

CYCLIC SIEVING AND CLUSTER MULTICOMPLEXES

BRENDON RHOADES

ABSTRACT. Reiner, Stanton, and White [10] proved results regarding the enumeration of polygon dissections up to rotational symmetry. Eu and Fu [2] generalized these results to Cartan-Killing types other than A by means of actions of deformed Coxeter elements on cluster complexes of Fomin and Zelevinsky [6]. The Reiner-Stanton-White and Eu-Fu results were proven using direct counting arguments. We give representation theoretic proofs of closely related results using the notion of noncrossing and seminoncrossing tableaux due to Pylyavskyy [9] as well as some geometric realizations of finite type cluster algebras due to Fomin and Zelevinsky [5].

1. INTRODUCTION AND BACKGROUND

Let X be a finite set and let $C = \langle c \rangle$ be a finite cyclic group acting on X with distinguished generator c . Let $X(q) \in \mathbb{N}[q]$ be a polynomial in q with nonnegative integer coefficients and let $\zeta \in \mathbb{C}$ be a root of unity with the same multiplicative order as c . Following Reiner, Stanton, and White [10] we say that the triple $(X, C, X(q))$ exhibits the *cyclic sieving phenomenon (CSP)* if for all integers $d \geq 0$, the cardinality of the fixed point set X^{c^d} is equal to the polynomial evaluation $X(\zeta^d)$. Given a finite set X equipped with the action of a finite cyclic group $C = \langle c \rangle$, since the cycle type of the image of c under the canonical homomorphism $C \rightarrow \mathfrak{S}_X$ is determined by the fixed point set sizes $|X^{c^d}|$ for $d \geq 0$, finding a polynomial $X(q)$ such that $(X, C, X(q))$ exhibits the CSP completely determines the enumerative structure of the action of C on X . It can be shown that given a finite set X with the action of a finite cyclic group $C = \langle c \rangle$, it is always possible to find a polynomial $X(q)$ such that $(X, C, X(q))$ exhibits the CSP: for example, if the order of C is n , we can take $X(q) = \sum_{i=0}^{n-1} a_i q^i$, where a_i is equal to the number of C -orbits in X whose stabilizer order divides i [10, Definition-Proposition, p. 1]. The interest in proving a CSP lies in finding a ‘nice’ formula for $X(q)$, ideally making no explicit reference to the action of C on X . These ‘nice’ formulas for $X(q)$ are typically either simple sums or products of q -analogues of numbers or binomial coefficients or are generating functions for some natural statistic

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$stat : X \rightarrow \mathbb{N}$, i.e., $X(q) = \sum_{x \in X} q^{stat(x)}$. In this paper, the set X will typically consist of objects related to noncrossing dissections of a regular n -gon and the group C will act by an appropriate version of rotation.

Let Φ be a root system and let $\Pi \subset \Phi$ be a system of simple roots within Φ . The choice of the simple system Π partitions Φ into two subsets $\Phi = \Phi_{>0} \uplus \Phi_{<0}$ of *positive* and *negative* roots. Let $\Phi_{\geq -1} := \Phi_{>0} \uplus -\Pi$ denote the set of *almost positive* roots, i.e., roots in Φ which are either positive or negatives of simple roots. In 2003 Fomin and Zelevinsky [6] introduced a simplicial complex $\Delta(\Phi)$ called the *cluster complex* whose ground set is $\Phi_{\geq -1}$. A certain deformed Coxeter element τ in the Weyl group of Φ arising from a bipartition of the associated Dynkin diagram acts on the set $\Phi_{\geq -1}$ of almost positive roots and induces a (simplicial) action on the complex $\Delta(\Phi)$ which preserves dimension. When Φ is of type ABCD, the action of τ on the set of faces of $\Delta(\Phi)$ of a fixed dimension is isomorphic to the action of the rotation operator on a certain set of noncrossing polygon dissections with a fixed number of edges (where in type D the definitions of ‘rotation’ and ‘noncrossing’ differ slightly from those for classical polygon dissections). We outline the corresponding actions on polygon dissections in types ABCD. For $n \geq 3$, we denote by \mathbb{P}_n the regular n -gon.

For $n \geq 3$, let $\Phi_{A_{n-3}}$ be a root system of type A_{n-3} . For $k \geq 0$ fixed, the action of the Coxeter element τ on the set of $(k-1)$ -dimensional faces of the cluster complex $\Delta(\Phi_{A_{n-3}})$ is isomorphic to the action of rotation on the set of noncrossing dissections of \mathbb{P}_n with exactly k diagonals.

