Exploring Hyperplane Arrangements and their Regions

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Abstract

This paper studies the Linial, Coxeter, Shi, and Nil hyperplane arrangements and explores ideas of how to count their regions. We give a conjecture concerning the regions of the Linial arrangement (motivated by computational evidence) and a proof of the characteristic polynomial is given for $Nil(n)$ using the Finite Fields method.

1 Introduction

The Linial arrangement, $Lin(n)$, the Coxeter arrangement, $Cox(n)$, the Shi arrangement, $Shi(n)$, and the Nil arrangement, $Nil(n)$, are defined by:

$$\begin{aligned} Lin(n) &= \{x_i - x_j = 1 \,|\, 1 \leq i < j \leq n\} \\ Cox(n) &= \{x_i - x_j = 0 \,|\, 1 \leq i < j \leq n\} \\ Shi(n) &= Cox(n) \cup Lin(n) \\ Nil(n) &= \{x_1 - x_j = i \,|\, 1 \leq i < j \leq n\} \end{aligned}$$

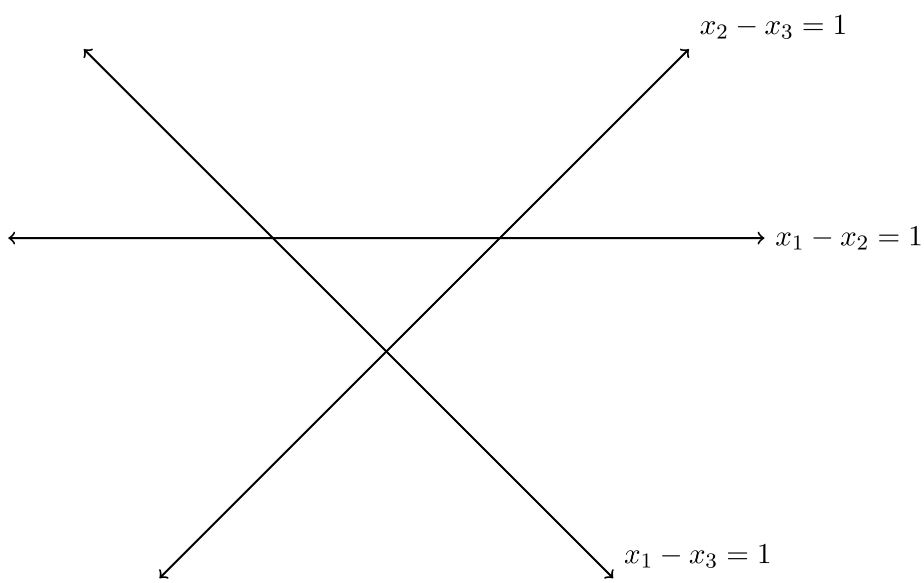


Figure 1: Cross section of $Lin(3)$ in the plane.

