG. SPACE	$\mathbb{R}^d$	?
STATE SPACE	$T^*\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d \ni (p,q)$	$K = PL^2(\mathbb{R}^d, dm)$ $\psi \in L^2(\mathbb{R}^d, dm) \ni \|\psi\|_K = 1.$
OBSERVABLES Examples	Functions on $T^*\mathbb{R}^d$ p_k q_k $H(q,p) = \frac{1}{2m}p^2 + V(q)$	S.A. ops. on K $\hat{p}_k = \frac{\hbar}{i}\frac{\partial}{\partial q_k}$ $\hat{q}_k = M_{q_k}$ $\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(q)$
DYNAMICS	Newtons Equations of Motion $\ddot{q}(t) = -\nabla V(q(t))$	Schrödinger, Eq. $i\hbar\dot{\psi}(t) = \hat{H}\psi(t), \quad \psi(t) \in K$
MEASURMENTS	Evaluation $f(q,p)$	$\langle \psi, \theta\psi \rangle$ – expected value.

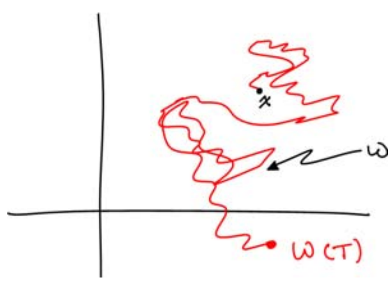
The Path Integral Prescription on $\mathbb{R}^d$

Notation 1. For $x \in \mathbb{R}^d$ and $T > 0$, let

$$W\left(\mathbb{R}^d; x, T\right) := \left\{\omega \in C\left([0,T] \to \mathbb{R}^d\right) : \omega(0) = x\right\}$$

and let

$$H(\mathbb{R}^d; T) := \left\{\omega \in W\left(\mathbb{R}^d; T\right) : \int_0^T |\omega'(s)|^2 ds < \infty\right\}.$$

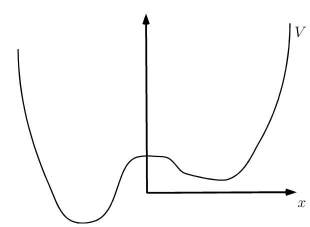


Theorem 2 (Meta-Theorem – Feynman (Kac) Quantization)**.** *Let $V : \mathbb{R}^d \to \mathbb{R}$ be a nice function. Then*

$$e^{-T\hat{H}} f(x) = \text{“}\frac{1}{Z_T}\int_{H\left(\mathbb{R}^d; x, T\right)} e^{-\int_0^T E(\omega(t), \dot{\omega}(t))dt} f(\omega(T))\mathcal{D}\omega\text{”} \tag{1}$$

where $E(x,v) = \frac{1}{2}m|v|^2 + V(x)$ is the classical energy and

$$\text{“}Z_T := \int_{H\left(\mathbb{R}^d; x, T\right)} e^{-\frac{1}{2}\int_0^T |\dot{\omega}(t)|^2 dt}\mathcal{D}\omega\text{”}.$$



Kac' s Formula (1949) (A Rigorous Interpretation)

Theorem 3 (Kac's Formula)**.**

$$e^{-T\hat{H}}f(x) = \int_{W\left(\mathbb{R}^d;T\right)} e^{-\int_0^T V(x+\omega(t))dt} f(x+\omega(T))d\mu\left(\omega\right)$$

where μ is Wiener measure (1923).

Informally,

$$d\mu(\omega)\text{“} = \text{”}\frac{1}{Z}e^{-\frac{1}{2}\int_0^1 |\omega'(s)|^2 ds}\,\mathcal{D}\omega.$$

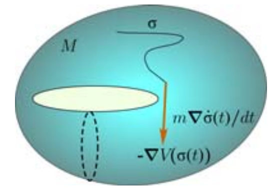
Formally, μ is the unique measure on $W\left(\mathbb{R}^d;T\right)$ such that

$$\int_{W\left(\mathbb{R}^d;T\right)} e^{i\varphi(\omega)}d\mu\left(\omega\right) = \exp\left(-\frac{1}{2}\left(\varphi,\varphi\right)_{H\left(\mathbb{R}^d;T\right)^*}\right)$$

for all $\varphi \in W\left(\mathbb{R}^d;T\right)^*$.

