Math 103A: Winter 2014
Practice Final Exam Solutions

Instructions: Please write your name on your blue book. Make it clear in your blue book what problem you are working on. Write legibly and explain your reasoning. This exam is graded out of 100 points. Following these instructions is worth 5 points.

Problem 1: Which of the following sets with the following operations are groups? Justify your answers.

(1) $\{A \in GL(3, \mathbb{R}) : \det(A) = 2^a \text{ for some } a \in Z\}$, under matrix multiplication.
(2) $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$, under vector addition.
(3) $\{A \in GL(4, \mathbb{R}) : \det(A) \geq 1\}$, under matrix multiplication.

Solution:

(1) Yes. If $\det(A) = 2^r$ and $\det(B) = 2^s$, then $\det(AB) = \det(A)\det(B) = 2^r 2^s = 2^{r+s}$, so we are closed under multiplication. Also, $\det(I) = 1 = 2^0$, so we have the identity element. Finally, $\det(A^{-1}) = 1/\det(A) = 2^{-r}$, so we are closed under taking inverses.
(2) No. $(1, 0) + (0, 1) = (1, 1)$, so this set is not closed under vector addition.
(3) No. If $A = 2I_4$, then $\det(A) = 16$ but $\det(A^{-1}) = 1/16$, so this set is not closed under taking matrix inverses.

Problem 2: Let G be a group and let $g \in G$. If $|g| = 28$, what is $|g^{16}|$?

Solution: We have that $|g^{16}| = \frac{28}{\gcd(28,16)} = \frac{28}{4} = 7$.

Problem 3: Let $\beta = (1, 2, 3)(4, 5, 6)(7, 8)(9, 10) \in S_{10}$. Find an element $\alpha \in S_{10}$ which is not a power of β such that $\alpha\beta = \beta\alpha$.

Solution: Let $\alpha = (1, 4)(2, 5)(3, 6)$. Then α is not a power of β (since no power of β sends 1 to 4) but $\alpha\beta = (1, 5, 3, 4, 2, 6)(7, 8)(9, 10) = \beta\alpha$.

Problem 4: Prove that Q is not isomorphic to Z.

Solution: Suppose that $\phi : Z \to Q$ were an isomorphism. Since ϕ is a homomorphism, $\phi(Z) = \langle \phi(1) \rangle = \{n\phi(1) : n \in Z\}$. In particular $\frac{\phi(1)}{2} \notin \phi(Z)$, so ϕ is not surjective, which is a contradiction.

Problem 5: How many elements of order 7 are there in $Z_{49} \oplus Z_{49}$?

Solution: Suppose $(a, b) \in Z_{49} \oplus Z_{49}$ has order 7. Then $\gcd(|a|, |b|) = 7$. This means that $a, b \in \langle 7 \rangle$. The only pair not allowed is $(0, 0)$. The number of allowable pairs is hence $7(7) - 1 = 48$.

Problem 6: Prove $D_4/Z(D_4) \approx Z_2 \oplus Z_2$.

Solution: We know that $Z(D_4) = \{R_0, R_{180}\}$, so that $D_4/Z(D_4)$ has order $8/2 = 4$. This means that $D_4/Z(D_4)$ is isomorphic to Z_4 or $Z_2 \oplus Z_2$. We compute that

$|R_0Z(D_4)| = 1$ while $|R_{90}Z(D_4)| = |DZ(D_4)| = |VZ(D_4)| = 2$, so order considerations force $D_4/Z(D_4) \approx Z_2 \oplus Z_2$.

Problem 7: Let G be a group and let H and K be subgroups of G. Prove $H \cap K$ is also a subgroup of G.

Solution: We have that $e \in H \cap K$, so that $H \cap K \neq \emptyset$. Let $g_1, g_2 \in H \cap K$. We know that $g_1^{-1} \in H \cap K$ because H and K are closed under taking inverses. Moreover $g_1^{-1}g_2 \in H \cap K$ because H and K are closed under multiplication. We conclude that $H \cap K$ is a subgroup of G.

Problem 8: List seven non-isomorphic groups of order 16.

Solution: By the Fundamental Theorem for Finite Abelian Groups, the non-isomorphic Abelian groups of order 16 are $Z_{16}, Z_8 \oplus Z_2, Z_4 \oplus Z_4, Z_4 \oplus Z_2 \oplus Z_2, Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$. We need to find two more groups. We take $D_4 \oplus Z_2$ and D_8. These groups are not isomorphic to eachother because D_8 has an element of order 8 whereas $D_4 \oplus Z_2$ does not.

Problem 9: Let G be a group, let $\mathrm{Aut}(G)$ be the automorphism group of G and let $\mathrm{Inn}(G)$ be the inner automorphism group of G. Prove that $\mathrm{Inn}(G) \triangleleft \mathrm{Aut}(G)$. (You may assume without proof that $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)$.)

Solution: Let $g \in G$ and let $\phi_g : x \to gxg^{-1}$ be the associated element of $\mathrm{Inn}(G)$. Let $\psi \in \mathrm{Aut}(G)$. For $x \in G$ we have $\psi(\phi_g(\psi^{-1}(x))) = \psi(g\psi^{-1}(x)g^{-1}) = \psi(g)x\psi(g)^{-1} = \phi_{\psi(g)}(x)$. We conclude that $\psi \circ \phi_g \circ \psi^{-1} = \phi_{\psi(g)} \in \mathrm{Inn}(G)$, so that $\mathrm{Inn}(G) \triangleleft \mathrm{Aut}(G)$.

Problem 10: Prove or give a counterexample: Every infinite group contains an element of infinite order.

Solution: This is false. Let G be the group of all infinite binary sequences $(a_1, a_2, \dots)$ (with $a_n = 0$ or 1 for all $n \geq 1$) under componentwise addition modulo 2. Every non-identity element of G has order 2, but G has infinite order.