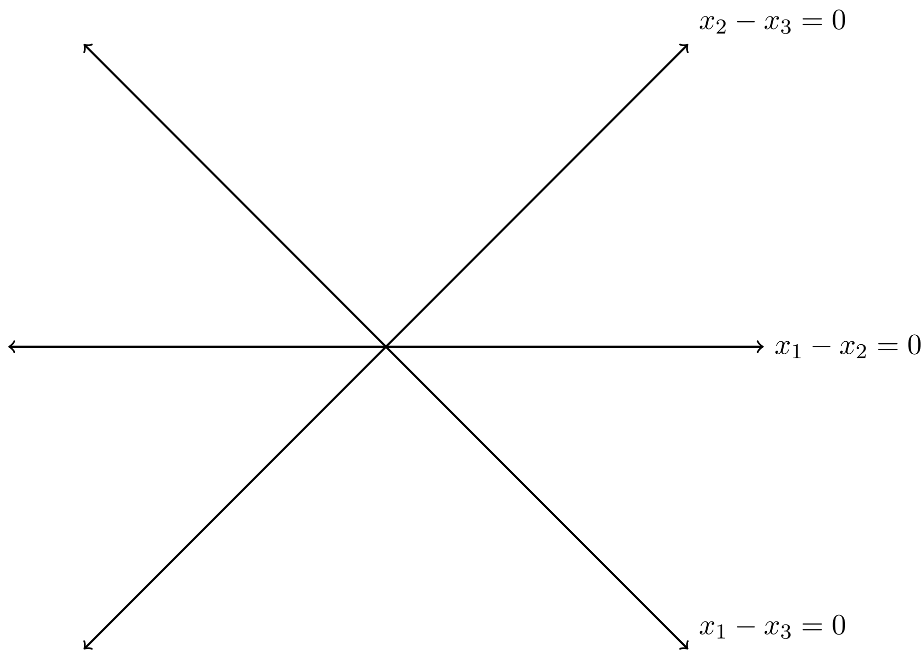


Figure 2: Cross section of $Cox(3)$ in the plane.

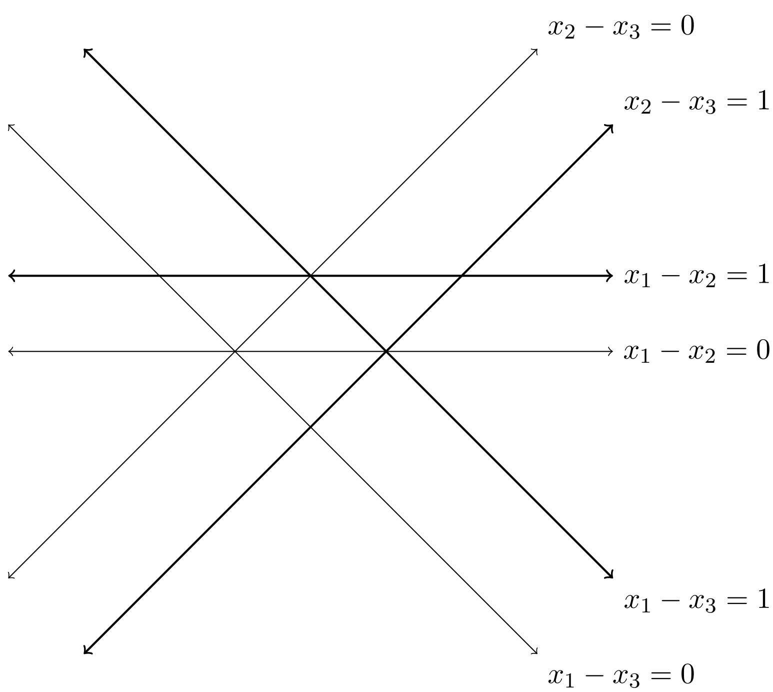


Figure 3: Cross section of $Shi(3)$ in the plane.

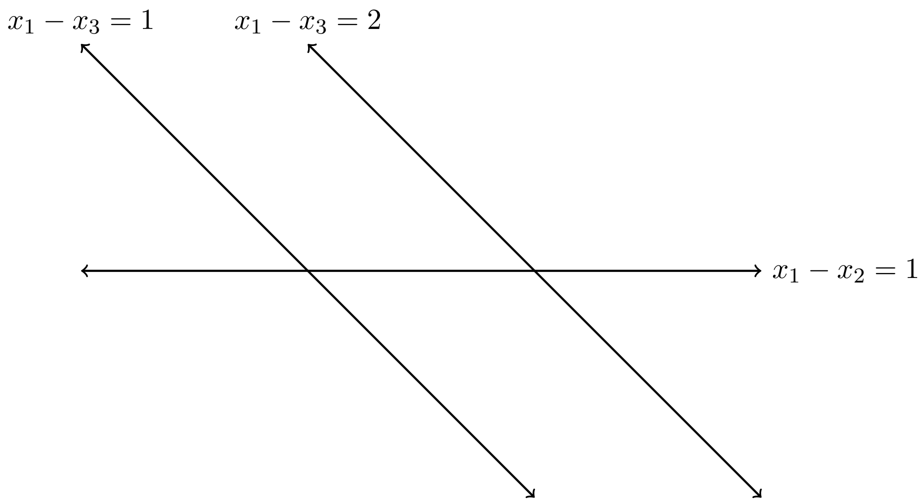


Figure 4: Cross section of $Nil(3)$ in the plane.