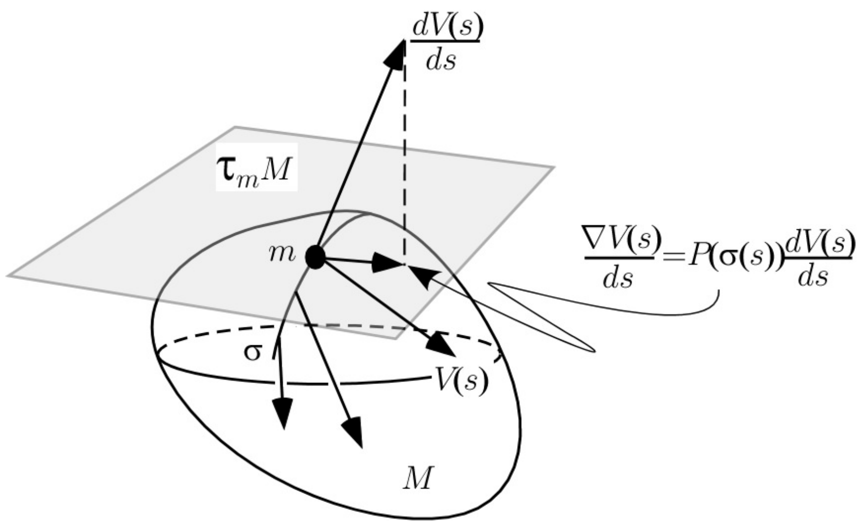
Classical Mechanics on a Manifold

- Let (M, g) be a Riemannian manifold.



- Newton's Equations of motion

$$m\frac{\nabla\dot{\sigma}\left(t\right)}{dt} = -\nabla V(q(t)). \tag{2}$$



Classical Energy and Hamiltonian

- $L(x,v) := \frac{1}{2}m\left|v\right|_g^2 - V(x)$ is the **Lagrangian**.
- $E\left(x,v\right) := \frac{1}{2}m\left|v\right|_g^2 + V\left(x\right)$ is the **energy**.
- $p = \frac{\partial L(x,v)}{\partial v} = mg\left(v,\cdot\right)$ is the conjugate momentum to v.
- $H(x,p) = \frac{1}{2m}\left|p\right|_{g^*}^2 + V(x)$ is the **Hamiltonian.**
- $H(x,p) = E\left(x,v\right) = p\left(v\right) - L\left(x,v\right)$ where $p = \frac{\partial L(x,v)}{\partial v} = mg\left(v,\cdot\right)$.

.

“Canonical” Quantization

We now set $m = 1$, $\sqrt{g} = \sqrt{\det(g_{ij})}$, and $d\,\mathrm{Vol} := \sqrt{g}dx^1 \dots dx^d$.

- In local coordinates,
$$H = \frac{1}{2}g^{ij}(q)\,p_i p_j + \tilde{V}(q)$$
$$= \frac{1}{2}\frac{1}{\sqrt{g}}p_i\sqrt{g}g^{ij}(q)p_j + \tilde{V}(q)\,.$$
- Quantize:
$$p_i \to \hat{p}_i := \frac{1}{i}\frac{\partial}{\partial q^i} \text{ and } q_i \to \hat{q}_i := M_{q^i}.$$
- Then $H \to \hat{H}$ acting on $L^2(M, d\,\mathrm{Vol})$ by
$$\hat{H} = -\frac{1}{2}g^{ij}(q)\frac{\partial^2}{\partial q^i \partial q^j} + v(q).$$
or
$$\hat{H} = -\frac{1}{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q^i}\left(\sqrt{g}g^{ij}(q)\frac{\partial}{\partial q^j}\right) + v(q) = -\frac{1}{2}\Delta_M + M_V.$$

Path Integral Quantization of $\hat{H}$

$$\left(e^{-T\hat{H}}f\right)(x_0) = \text{“}\frac{1}{Z_T}\int_{\sigma(0)=x_0} e^{-\int_0^T E(\sigma(t),\dot{\sigma}(t))dt}f(\sigma(T))\mathcal{D}\sigma\text{”} \tag{3}$$

where $E(x, v)$ is the classical energy as above;
$$E(x,v) := \frac{1}{2}g(v,v) + V(x)$$

We now set $T = 1$.

Goal

Make sense out of the measure ν, “defined” by
$$d\nu(\sigma)\text{“} = \text{”}\frac{1}{Z}e^{-\int_0^1\left[\frac{1}{2}|\dot{\sigma}(t)|^2 + V(\sigma(t))\right]dt}\mathcal{D}\sigma.$$

A Motivation: Yang – Mills Equations

- The Yang – Mills equations are the Euler Lagrange equations for
$$I(A) = \int_{\mathbb{R}\times\mathbb{R}^d}\langle F^A\rangle_L^2 dt dx.$$
- $\mathfrak{g} = \mathrm{Lie}(G)$ and G is a compact Lie group.
- $A : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \otimes \mathfrak{g}$ is a connection one form.
- $F^A = dA + A \wedge A$ is the curvature tensor.
- $\langle\cdot\rangle_L^2$ is a non-degenerate quadratic form determined by the Lorentzian metric on $\mathbb{R}^{d+1}$ and an inner product on $\mathfrak{g}$.
- Path integral quantization measure is
$$d\mu(A)\text{“} = \text{”}\frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^d}\left|F^A\right|^2 dt dx\right)\mathcal{D}A. \tag{4}$$
- μ is to be interpreted on $M := \mathcal{M}/\mathcal{G}$. (See http://www.claymath.org.)
- When $d = 1$ and $\mathbb{R}^d = \mathbb{R}^1$ is replace by S^1 the space $\mathcal{M}/\mathcal{G}_0$ simply becomes G itself and the path integral in (4) reduces to the one like that in Eq. (3) with $M = G$ and $V = 0$. See Driver and Hall [Comm. Math. Phys. 201 (1999).]