It turns out that the action of the Coxeter element τ on the cluster complex $\Delta(\Phi)$ is the same for Φ of type B or C. For reasons related to geometric realizations of the types B and C cluster algebras [5] we will use the descriptor ‘C’ in this paper. For $n \geq 2$, let $\Phi_{C_{n-1}}$ be a root system of type C_{n-1} . For $k \geq 0$, the action of τ on the $(k-1)$ -dimensional faces of $\Delta(\Phi_{C_{n-1}})$ is isomorphic to the action of rotation on the set of centrally symmetric dissections of \mathbb{P}_{2n} with exactly k diagonals, where a pair of centrally symmetric nondiameter diagonals of \mathbb{P}_{2n} counts as a *single* diagonal.

Let Φ_{D_n} denote a root system of type D_n for $n \geq 2$. To realize the action of the Coxeter element τ on the cluster complex $\Delta(\Phi_{D_n})$ as an action on dissection-like objects, we must slightly modify our definitions of noncrossing dissections and rotation. A *D-diagonal* in \mathbb{P}_{2n} is either a pair of centrally symmetric nondiameter diagonals or a diameter colored one of two colors, solid/blue or dotted/red. Two D-diagonals are said to *cross* if they cross in the classical sense, except that distinct diameters of the same color do not cross and identical diameters of different colors do not cross. A *D-dissection* of \mathbb{P}_{2n} is a collection of pairwise noncrossing D-diagonals in \mathbb{P}_{2n} . *D-rotation* acts on D-dissections by classical rotation, except that D-rotation switches the color of diameters. In particular, the action of D-rotation on D-dissections of \mathbb{P}_{2n} has order n if n is even and order $2n$ if n is odd. The action of the Coxeter element τ on the set of $(k-1)$ -dimensional faces of the cluster complex $\Delta(\Phi_{D_n})$ is isomorphic to the action

of D-rotation on the set of D-dissections of \mathbb{P}_{2n} with exactly k D-diagonals (where a pair of uncolored centrally symmetric nondiameters counts as a *single* D-diagonal).

The following CSPs involving the actions of deformed Coxeter elements on cluster complexes were proven by Reiner, Stanton, and White [10] in the case of type A and by Eu and Fu [2] in the cases of types B/C and type D. We use the standard q -analog notation

$$\begin{aligned} [m]_q &:= 1 + q + \cdots + q^{m-1}, \\ [m]!_q &:= [m]_q [m-1]_q \cdots [2]_q [1]_q, \\ \begin{bmatrix} m \\ r \end{bmatrix}_q &:= \frac{[m]!_q}{[r]!_q [m-r]!_q}, \end{aligned}$$

for $m \geq r > 0$.

Theorem 1.1. *Fix $k \geq 0$.*

1. [10, Theorem 7.1] *For $n \geq 3$ let X be the set of noncrossing dissections of \mathbb{P}_n with exactly k diagonals. Let $C = \mathbb{Z}_n$ act on X by rotation. The triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where*

$$\begin{aligned} X(q) &= \frac{1}{[n+k]_q} \begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q \begin{bmatrix} n-3 \\ k \end{bmatrix}_q \\ &= \frac{[n+k-1]!_q}{[k]!_q [k+1]!_q [n-k-3]!_q [n-1]_q [n-2]_q}. \end{aligned}$$

2. [2, Theorem 4.1, $s = 1$] *For $n \geq 2$ let X be the set of centrally symmetric dissections of \mathbb{P}_{2n} with exactly k noncrossing diagonals, where a pair of centrally symmetric nondiameter diagonals counts as a single diagonal. Let the cyclic group $C = \mathbb{Z}_n$ of order n act on X by rotation. The triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where*

$$X(q) = \begin{bmatrix} n+k+1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2}.$$

3. [2, Theorem 5.1, $s = 1$] *For $n \geq 2$ let X be the set of D-dissections of \mathbb{P}_{2n} with exactly k D-diagonals. Let the cyclic group $C = \mathbb{Z}_{2n}$ of order $2n$ act on X by D-rotation. The triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where*

$$\begin{aligned} X(q) &= \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q^2} \cdot q^n \\ &\quad + \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q^2} + \begin{bmatrix} n+k-2 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q^2} \cdot q^n. \end{aligned}$$

Specializing Part 1 of Theorem 1.1 to $q = 1$, we get that the number of dissections of \mathbb{P}_n with k noncrossing diagonals is

$$\frac{1}{n+k} \binom{n+k}{k+1} \binom{n-3}{k} = \frac{(n+k-1)!}{k!(k+1)!(n-k-3)!(n-1)(n-2)}.$$

This enumeration was proven first by Cayley [1]. O'Hara and Zelevinsky noted that the Frame-Robinson-Thrall hook length formula [7] implies that this expression is also equal to the number of standard Young tableaux of shape $(k+1)^2 1^{n-k-3}$. When $k = n-3$, the dissections in question are in fact triangulations and the above expression specializes to the Catalan number $C_{n-3} = \frac{1}{n-2} \binom{2n-6}{n-3}$. Eu and Fu proved Parts 1-3 of Theorem 1.1 in the more general context of *s-divisible* polygon dissections, and hence in the context of the actions of deformed Coxeter elements on the generalized cluster complexes of Fomin and Reading [3]. The main purpose of this paper is to use representation theoretic methods motivated by the theory of cluster algebras to prove CSPs which are related to the CSPs in Theorem 1.1.