The next section analyzes $Lin(n)$ by discussing a way of labeling each region with arc diagrams. The arc diagrams are determined by defining a region by its position relative to each hyperplane in the arrangement. With this labeling, it is possible to generalize certain arc patterns as "bad" (i.e. these patterns do not correspond to Linial regions). These occur when the diagram attempts to specify a region as being on both sides of at least one hyperplane.

Section 4 also studies the regions of $Lin(n)$, this time by recognizing the existence of some equivalence relation between Shi regions and Linial regions. Some ideas of equivalence relations are discussed, but the question remains open. The Shi regions are labeled using the Athanasiadis-Linnuson [1] labeling of decorated permutations.

The final section looks at the related Nil arrangement, and counts the number of regions in $Nil(n)$ by constructing its characteristic polynomial, $\chi_{Nil(n)}(p)$, and taking the absolute value of the polynomial evaluated at $p = -1$. This is valid by the following theorem.

> **Theorem 1: (Zaslavsky's Theorem [2])** For a given hyperplane arrangement, $\mathcal{A}$, and its characteristic polynomial, $\chi_{\mathcal{A}}(p)$, the number of regions in $\mathcal{A}$ is given by $|\chi_{\mathcal{A}}(-1)|$, and the number of relatively bounded regions is given by $|\chi_{\mathcal{A}}(1)|$.

To understand how the characteristic polynomial of a hyperplane arrangement, $\mathcal{A}$ lying in some field, $\mathcal{F}^n$, is constructed, we define the regions of $\mathcal{A}$ as the connected components of $\mathcal{F}^n - \cup_{\mathcal{A}} H$, for each hyperplane, $H \in \mathcal{A}$ (the space minus the hyperplanes) [2, Definition 2]. For the purposes of this paper, $\mathcal{F} = \mathbb{Z}/p\mathbb{Z} = \{0, ..., p-1\}$ for a sufficiently large prime number, p (Note: $(\mathbb{Z}/p\mathbb{Z})^n$ is the Cartesian product of $\mathbb{Z}/p\mathbb{Z}$ with itself n times), and $\mathcal{A} = Nil(n)$. Let $\mathbb{S}_{Nil(n)} \subset (\mathbb{Z}/p\mathbb{Z})^n$ be the solution space of $Nil(n)$. By using the finite field $(\mathbb{Z}/p\mathbb{Z})^n$, we are able to count the number of points in that field (p^n here) and subtract away the finite number of points in the solution space. Let's look at the following example to see the characteristic polynomial for $Nil(3)$ (Figure 4).

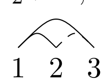
> **Example 1:** $Nil(3) = \{x_1 - x_2 = 1,\ x_1 - x_3 = 1,\ x_1 - x_3 = 2\}$ over the field $(\mathbb{Z}/p\mathbb{Z})^3$
>
> $$\chi_{Nil(3)}(p) = p^3 - |\mathbb{S}_{Nil(3)}|$$
>
> The number of points that satisfy each hyperplane individually is p^2 (choose x_1 and either x_2 or x_3 depending on which hyperplane you are considering) and the number of points of

intersection between each pair is p (only x_1 is free) except for the last two which are parallel. There are no points of intersection for all three, so $|\mathbb{S}_{Nil(3)}| = 3p^2 - 2p$.

2 Labeling the regions of $Lin(n)$

Given a hyperplane arrangement, $\mathcal{A}$, with k hyperplanes, we can naturally define a positive side and a negative side for each hyperplane by making the equations inequalities. A region in $\mathcal{A}$ can, then, be uniquely described as an ordered string of 1's and 0's of length k. This effectively acts as the coordinates of the region. However, it is not always the case that the number of regions created by these k hyperplanes will be 2^k. Take, as example, $Lin(3)$ (Figure 1); there are only 7 regions created by the three hyperplanes, but 8 binary string possibilities. This raises the question of how to tell which strings are "bad". In the case of our example, let $a_{12}a_{13}a_{23}$ be a binary string such that a_{ij} corresponds to a side of $x_i - x_j = 1$. We get the regions labeled as in Figure 5 below. From this picture we can see 101 does not correspond to a region. This makes sense, because it corresponds to a region that is supposed to satisfy $x_1 - x_3 < 1$ and $x_1 - x_3 > 2$ (derived from the sum of the other two equations). The contradiction can also be seen geometrically by looking at the picture. This method of labeling is useful for having a computer find "bad" patterns, but it is not so easy to visually identify nor generalize these "bad" patterns. To do this, we need to define an arc diagram for a region created by $Lin(n)$.