Some Background

If $\hat{H}$ is “defined” by
$$e^{-T\hat{H}}f(x_0) = \frac{1}{Z_T}\int_{\sigma(0)=x_0} e^{-\int_0^T E(\sigma(t),\dot{\sigma}(t))dt}f(\sigma(T))\mathcal{D}\sigma \tag{5}$$
then various rigorous and not so rigorous results indicate:
$$\hat{H} = -\frac{1}{2}\Delta + \frac{1}{\kappa}\mathrm{S}$$
where

- S is the scalar curvature of M, and
- $\kappa \in \{6, 8, 12, \infty\}$.
- For example, see Cheng 72 with $\kappa = 6$. Um 73, Atsuchi & Maeda 85, and Darling 85. Geo. Quant. gives $\kappa = 12$. Also see Kärki, Topi, Niemi, Antti J, Phys. Rev. D (3) 56 (1997) – quoted below.

Remark 4 (Scalar Curvature)**.**
$$\mathrm{Vol}(B_\epsilon(m)) = |B_\epsilon(0)|\left(1 - \frac{\epsilon^2}{6(d+2)}\mathrm{S}(m) + O(\epsilon^3)\right)$$

Path Spaces

Notation 5 (Path Spaces)**.** Given a pointed Riemannian manifold (M, g, o), let

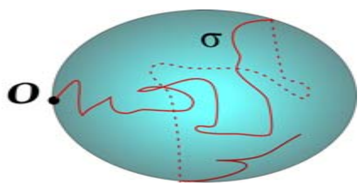
$$W(M) = \{\sigma \in C([0,1] \to M) \,|\, \sigma(0) = o\}.$$

For those $\sigma \in W(M)$ which are absolutely continuous, let

$$E_M(\sigma) := \int_0^1 |\sigma'(s)|_g^2 \, ds$$

denote the **energy of** σ. The space of **finite energy paths** $H(M)$ is given by

$$H(M) := \left\{ \begin{array}{c} \sigma \in W(M) | \sigma \text{ is absolutely continuous} \\ \text{and } E_M(\sigma) < \infty \end{array} \right\}.$$



Wiener Measure on $W(M)$

Notation 6. Let M be a Riemannian manifold with base point $o \in M$.

Theorem 7 (Wiener measure)**.** *There exists a unique probability measure $\nu_{W(M)}$ on $W(M)$ such that*

$$\int_{W(M)} F(\sigma(s_1), \ldots, \sigma(s_n)) d\nu_{W(M)}(\sigma) = \int_{M^n} F(x_1, \ldots, x_n) \prod_{i=0}^{n-1} p_{\Delta_i s}(x_i, x_{i+1}) dx_1 \cdots dx_n.$$

where, $\Delta_i s := s_i - s_{i-1}$, $x_0 = o$, dx *denotes the volume measure on* M, *and* $p_t(x,y) = \ker e^{t\Delta/2}(x,y)$.

Example 1. When $M = \mathbb{R}^d$,

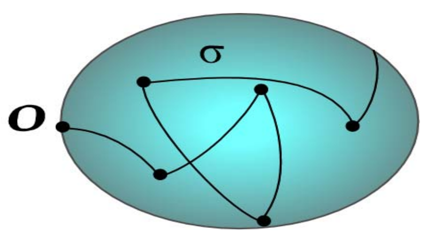
$$p_t(x,y) = \left(\frac{1}{2\pi t}\right)^{\frac{d}{2}} \exp\left(-\frac{1}{2t}|x-y|^2\right).$$

We call, $\mu := \nu_{W(\mathbb{R}^d)}$, classical *Wiener Measure.*

Piecewise Geodesics

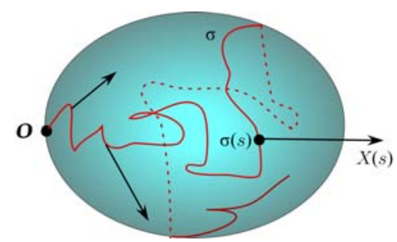
- $\mathcal{P} := \{0 = s_0 < s_1 < s_2 < \ldots < s_n = 1\}$
- $\Delta_i s := s_i - s_{i-1}$
- **Piecewise geodesics:**

$$H_{\mathcal{P}}(M) = \{\sigma \in H(M) : \nabla\sigma'(s)/ds = 0 \text{ off } \mathcal{P}\}$$



