Math 103A: Winter 2014
Practice Midterm 2 Solutions

Problem 1: Let $G = Z_2 \oplus Z_2$. Then G is Abelian, but G does not contain an element of order 4 (indeed $|(0,0)| = 1$ and $|(1,0)| = |(0,1)| = |(1,1)| = 2$). Since $|G| = 4$, G is not cyclic.

Problem 2: We know that $|g|$ divides $|G| = 16$. Since $g^8 \neq e$, we also know that $|g|$ does not divide 8. This forces $|g| = 16$, which implies that $|g^2| = 8$.

Problem 3: We know that $|A_4| = \frac{4!}{2} = 12$ and that $\langle(1,2,3)\rangle$ has order 3. Therefore, the index of $\langle(1,2,3)\rangle$ in A_4 is $\frac{12}{3} = 4$, so $\langle(1,2,3)\rangle$ has 4 distinct right cosets in A_4.

Problem 4: The order of a permutation written in disjoint cycle notation is the least common multiple of its cycle lengths. Therefore, $(1,2,3)(4,5)$ has order 6 in S_5. We also know that R_{90} has order 4 in D_4. The order of $((1,2,3)(4,5), R_{90})$ in $S_5 \oplus D_4$ is therefore $\text{lcm}(6,4) = 12$.

Problem 5: Suppose $\text{Inn}(G)$ has order 1. Let $x, y \in G$. We know that the map $g \mapsto xgx^{-1}$ is the identity function on G. In particular, $y = xyx^{-1}$, or $yx = xy$. Therefore, G is Abelian.

Suppose G is Abelian. Let $x \in G$. For any $g \in G$, $xgx^{-1} = gxx^{-1} = g$, so the inner automorphism $\phi_g : G \to G$ given by $\phi_g(x) = gxg^{-1}$ is the identity function on G. It follows that $\text{Inn}(G)$ has order 1.

Problem 6: We claim that such a list is given by $Z_{12}, Z_6 \oplus Z_2, A_4, D_6$. The Abelian groups Z_{12} and $Z_6 \oplus Z_2$ are not isomorphic because $\gcd(2,6) \neq 1$. No Abelian group is isomorphic to a non-Abelian group, so it is enough to show that A_4 and D_6 are not isomorphic. Indeed, A_4 contains only elements of orders $1, 2$, and 3 whereas D_6 contains an element of order 6 (namely, rotation of the regular hexagon by one notch).