Definition 1: An *arc diagram* for $Lin(n)$ is a line of n points, where two points, i and j, are connected with a solid arc if and only if $x_i - x_j > 1$ and connected with a dashed arc if and only if $x_i - x_j < 1$ (refer to Figure 6 for a translation of every region).

Example 2: Take the region $\{x_1 - x_2 > 1,\, x_1 - x_3 > 1,\, x_2 - x_3 < 1\}$ (110 from Figure 5). Its corresponding arc diagram is 

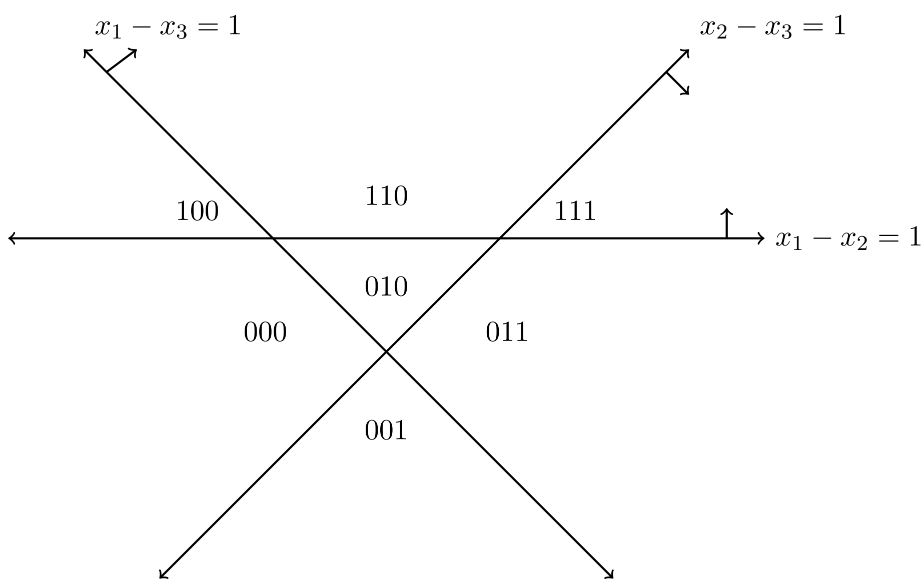


Figure 5: $Lin(3)$ with each hyperplane's positive side noted with a small perpendicular arrow. Each region is labeled according to this orientation.

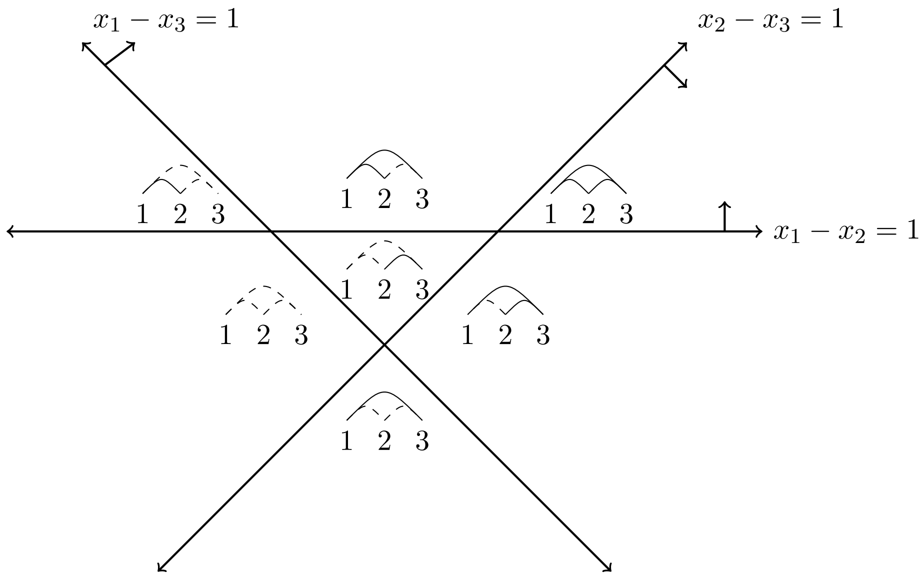
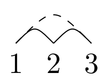
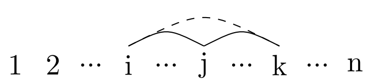