Tangent Spaces

$$T_\sigma \mathrm{H}(M) = \left\{ \begin{array}{c} X : [0,1] \to TM : X(s) \in T_{\sigma(s)}M, \;\; X(0) = 0, \\ \& \int_0^1 \left|\frac{\nabla X(s)}{ds}\right|^2 ds < \infty \end{array} \right\}.$$

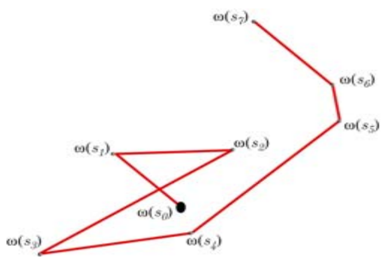


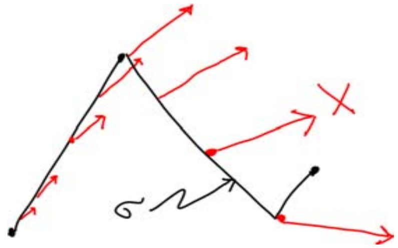
$$T_\sigma \mathrm{H}_{\mathcal{P}}(M) = \{X \in T_\sigma \mathrm{H}(M) : \; X \text{ satisfies (Jacobi)}\}$$

$$\frac{\nabla^2 X(s)}{ds^2} = R(\sigma'(s), X(s))\sigma'(s) \text{ for } s \notin \mathcal{P}. \qquad \text{(Jacobi)}$$

Example: $M=\mathbb{R}^d$

$$H_{\mathcal{P}}(\mathbb{R}^d) := \left\{\omega \in H(\mathbb{R}^d) : \omega''(s) = 0 \text{ if } s \notin \mathcal{P}\right\}.$$





A tangent vector, $X \in T_\sigma H_{\mathcal{P}}\left(\mathbb{R}^d\right)$.

Metrics

Let $\sigma \in \mathrm{H}_{\mathcal{P}}(M)$, and $X, Y \in T_\sigma \mathrm{H}_{\mathcal{P}}(M)$. **Metrics:**

- H^1–**Metric on** $H(M)$

$$G^1(X,X) := \int_0^1 \left\langle \frac{\nabla X(s)}{ds}, \frac{\nabla X(s)}{ds} \right\rangle ds,$$

- H^1–**Metric on** $H_{\mathcal{P}}(M)$ (Riemannian Sum Approximation)

$$G^1_{\mathcal{P}}(X,Y) := \sum_{i=1}^n \langle \frac{\nabla X(s_{i-1}+)}{ds}, \frac{\nabla Y(s_{i-1}+)}{ds} \rangle \Delta_i s,$$

- H^0–**Metric on** $H_{\mathcal{P}}(M)$ (Riemannian Sum Approximation)

$$G^0_{\mathcal{P}}(X,Y) := \sum_{i=1}^n \langle X(s_i), Y(s_i) \rangle \Delta_i s,$$

- H^1–**Metric restricted to** $H_{\mathcal{P}}(M)$ – $G^1|_{TH_{\mathcal{P}}(M)}$ (the hardest case).

Approximating Measures

Definition 8 (Approximates to Wiener Measure to $\mu_{W(M)}$)**.** For each partition $\mathcal{P} = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$ of $[0,1]$, let $\nu^0_{\mathcal{P}}$ and $\nu^1_{\mathcal{P}}$ denote measures on $H_{\mathcal{P}}(M)$ defined by

$$d\nu^0_{\mathcal{P}} := \frac{1}{Z^0_{\mathcal{P}}} e^{-\frac{1}{2}E_M} \cdot d\mathrm{Vol}_{G^0_{\mathcal{P}}},$$

$$d\nu^1_{\mathcal{P}} = \frac{1}{Z^1_{\mathcal{P}}} e^{-\frac{1}{2}E_M} \cdot d\mathrm{Vol}_{G^1_{\mathcal{P}}}, \text{ and}$$

$$d\nu_{\mathcal{P}} := \frac{1}{Z^1_{\mathcal{P}}} e^{-\frac{1}{2}E_M} \cdot d\mathrm{Vol}_{G^1|_{TH_{\mathcal{P}}(M)}}$$

where $E_M : H(M) \to [0,\infty)$ is the energy functional

$$E_M(\sigma) := \int_0^1 |\sigma'(s)|_g^2 \, ds$$

and $Z^0_{\mathcal{P}}$ and $Z^1_{\mathcal{P}}$ are normalization constants given by

$$Z^0_{\mathcal{P}} := \prod_{i=1}^n \left(\sqrt{2\pi}\,(s_i - s_{i-1})\right)^d \text{ and } Z^1_{\mathcal{P}} := (2\pi)^{dn/2}. \tag{6}$$

