Math 103A: Winter 2014
Homework 7
Due 5:00pm on Friday 3/7/2014

Problem 1: (Exercise 9.6 in Gallian) Let $H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R} \right\}$. Is H a normal subgroup of $GL(2, \mathbb{R})$?

Problem 2: (Exercise 9.22 in Gallian) Determine the order of $(Z \oplus Z)/\langle(2,2)\rangle$. Is this group cyclic?

Problem 3: Give an example of a group G and subgroups $K < H < G$ such that $K \triangleleft H$ and $H \triangleleft G$ but K is not a normal subgroup of G.

Problem 4: (Exercise 9.40 in Gallian) Let $\phi : G \to H$ be an isomorphism of groups. Suppose that K is a normal subgroup of G. Prove that $\phi(K)$ is a normal subgroup of H.

Problem 5: (Exercise 9.62 in Gallian) Let G be a group and let G' be the subgroup of G generated by the set of all elements of the form $xyx^{-1}y^{-1}$, where $x, y \in G$. (G' is called the *commutator subgroup of* G.)

(1) Prove G' is normal in G.
(2) Prove G/G' is Abelian.
(3) If $N \triangleleft G$ and G/N is Abelian, prove that $G' \leq N$.
(4) Prove that if $G' \leq H \leq G$, then H is normal in G.

Problem 6: (Exercise 10.14 in Gallian) Prove that the mapping $\phi : Z_{12} \to Z_{10}$ given by $\phi(x) = 3x$ is not a homomorphism.

Problem 7: If $\phi : G \to H$ and $\psi : H \to K$ are group homomorphisms, prove that $\psi \circ \phi : G \to K$ is also a homomorphism.

Problem 8: (Exercise 10.26 in Gallian) Determine all homomorphisms from Z_4 to $Z_2 \oplus Z_2$.

Problem 9: Let G and H be groups. Prove that the projection map $\pi : G \oplus H \to G$ given by $\pi(g, h) = g$ is a group homomorphism. Deduce that $\{e\} \oplus H = \{(e, h) : h \in H\}$ is a normal subgroup of $G \oplus H$ and $(G \oplus H)/(\{e\} \oplus H) \approx G$.