Figure 6: $Lin(3)$ regions labeled with arc diagrams

This leaves  as the "bad" arc diagram for $Lin(3)$. This is visually contradictory, because solid arcs mean the distance between those two points is greater than 1 and the dashed arc means that the total distance between the three points is less than 1. Clearly, this pattern of solid and dashed lines will always lead to a contradiction. We generalize this pattern to $Lin(n)$.

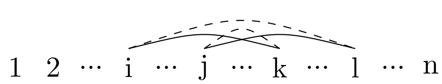
Pattern 1:



for all $1 \leq i < j < k \leq n$

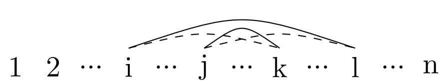
Let's take a look at a few more "bad" patterns, this time on 4 vertices.

Pattern 2:



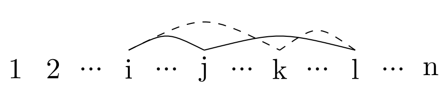
for all $1 \leq i < j < k < l \leq n$

Pattern 3:



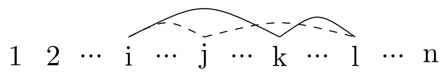
for all $1 \leq i < j < k < l \leq n$

Pattern 4:



for all $1 \leq i < j < k < l \leq n$

Pattern 5:



for all $1 \leq i < j < k < l \leq n$

In order to be convinced that these are, indeed, "bad", use the same visual test mentioned for Pattern 1, or use the inequality argument discussed in the outset of this section. While other "bad" patterns on 4 or more vertices do exist, we believe that only these 5 are necessary and sufficient for finding all arc diagrams that do not map to Linial regions. Using a computer program designed to exclude all arc diagrams that fit these 5 "bad" patterns and then report back the remaining number of arc diagrams, we found the number leftover to be equal to the known number of regions for $n \leq 8$. Due to the speed limits of the computer used, computational evidence for $n > 8$ was not verified. We, therefore, arrive at the following conjecture.

Conjecture 1: All arc diagrams on n points that do not contain these 5 "bad" patterns map to regions in $Lin(n)$.

3 Linial Equivalence Relation

Another approach to understand the Linial regions is to cut up the regions into pieces by introducing more hyperplanes. The Shi arrangement is made up of the Linial hyperplanes and the Coxeter hyperplanes, and has a nice region labeling provided by Athanasiadis-Linnuson [1] (A-L). This labeling begins by filling each region in $Cox(n)$ with a permutation $w = w_1 \cdots w_n$, where for all k $w_k \in \{1, ..., n\}$ and $w_p \neq w_q$. The permutation is determined by the following rule.

Rule 1: If $x_i - x_j > 0$, then i comes before j in the permutation w, otherwise i comes after j in w.

After determining w, it is decorated with arcs by this next rule.

Rule 2: If $x_i - x_j < 1$, then connect i and j with a solid arc in w.

These two rules specify a Shi region, because, together, they imply which side of each hyperplane the region is on. $Shi(3)$ is filled out with its A-L labels below in Figure 7. The picture is also color coded to highlight the existence of some equivalence relation between Shi regions and Linial regions. Let $\mathcal{R}_{Shi(n)}$ be the set of A-L region labels for $Shi(n)$. Also, let $\mathcal{R}_{Lin(n)}$ be the set of regions $\{r_1, r_2, ..., r_k\}$ in $Lin(n)$, where $k = \#$ of regions in $Lin(n)$. Then for every $r_i \in \mathcal{R}_{Lin(n)}$, $r_i \subset \mathcal{R}_{Shi(n)}$, and $r_i \cap r_j = \emptyset$ for all $i \neq j$.

Definition 2: Let two Shi regions be *equivalent* if and only if the are in the same Linial region, r_i.