Flat Case $(M = \mathbb{R}^d)$ Example

- H^1 and H^0 – **Metrics on** $H(\mathbb{R}^d)$

$$G^1(h,k) := \int_0^1 \langle h'(s), k'(s)\rangle ds \text{ and } G^0(h,k) := \int_0^1 \langle h(s), k(s)\rangle ds$$

- H^1–**Metric on** $H_{\mathcal{P}}(\mathbb{R}^d)$

$$G^1_{\mathcal{P}}(h,k) := \sum_{i=1}^{n} \langle h'(s_{i-1}+), k'(s_{i-1}+)\rangle \Delta_i s$$

- H^0–**Metric on** $H_{\mathcal{P}}(\mathbb{R}^d)$

$$G^0_{\mathcal{P}}(h,k) := \sum_{i=1}^{n} \langle k(s_i), h(s_i)\rangle \Delta_i s$$

Limiting Measures for $M = \mathbb{R}^d$

Theorem 9 (Wiener 1923)**.** *Let*

$$\mu^1_{\mathcal{P}} = \frac{1}{Z^1_{\mathcal{P}}} e^{-\frac{1}{2}E_{\mathbb{R}^d}} \mathrm{Vol}_{G^1_{\mathcal{P}}}, \quad \text{and}$$
$$\mu^0_{\mathcal{P}} = \frac{1}{Z^0_{\mathcal{P}}} e^{-\frac{1}{2}E_{\mathbb{R}^d}} \mathrm{Vol}_{G^0_{\mathcal{P}}},$$

where $Z^1_{\mathcal{P}}$ *and* $Z^0_{\mathcal{P}}$ *are normalization constants;*

$$Z^1_{\mathcal{P}} := (2\pi)^{dn/2}, \qquad Z^0_{\mathcal{P}} := \prod_{i=1}^{n} (\sqrt{2\pi}\Delta_i s)^d.$$

Then

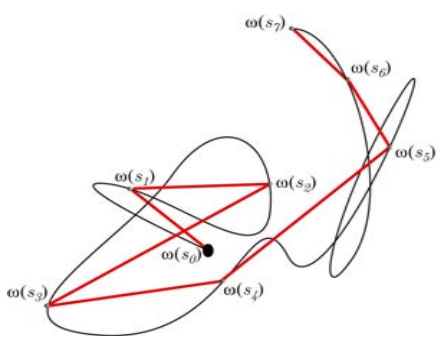
$$\mu = \lim_{|\mathcal{P}|\to 0} \mu^1_{\mathcal{P}} = \lim_{|\mathcal{P}|\to 0} \mu^0_{\mathcal{P}},$$

where μ *is standard Wiener measure on* $W(\mathbb{R}^d)$.

Proof

Let $* \in \{0,1\}$. For $\omega \in H_{\mathcal{P}}\left(\mathbb{R}^d\right)$, let $x_i := \omega(s_i)$. Then one shows;

$$\int_{H_{\mathcal{P}}(\mathbb{R}^d)} f(\omega)\, d\mu^*_{\mathcal{P}}(\omega) = \int_{W(\mathbb{R}^d)} f(\omega_{\mathcal{P}})\, d\mu(\omega)$$



$\omega_{\mathcal{P}}$ in red where
$\mathcal{P} = \{0 = s_0 < s_1 < \ldots\}$.

- Now suppose f is a bounded and continuous on $W(\mathbb{R}^d)$.
- Apply the dominated convergence theorem and uniform continuity to show

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(\mathbb{R}^d)} f(\omega)\, d\mu^*_{\mathcal{P}}(\omega) = \lim_{|\mathcal{P}|\to 0} \int_{W(\mathbb{R}^d)} f(\omega_{\mathcal{P}})\, d\mu(\omega)$$
$$= \int_{W(\mathbb{R}^d)} f(\omega)\, d\mu(\omega).$$

Limits in the Manifold Case

Theorem 10 (Andersson and D. 1999.)**.** *Suppose that $f : W(M) \to \mathbb{R}$ is a bounded and continuous, then*

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma) d\nu^1_{\mathcal{P}}(\sigma) = \int_{\mathrm{W}(M)} f(\sigma) d\nu_{\mathrm{W}(M)}(\sigma) \tag{7}$$

and

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma) d\nu^0_{\mathcal{P}}(\sigma) = \int_{\mathrm{W}(M)} f(\sigma) e^{-\frac{1}{6}\int_0^1 \mathrm{S}(\sigma(s))ds} d\nu_{\mathrm{W}(M)}(\sigma), \tag{8}$$

where S is the scalar curvature of (M, g).

