Math 103A: Winter 2014
Practice Final Exam

Instructions: Please write your name on your blue book. Make it clear in your blue book what problem you are working on. Write legibly and explain your reasoning. This exam is graded out of 100 points. Following these instructions is worth 5 points.

Problem 1: Which of the following sets with the following operations are groups? Justify your answers.

(1) $\{A \in GL(3, \mathbb{R}) : \det(A) = 2^a \text{ for some } a \in Z\}$, under matrix multiplication.
(2) $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$, under vector addition.
(3) $\{A \in GL(4, \mathbb{R}) : \det(A) \geq 1\}$, under matrix multiplication.

Problem 2: Let G be a group and let $g \in G$. If $|g| = 28$, what is $|g^{16}|$?

Problem 3: Let $\beta = (1, 2, 3)(4, 5, 6)(7, 8)(9, 10) \in S_{10}$. Find an element $\alpha \in S_{10}$ which is not a power of β such that $\alpha\beta = \beta\alpha$.

Problem 4: Prove that Q is not isomorphic to Z.

Problem 5: How many elements of order 7 are there in $Z_{49} \oplus Z_{49}$?

Problem 6: Prove $D_4/Z(D_4) \approx Z_2 \oplus Z_2$.

Problem 7: Let G be a group and let H and K be subgroups of G. Prove $H \cap K$ is also a subgroup of G.

Problem 8: List seven non-isomorphic groups of order 16.

Problem 9: Let G be a group, let $\mathrm{Aut}(G)$ be the automorphism group of G and let $\mathrm{Inn}(G)$ be the inner automorphism group of G. Prove that $\mathrm{Inn}(G) \triangleleft \mathrm{Aut}(G)$. (You may assume without proof that $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)$.)

Problem 10: Prove or give a counterexample: Every infinite group contains an element of infinite order.