The question of how to determine which Shi regions are equivalent, i.e. in the same Linial region, is still open. However, with this labeling it is possible to define an equivalence relation for the unbounded Linial regions that include an un-decorated permutation (e.g. 312 in the "blue" Linial region of Figure 7). Given an A-L region label, if you can swap the order of the numbers connected by the arcs and still maintain the relative positions of any numbers outside the arcs, then doing so and, finally, removing the arcs will result in an un-decorated permutation. If any numbers exist underneath the arc, take the whole group of numbers in that arc, and put them in decreasing order.

All A-L region labels with the same transformation permutation are equivalent. Let's look at some examples to clear things up.

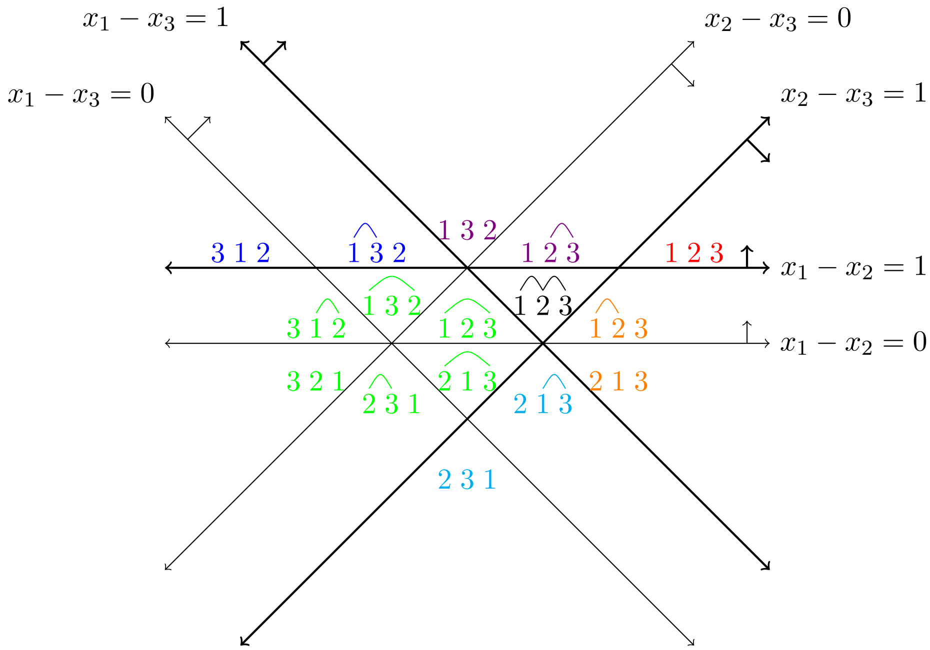


Figure 7: $Shi(3)$ regions labeled according to the two rules.

Example 3: Let's look at the case where there are no numbers between two numbers connected by an arc: $3\,\overparen{1\,2} \mapsto 321$ by swapping 1 and 2. Note that their relative position to 3 is kept.

Example 4: For the case where there are numbers underneath the arc, we have: $\overparen{1\,3\,2} \mapsto$ 321 by putting all numbers involved with the arc in decreasing order.

Example 5: For the case of multiple arcs that don't overlap or connect at a common point, treat each arc individually and apply the rules: $\overparen{2\,4}\,\overparen{1\,3} \mapsto 4231$.

Example 6: For the case of multiple arcs that overlap, do one arc at a time and double check that the resulting permutation maintains the relative order of numbers that don't "interact" with each other via arc connection: $2\,1\,4\,3 \mapsto 4\,2\,\overparen{1\,3} \mapsto 4231$. Notice that 2 and 3 are still in increasing order, because in the original labeling they were increasing and shared no arc connection.

It is important to note that these guidelines are not always able to be followed.

Example 7: For the case where arcs don't overlap but share a common point, it is impossible to swap them and still maintain all of the relative orders that need to be kept: $1\,2\,3 \mapsto \overparen{2\,1\,3} \mapsto 321$, but this does not maintain the order of 1 and 3.