There is a large literature pertaining to results of the type in Theorem 10, see for example Cheng72, Um74, Pinsky78, Fujiwara 80, Darling84, A. Inoue and Y. Maeda 85, W. Ichinose 97 and Jyh-Yang Wu 98. The version given here is contained in Andersson and Driver 98.

Notation 11. Let R_p be the curvature tensor at $p \in M$ and $\{e_i\}_{i=1,2,\ldots,d}$ is any orthonormal basis in $T_p(M)$.

Adrian Lim's Theorem

Theorem 12 (Adrian Lim 2006)**.** *Let (M^d, g) be a d – dimensional compact Riemannian manifold such that*

$$0 \le \textit{Sectional-Curvatures} \le \varepsilon(d) = \frac{3}{17d},$$

and $\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$.

If $f : W(M) \to \mathbb{R}$ is a bounded and continuous function, then

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma)\, d\nu_{\mathcal{P}}(\sigma) = \int_{W(M)} f(\sigma) e^{-\frac{1}{6}\int_0^1 \mathrm{S}(\sigma(s))\, ds} \sqrt{\det\left(I + \frac{1}{12}K_\sigma\right)}\, d\nu(\sigma).$$

where, for $\sigma \in H(M)$, K_σ is the integral operator acting on $L^2([0,1]; \mathbb{R}^d)$ defined by

$$(K_\sigma f)(s) = \int_0^1 (s \wedge t)\, \Gamma_{\sigma(t)} f(t)\, dt$$

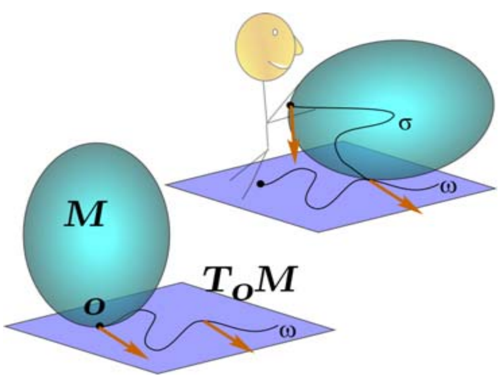
with

$$\Gamma_p = \sum_{i,j=1}^d \left(\begin{array}{c} R_p(e_i, R_p(e_i, \cdot)e_j)\, e_j + R_p(e_i, R_p(e_j, \cdot)e_i)\, e_j \\ +R_p(e_i, R_p(e_j, \cdot)e_j)\, e_i \end{array} \right).$$

On the proofs.

Notation 13. To each $\sigma \in H(M)$ and $s \in [0,1]$ let

- **Parallel translation:** $//_s(\sigma) : T_oM \to T_{\sigma(s)}M$
  $\frac{\nabla}{ds}//_s(\sigma) = 0$ with $//_0(\sigma) = Id_{T_oM}$.
- **Cartan's rolling map:** $\psi : H(T_oM) \longrightarrow H(M)$ given by $\sigma = \psi(\omega)$ where
  $\sigma'(s) = //_s(\sigma)\omega'(s)$ with $\sigma(0) = o$. (9)



Proof of the $G^1_{\mathcal{P}}$ – Theorem

On $H_{\mathcal{P}}(M)$, let

$$\nu^1_{\mathcal{P}} = \frac{1}{Z^1_{\mathcal{P}}} \exp(\frac{1}{2}E_M)\mathrm{Vol}_{G^1_{\mathcal{P}}}.$$

Then $\lim_{|\mathcal{P}|\to 0} \nu^1_{\mathcal{P}} = \nu_{W(M)}$.

*Proof Sketch***:** Although the rolling map $\psi : H(\mathbb{R}^d) \to H(M)$ is not an isomorphism, we do have (with $\psi_{\mathcal{P}} := \psi|_{\mathcal{H}_{\mathcal{P}}}(\mathbb{R}^d)$):

1. $\det(D\psi_{\mathcal{P}}) = \det(I + T_{\mathcal{P}})^2 = 1$ because one shows that $T_{\mathcal{P}}$ is nilpotent.
2. Equivalently: $\psi^*_{\mathcal{P}}\mathrm{Vol}^M_{G^1_{\mathcal{P}}} = \mathrm{Vol}^{\mathbb{R}^d}_{G^1_{\mathcal{P}}}$
3. $E_{\mathbb{R}^d}(\omega) = E_M(\psi(\omega))$ for $\omega \in H(\mathbb{R}^d)$.
4. 2 & 3 imply that
   $$\psi_*\mu^1_{\mathcal{P}} = \nu^1_{\mathcal{P}}.$$
5. Eelles & Elworthy (Gangolli) show
   $$\tilde{\psi}_*\mu = \nu,$$
   where $\tilde{\psi} : W(\mathbb{R}^d) \to W(M)$ is the stochastic version of ψ.
6. 4 & 5 along with Wong and Zakai approximation theorem shows $\lim_{|\mathcal{P}|\to 0} \nu^1_{\mathcal{P}} = \nu$.

