

Math 103A: Winter 2014
Midterm 1 Solutions and Comments

Instructions: Please write your name on your blue book. Make it clear in your blue book what problem you are working on. Write legibly and justify your answers. This exam is graded out of 100 points. Following these instructions is worth 5 points.

Problem 1: [5 + 10 pts.] (a) Carefully define the “special linear group” $SL(2, \mathbb{R})$. (b) Is $SL(2, \mathbb{R})$ Abelian? (Be sure to justify your answer.)

Solution: (a) $SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$ (under matrix multiplication).

(b) No. If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $A, B \in SL(2, \mathbb{R})$ but $AB \neq BA$.

Comments: Most students remembered the definition of the special linear group and got part (a) correct. Part (b) was more challenging. Some students seemed confused about what ‘Abelian’ means. Others gave an example of non-commuting matrices in $GL(2, \mathbb{R})$ which do not both lie in $SL(2, \mathbb{R})$.

Problem 2: [15 pts.] Recall that Z denotes the set of integers. Define a relation \sim on Z by $a \sim b$ if and only if $ab \geq 0$. Is \sim an equivalence relation on Z ? (Be sure to justify your answer.)

Solution: No. We have that $1 \sim 0$ and $0 \sim -1$, but $1 \sim -1$ does not hold. Therefore \sim is not transitive and not an equivalence relation.

Comments: This was a problem from homework. Unfortunately, many students seemed to think that \sim is an equivalence relation (and some students were confused about the definition of an equivalence relation - a few seemed to mix this up with the group axioms). There were students who got the correct answer, but “wasted time” on this problem; once you know that \sim is not transitive, you already know that \sim is not an equivalence relation, so there is no need to decide whether \sim is symmetric or reflexive.

Problem 3: [15 pts.] Recall that $U(10)$ denotes the group $\{1 \leq i \leq 10 : \gcd(i, 10) = 1\}$ under multiplication mod 10. Draw the Cayley table (i.e., the multiplication table) of $U(10)$.

Solution: (See Example 11 on Page 46 of your textbook.)

Comments: I did this in class. Virtually everyone got this right. There were some students who seemed confused about the definition of $U(10)$.

Problem 4: [15 pts.] Is the set $\text{Mat}_{2 \times 2}(\mathbb{R})$ of 2×2 real matrices a group under matrix multiplication? (Be sure to justify your answer.)

Solution: No. If $0 \in \text{Mat}_{2 \times 2}(\mathbb{R})$ denotes the 2×2 zero matrix, then $0A = 0$ for all $A \in \text{Mat}_{2 \times 2}(\mathbb{R})$. In particular, there does not exist a matrix A such that $0A = I$ and 0 does not have an inverse.

Comments: This is closely related to what I did in class. There were students who seemed to think that binary operations in groups are required to be Abelian. There were students who misremembered a formula for the inverse of a 2×2 matrix and used it to conclude that every 2×2 matrix was invertible. There was a fair amount of confusion here.

Problem 5: [20 pts.] Give an example of a group G and a subgroup $H < G$ such that G has infinite order and H has order six.

Solution: Let G be the group of nonzero complex numbers \mathbb{C}^* under multiplication. Let $H = \{\zeta \in \mathbb{C}^* : \zeta^6 = 1\}$ be the group of sixth roots of unity. We showed in class that $H < G$ and it is clear that $|H| = 6$ while $|G| = \infty$.

Comments: The subgroup H in the solution was presented in class; I expected that most students would think of the example I gave here. There are other examples, though. I particularly liked letting G be the group of symmetries of a circle and letting H be either (i) the symmetries of an inscribed regular triangle or (ii) the group of 6-fold rotations of the circle.

Most students did very poorly on this problem. Many gave the answer of $G = Z$ and $H = Z_6$. While (as defined by our book), $Z_6 = \{0, 1, 2, 3, 4, 5\}$ is a subset of Z , the binary operations on these sets are different (ordinary addition on Z vs. addition mod 6 on Z_6). By definition, the binary operation on any subgroup has to be the same as the binary operation on the big group. Other students gave examples that were not groups, or did not satisfy the required properties on order.

Problem 6: [15 pts.] Prove or give a counterexample: Every finite group G of order n contains an element $g \in G$ of order n .

Solution: This is false. Let $G = D_4$, the symmetry group of the square. The $|G| = 8$. The rotations in G have orders $|R_0| = 1, |R_{90}| = 4, |R_{180}| = 2$, and $|R_{270}| = 4$. The reflections in G all have order 2. In particular, no element of G has order 8.

Comments: Some students used the example of $U(8)$. I'd say that more than half of the responses said that this statement was true. In order to disprove a statement such as this, it is very helpful to have some knowledge of the basic examples of groups. In fact, a group G satisfies the conditions of this problem if and only if it is cyclic. (Can you see why?)