Example 8: Finally, there are some cases where arcs overlap, and following the rules is not possible: $1\,2\,3\,4 \mapsto 3\,2\,1\,4 \mapsto 3421$. This does not maintain the order of the 1 and 4. Also, if you start with the second arc first, you get 4312 which is not only inconsistent, but it also does not maintain the order of the 1 and 4.

These rules make sense, because an arc connecting i to j indicates that the $x_i - x_j < 1$. By swapping their order, you are saying that $x_i - x_j < 0$. Those two are in the same Linial region, because $x_i - x_j < 0 \Rightarrow x_i - x_j < 1$. Moreover, any numbers, $k < m$, that don't "interact" in any way via arc connection imply that $x_k - x_m > 1$, because otherwise they would have been joined with an arc. Finally, the numbers underneath arcs must be put in a decreasing order, because if i is connected to j, effectively the distance between i and j is less than 1. We can conclude, then, that the distances between any two numbers between i and j in the diagram must be less than 1. By putting them in decreasing order, we are saying that they are all less than 0, which surely makes them less than 1.

As for the remaining A-L labels, it is clear that they aren't equivalent to any permutation region, but beyond this there is no clear equivalence relation to be made. You can translate each A-L label into a Linial labeling by following the rules above. From the A-L labeling, you can deduce if $x_i - x_j < 1$ or $x_i - x_j > 1$ for all $i < j$. Then by observing which A-L labels map to the same Linial label, you can verify that two A-L labels are equivalent.

4 Solving for the Characteristic Polynomial of the $Nil(n)$ Hyperplane Arrangement

The final arrangement considered is $Nil(n)$. This arrangement was easier to grasp, and turned out to have some nice properties. We begin by proving the following lemma.

Lemma 1: $\chi_{n+1}(p) = \chi_n(p) \cdot (p - n)$

Proof: We know $\chi_n(p) = p^n - |\mathbb{S}_n|$, so it suffices to show $|\mathbb{S}_{n+1}| = p \cdot |\mathbb{S}_n| + n \cdot \chi_n(p)$. To understand how we break down $|\mathbb{S}_{n+1}|$ we look at difference between $Nil(n+1)$ and $Nil(n)$. To illustrate this difference, let's look at $Nil(3)$ and $Nil(4)$.

Example 9:

$$
\begin{array}{ll|l}
Nil(3) = \{ & \begin{array}{l} x_1 - x_2 = 1 \\ x_1 - x_3 = 1 \\ x_1 - x_3 = 2 \quad \} \end{array} & Nil(4) = \{ \quad \begin{array}{l} x_1 - x_2 = 1 \\ x_1 - x_3 = 1 \\ x_1 - x_3 = 2 \end{array} \\
\hline
 & & \qquad\qquad \begin{array}{l} x_1 - x_4 = 1 \\ x_1 - x_4 = 2 \\ x_1 - x_4 = 3 \quad \} \end{array}
\end{array}
$$

From this we observe that $Nil(4)$ retains all of the "old" hyperplanes from $Nil(3)$ in the "upper half" and adds 3 "new" hyperplanes in the "lower half". Likewise, in the general case, $Nil(n+1)$ retains all of the "old" hyperplanes from $Nil(n)$ in what we'll call the "upper half" of the arrangement and adds n "new" hyperplanes in what we'll call the "lower half" of the arrangement. This observation is what we will use to break up $|\mathbb{S}_{n+1}|$ into smaller pieces we can count separately.

The first piece to count will be the number of points that satisfy the "old" hyperplanes in "upper half" of the arrangement. Because these "old" hyperplanes are the same equations as those in $Nil(n)$, it seems reasonable to use the solution space $\mathbb{S}_n$ as solutions to the "old" hyperplanes in $Nil(n+1)$. However, we must account for the dimension change; we need a way of mapping points from $(\mathbb{Z}/p\mathbb{Z})^n$ to points in $(\mathbb{Z}/p\mathbb{Z})^{n+1}$. Consider the onto function $\Pi(x_1, ..., x_{n+1})$, such that

$$\Pi : (\mathbb{Z}/p\mathbb{Z})^{n+1} \to (\mathbb{Z}/p\mathbb{Z})^n, \text{ where } \Pi(x_1, ..., x_{n+1}) \mapsto (x_1, ..., x_n)$$

If $(s_1, ..., s_n) \in \mathbb{S}_n$, then $\Pi^{-1}(s_1, ..., s_n) \subset \mathbb{S}_{n+1}$ and $|\Pi^{-1}(s_1, ..., s_n)| = p$. Applying this map to all points $(s_1, ..., s_n) \in \mathbb{S}_n$ means the number of points in the "upper half" of the arrangement is counted by $p \cdot |\mathbb{S}_n|$.