Proof of the $G^0_{\mathcal{P}}$ – Theorem

On $H_{\mathcal{P}}(M)$, let

$$\nu^0_{\mathcal{P}} = \frac{1}{Z^0_{\mathcal{P}}} e^{-\frac{1}{2}E_{\mathbb{M}}} \mathrm{Vol}_{G^0_{\mathcal{P}}}.$$

Then

$$\lim_{|\mathcal{P}|\to 0} d\nu^0_{\mathcal{P}}(\sigma) = \exp\left(-\frac{1}{6}\int_0^1 \mathrm{S}(\sigma(s))ds\right) d\nu(\sigma)$$

where S is the scalar curvature of M.

Proof: One shows that

$$d\nu^0_{\mathcal{P}} = \rho_{\mathcal{P}} d\nu^1_{\mathcal{P}}$$

and that

$$\lim_{|\mathcal{P}|\to 0} \rho_{\mathcal{P}}(\sigma) = \exp\left(-\frac{1}{6}\int_0^1 \mathrm{S}(\sigma(s))ds\right)$$

See De Witt (57), Cheng (72), Um (73), Pinski(78), Darling (84), Atsushi(85), ...

Proof of Adrian Lim's Theorem

Theorem 14 (Adrian Lim 2006)**.**

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma)\, d\nu_{\mathcal{P}}(\sigma)$$
$$= \int_{W(M)} f(\sigma) e^{-\frac{1}{6}\int_0^1 \mathrm{S}(\sigma(s))\,ds} \sqrt{\det\left(I + \frac{1}{12}K_\sigma\right)}\, d\nu(\sigma).$$

where, for $\sigma \in H(M)$, K_σ is the integral operator acting on $L^2([0,1];\mathbb{R}^d)$ defined by

$$(K_\sigma f)(s) = \int_0^1 (s \wedge t)\, \Gamma_{\sigma(t)} f(t)\, dt$$

with

$$\Gamma_p = \sum_{i,j=1}^{d} \begin{pmatrix} R_p(e_i, R_p(e_i, \cdot)e_j)\, e_j + R_p(e_i, R_p(e_j, \cdot)e_i)\, e_j \\ +R_p(e_i, R_p(e_j, \cdot)e_j)\, e_i \end{pmatrix}.$$

Proof of Adrian Lim's Theorem

Let $\mathcal{P} = \mathcal{P}_n = \left\{s_l = \frac{l}{n} : l = 0, \dots, n\right\}$,

$$b_i' := \frac{b(s_i) - b(s_{i-1})}{1/n} = n \cdot \Delta_i b.$$

Define $\rho_{\mathcal{P}}(\sigma)$ so that

$$d\nu_{\mathcal{P}_n}(\sigma) = \rho_n(\sigma)\, d\nu^1_{\mathcal{P}_n}(\sigma).$$

Two Steps

1. Show $\{\rho_n\}_{n=1}^\infty$ is a uniformly integrable sequence, by showing there exists $p > 1$ such that
$$\sup_n \int_{H_{\mathcal{P}_n}(M)} \rho_n^p(\sigma)\, d\nu^1_{\mathcal{P}_n}(\sigma) < \infty.$$
2. Show $\lim_{n\to\infty} \rho_n$ exists a.s. and identify the limit.

Proposition 15 (Formula for ρ_n)**.** *Let $h_{i,a}(s)$ solve*

$$\frac{d^2 h(s)}{ds^2} = \Omega_{u(s)}(b_i', h(s)) b_i' \text{ with} \tag{10}$$

$$h_{i,a}(0) = 0, \text{ and}$$
$$h'_{i,a}(s_{j-1}+) = \delta_{ij} e_a \text{ for } j = 1, \dots, n. \tag{11}$$

Let $\mathcal{Q}^n$ denote the $dn \times dn$ matrix which is given in $d \times d$ blocks, $\mathcal{Q}^n := \{(\mathcal{Q}^n_{mk})\}_{m,k=1}^n$, with

$$(\mathcal{Q}^n_{mk} e_a, e_c) := \int_0^1 \langle h'_{ma}(s), h'_{kc}(s)\rangle\, ds \text{ for } a, c = 1, 2, \dots, d.$$

Then

$$\rho^2_{\mathcal{P}} = \det(n\mathcal{Q}^n).$$

Proposition 16. *Suppose that M is a symmetric positive definite $N \times N$ matrix and $\alpha \geq 1$. Then*

$$\det(M) \leq \alpha^N e^{\mathrm{tr}\left(\alpha^{-1}M - I\right)} \leq \alpha^N e^{\alpha^{-1}\mathrm{tr}(M-I)}. \tag{12}$$

- Now do 60+ pages of analysis!

