Math 103A: Winter 2014
Midterm 2 Solutions and Comments

Instructions: Please write your name on your blue book. Make it clear in your blue book what problem you are working on. Write legibly and explain your reasoning. This exam is graded out of 100 points. Following these instructions is worth 5 points.

Problem 1: [15 points] (a) Carefully define what it means for a group G to be "cyclic". (b) Prove or give a counterexample: If G and H are cyclic groups, then $G \oplus H$ is also a cyclic group.

Solution: (a) G is cyclic if there exists $g \in G$ such that $G = \langle g \rangle$.
(b) This is false. For example the group $Z_2 \oplus Z_2$ not cyclic. To see this, observe that $|(0,0)| = 1$ but $|(1,0)| = |(1,1)| = |(0,1)| = 2$, so that $Z_2 \oplus Z_2$ does not contain any elements of order 4.

Comments: There was a lot of confusion about what it means for a group to be cyclic. Many of you wrote that G is cyclic if there exists $g \in G$ such that $G = \{g^i : 1 \leq i \leq |G|\}$. This only works for finite cyclic groups (it is not true for Z). There were also many students who did not get Part (b) of this problem – breaking up direct products cyclic groups according to prime factorization was covered extensively in Chapter 8.

Problem 2: [15 points] Let G be a group of order 63. Prove that G contains an element of order 3.

Solution: By Lagrange's Theorem, the possible orders of non-identity elements of G are $3, 7, 9, 21$, or 63. If $g \in G$ has an order which is divisible by 3, then $g^{\frac{|g|}{3}}$ will have order 3. We are therefore reduced to the case where every non-identity element of G has order 7. In this case, we have that G contains $63 - 1 = 62$ elements of order 7. But by a result proved in your book, we know that the number of elements of order 7 must be divisible by $\phi(7) = 6$. Since 6 does not divide 62, we have a contradiction.

Comments: This was a problem from the homework. There were a variety of problems here. Some of you "proved" that G cannot contain *any* elements of order 7 at all. This is not the case (for example, 9 has order 7 in Z_{63}). When problems are repeated from the homework, it's important that you understand their solutions well.

Problem 3: [15 points] Let S be the square in the plane $\mathbb{R}^2$ which is centered at the origin and has side length 4. Let $P = (0,1) \in \mathbb{R}^2$. Let the dihedral group D_4 act on S by permutations. (a) What is the orbit $\text{orb}_{D_4}(P)$? (b) What is the stabilizer $\text{stab}_{D_4}(P)$?

Solution: (a) $\text{orb}_{D_4}(P) = \{(1,0), (-1,0), (0,1), (0,-1)\}$.
(b) $\text{stab}_{D_4}(P) = \{R_0, V\}$ (where V is the reflection through the vertical line $x = 0$).

Comments: Some students drew the square and/or the point incorrectly. Others gave a list of group elements for the orbit rather than points in the square.

Problem 4: [15 points] Let $\alpha = (2, 1, 4, 5, 6, 3) \in S_6$ and let $\beta = (1, 3, 2)(5, 6) \in S_6$. Express the product $\alpha\beta$ as a product of disjoint cycles and determine the order $|\alpha\beta|$.

Solution: We have $\alpha\beta = (2, 1, 4, 5, 6, 3)(1, 3, 2)(5, 6) = (1, 2, 4, 5, 3)(6) = (1, 2, 4, 5, 3)$. The order of a permutation is the least common multiple of the cycle lengths of that permutation in disjoint cycle form. We conclude that $|\alpha\beta| = 5$.

Comments: There was a fair amount of confusion about how to multiply permutations using cycle notation. This is a key computational skill of any undergraduate group theory course. There also seemed to be some confusion about what 'disjoint cycle notation' actually is.

Problem 5: [20 points] Give an example of a group G and an automorphism $\phi \in \mathrm{Aut}(G)$ such that $\phi \notin \mathrm{Inn}(G)$.

Solution: Let $G = Z$ be the group of integers under addition. Then $\mathrm{Inn}(Z) = \{\mathrm{Id}_Z\}$ because Z is Abelian. Therefore, any non-identity automorphism of Z is not an inner automorphism. We take $\phi : Z \to Z$ to be the automorphism $\phi(x) = -x$.

Comments: There are many possible answers here. This was intended to be the most difficult problem on the exam; many people struggled here. A few gave examples of maps ϕ which were not automorphisms (the function $\phi : S_n \to S_n$, $\phi(\alpha) = (12)\alpha$ was very popular – can you see why this doesn't work?). A few students got the idea that looking at an Abelian group was a good strategy (since any non-identity automorphism of an Abelian group is automatically not inner).

Problem 6: [15 points] Let G be the group of 2×2 real matrices of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. Prove that $G \approx \mathbb{R}$.

Solution: Define a function $\phi : \mathbb{R} \to G$ by $\phi(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. The function ϕ is obviously a bijection. Also, we have $\phi(a + b) = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \phi(a)\phi(b)$. Therefore, ϕ is an isomorphism and $G \approx \mathbb{R}$.

Comments: There was a great deal of confusion about the relevant group operations. The group G is a group under matrix multiplication. G is *not* a group under matrix addition (it has no identity) or componentwise multiplication ($\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ would have no inverse). Also, $\mathbb{R}$ is only a group under addition – not multiplication.