The second piece to count is the number of points that satisfy the "new" hyperplanes in the "lower half" of the arrangement. We need to be careful here, though, and remember to subtract away the points that are already counted by the first piece we counted. These are the points where the "old" hyperplanes and the "new" hyperplanes intersect each other. Only looking at the "new" hyperplanes, we see that each one has a solution space of p^n points and put them into $\mathbb{S}_{n+1}$. To see how many points we have double counted, we look back to our Π function. When s_{n+1} was allowed to vary, $|\Pi^{-1}(s_1, ..., s_n)| = p$. However, these "new" hyperplanes fix s_{n+1} making $|\Pi^{-1}(s_1, ..., s_n) \cap \mathbb{S}_{n+1}| = 1$. Applying this all points $(s_1, ..s_n) \in \mathbb{S}_n$, the number of points that each "new" hyperplane has in common with the current set of points in $\mathbb{S}_{n+1}$ is $1 \cdot |\mathbb{S}_n|$. Because the n "new" hyperplanes are parallel, there are no more points of intersection to consider. Applying this argument to each of the n "new" hyperplanes we get that the number of new points in the "lower half" of the arrangement is counted by $n \cdot (p^n - |\mathbb{S}_n|) = n \cdot \chi_n(p)$. Combining our results we get

$$|\mathbb{S}_{n+1}| = p \cdot |\mathbb{S}_n| + n \cdot \chi_n(p) \quad \blacksquare$$

With this lemma proved, we can now determine the coefficients of the polynomial with the following theorem.

Theorem 2: The coefficient of p^k in $\chi_n(p)$ is the Stirling number of the first kind, $s(n, k)$

Proof: The Stirling numbers of the first kind are given by the recurrence relation [2]

$$s(n+1, k) = s(n, k-1) - n \cdot s(n, k)$$

$$s(1, 1) = 1$$

$$s(n, k) = 0 \text{ for all } k > n$$

$$s(n, 0) = 0 \text{ for all } n > 0$$

Let $c(n, i)$ be the coefficient of p^i in $\chi_n(p)$. By the lemma, $\chi_{n+1}(p) = \chi_n(p) \cdot (p - n)$, and it is easy to show that $\chi_1(p) = p$. By distribution we see that

$$c(n+1, i) = c(n, i-1) - n \cdot c(n, i)$$

$$c(1, 1) = 1$$

$$c(n,i) = 0 \text{ for all } i > n$$

$$c(n,0) = 0 \text{ for all } n > 0$$

By comparison to the recurrence relation and initial conditions of the Stirling numbers of the first kind, it is clear that

$$c(n,i) = s(n,k) \text{ for all } i = k \quad \blacksquare$$

Corollary 1: The number of regions in $Nil(n)$ is $n!$.

Proof: By Theorem 2 the characteristic polynomial for $Nil(n)$ is

$$\chi_{Nil(n)}(p) = \sum_{k=1}^{n} s(n,k) \cdot p^k$$

By Theorem 1 (Zaslavsky's Theorem), the number of regions is $\chi_{Nil(n)}(-1) = n!$. This is easily seen when you look back at the recursion of $\chi_{Nil(n)}(p)$ which shows it to be equal to $p(p-1)\cdots(p-n+1)$. $\blacksquare$

5 References

1. Christos A. Athanasiadis and Svante Linnuson, "A Simple Bijection for the Regions of the Shi Arrangement of Hyperplanes," Discrete Mathematics, Volume 204, Issues 13, 6 June 1999, Pages 2739

2. Eric Samansky, "Zaslavsky's Theorem," http://www.math.rice.edu/ samans/ZaslavskyTheorem.pdf

3. http://math2.uncc.edu/ ghetyei/courses/old/S03.3166/stirling.pdf

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