Corollary 17. *For $\alpha \geq 1$,*

$$\begin{aligned}\det(n\mathcal{Q}^n) &\leq \alpha^{nd}\exp\left(\alpha^{-1}\operatorname{tr}\left(n\mathcal{Q}^n - I_{nd\times nd}\right)\right)\\ &= \alpha^{nd}\exp\left(\alpha^{-1}\sum_{m=1}^{n}\operatorname{tr}\left(n\mathcal{Q}^n_{m,m} - I_{d\times d}\right)\right)\\ &\leq \alpha^{nd}\exp\left(\alpha^{-1}d\sum_{m=1}^{n}\left\|n\mathcal{Q}^n_{m,m} - I_{d\times d}\right\|\right).\end{aligned}$$

$$\begin{aligned}\mathcal{Q}^n_{mm} &= \int_0^{1/n} S'_m(b,s)^T S'_m(b,s)\,ds\\ &+ \sum_{j=m+1}^{n} V^T_{mj}\left[\int_0^{1/n} C'_j(b,s)^T C'_j(b,s)ds\right]V_{mj}.\end{aligned}$$

where

$$V_{mj} := \left[\prod_{k=m+1}^{j-1} C_k(b,\Delta_k s)\right] S_m(b,\Delta_m s)$$

and C_j and S_j are certain fundamental solutions to Jacobi's equation,

$$\frac{d^2h(s)}{ds^2} = \Omega_{u(s)}(b'_i, h(s))b'_i.$$

Applications

Corollary 18 (Trotter Product Formula for $e^{t\Delta/2}$)**.** *For $s > 0$ let Q_s be the symmetric integral operator on $L^2(M, dx)$ defined by the kernel*

$$Q_s(x,y) = (2\pi s)^{-d/2}\exp\left(-\frac{1}{2s}d^2(x,y) + \frac{s}{12}\mathrm{S}(x) + \frac{s}{12}\mathrm{S}(y)\right)$$

for all $x, y \in M$. Then for all continuous functions $F : M \to \mathbb{R}$ and $x \in M$,

$$(e^{\frac{s}{2}\Delta}F)(x) = \lim_{n\to\infty}(Q^n_{s/n}F)(x).$$

See also Chorin, McCracken, Huges, Marsden (78) and Wu (98).

Proof. This is a special case of the L^2 – limit theorem. The main points are:

- $\nu^0_{\mathcal{P}}$ is essentially product measure on M^n.
- From this one shows that

$$(Q^n_{s/n}F)(x) \cong \int_{H_{\mathcal{P}}(M)} e^{\frac{1}{6}\int_0^1 \mathrm{S}(\sigma(s))ds}F(\sigma(s))\,d\nu^0_{\mathcal{P}}(\sigma)$$

Corollary 2: Integration by Parts for ν on $W(M)$

See Bismut, Driver, Enchev, Elworthy, Hsu, Li, Lyons, Norris, Stroock, Taniguchi, ...

Let $k \in PC^1$, and z solve:

$$z'(s) + \frac{1}{2}\mathrm{Ric}_{\tilde{//}_s(\sigma)}z(s) = k'(s), \qquad z(0) = 0.$$

and f be a cylinder function on $\mathrm{W}(M)$. Then

$$\int_{\mathrm{W}(M)} X^z f\,d\nu = \int_{\mathrm{W}(M)} f\int_0^1\langle k', d\tilde{b}\rangle\,d\nu,$$

where

$$\begin{aligned}(X^z f)(\sigma) &= \sum_{i=1}^{n}\langle(\nabla_i f)(\sigma), X^z_{s_i}(\sigma)\rangle\\ &= \sum_{i=1}^{n}\langle(\nabla_i f)(\sigma), \tilde{//}_{s_i}(\sigma)z(s_i,\sigma)\rangle\end{aligned}$$

and $(\nabla_i f)(\sigma)$ denotes the gradient F in the ith variable evaluated at $(\sigma(s_1), \sigma(s_2), \ldots, \sigma(s_n))$.

Proof

Integrate by parts in on $H_{\mathcal{P}}(M)$ and then pass to the limit as $|\mathcal{P}| \to 0$.

Quasi-Invariance Theorem for $\nu_{W(M)}$

Theorem 19 (D. 92, Hsu 95)**.** *Let* $h \in H(T_oM)$ *and* X^h *be the* $\nu_{W(M)}$ *– a.e. well defined vector field on* $W(M)$ *given by*

$$X_s^h(\sigma) = //_s(\sigma)h(s) \text{ for } s \in [0,1]. \tag{13}$$

Then X^h *admits a flow* e^{tX^h} *on* $W(M)$ *and this flow leaves* $\nu_{W(M)}$ *quasi-invariant. (**Ref:** D. 92, Hsu 95, Enchev-Strook 95, Lyons 96, Norris 95, ...)*