More precisely, given a finite set X acted on by a finite cyclic group $C = \langle c \rangle$ and a polynomial $X(q) \in \mathbb{N}[q]$, there are essentially two main methods that have been used to show that the triple $(X, C, X(q))$ exhibits the CSP. On its face, the statement that $(X, C, X(q))$ exhibits the CSP is purely enumerative. A direct enumerative proof of such a CSP consists of counting the fixed point sets X^{c^d} for all $d \geq 0$ and showing that these numbers are equal to the polynomial $X(q)$ specialized at appropriate roots of unity. This is how Theorem 1.1 was proven in [10] and [2]. A more algebraic approach dating back to Stembridge [14] in the context of the $q = -1$ phenomenon is as follows. Suppose we have a \mathbb{C} -vector space V with distinguished basis $\{e_x \mid x \in X\}$ indexed by elements of X . Suppose further that V is acted on by a group G and that an element $g \in G$ satisfies

$$g \cdot e_x = e_{c \cdot x}$$

for all $x \in X$. Then, for any $d \geq 0$, the fixed point set cardinality $|X^{c^d}|$ is equal to the character evaluation $\chi(g^d)$, where $\chi : G \rightarrow \mathbb{C}$ is the character of V . It is frequently the case that representation theoretic properties of V and/or group theoretic properties of G can be used to equate the character evaluation $\chi(g^d)$ with the specialization of the polynomial $X(q)$ at an appropriate root of unity. For example, in [11] this method is used to prove that $(X, C, X(q))$ exhibits the CSP, where X is the set of standard Young tableau of fixed rectangular shape $\lambda \vdash n$, the cyclic group $C = \mathbb{Z}_n$ acts on X by jeu-de-taquin promotion, and $X(q) = f^\lambda(q)$ is a q -shift of the generating function for major index on standard tableaux of shape λ . In the proof of this result, the module used is the irreducible \mathfrak{S}_n -module of shape λ taken with respect to its Kazhdan-Lusztig cellular basis and the group element which models the action of C on X is the long cycle $(1, 2, \dots, n) \in \mathfrak{S}_n$.

Since the definition of the CSP is entirely combinatorial, it is appealing to have a direct enumerative proof of a CSP. However, in some cases such as the action of jeu-de-taquin promotion on rectangular standard tableaux above only a representation theoretic proof is known. Moreover, many enumerative proofs of cyclic sieving phenomena involve tricky counting arguments and/or polynomial evaluations. Representation theoretic proofs of CSPs can be more elegant than their enumerative counterparts, as well as give algebraic insight into ‘why’ the CSP holds. It is the purpose of this paper to prove by representation theoretic means a pair of CSPs $(X, C, X(q))$ in Theorems 2.5 and 3.4 involving actions which are closely related to the actions in Parts 1 and 2 of Theorem 1.1 (roughly speaking, our sets X will be obtained by allowing edges to occur with multiplicity and allowing boundary edges to be omitted and our groups C will act by rotation). We will also prove by a hybrid of algebraic and enumerative means Theorem 4.6 which is a ‘multiplicity counting’ version of Part 3 of Theorem 1.1. One feature of our CSPs in Theorems 2.5, 3.4, and 4.6 is that the polynomials $X(q)$ involved will be more representation theoretically suggestive than the polynomials appearing in the CSPs of Theorem 1.1. In particular, our polynomials $X(q)$ will be (at least up to q -shift) the principal specializations of certain symmetric functions arising as Weyl characters of the modules involved in our proofs.

The representation theory involved in our proofs of Theorems 2.5 and 3.4 is motivated by the theory of finite type cluster algebras. Cluster algebras are a certain class of commutative rings introduced by Fomin and Zelevinsky [4]. Every cluster algebra comes equipped with a distinguished generating set of *cluster variables* which are grouped into finite overlapping sets called *clusters*, all of which have the same cardinality. (The common size of these clusters is called the *rank* of the cluster algebra.) The cluster algebras having only finitely many clusters enjoy a classification analogous to the Cartan-Killing classification of finite real reflection groups [5]. These cluster algebras are said to be of *finite type*. Any finite type cluster algebra has a linear basis consisting of *cluster monomials*, i.e., monomials in the cluster variables drawn from a fixed cluster together with a set of *frozen* or *coefficient variables* which only depends on the cluster algebra in question.

It turns out that the cluster algebras of types ABCD are ‘naturally occurring’. More precisely, in [5] Fomin and Zelevinsky endow rings related to the coordinate ring of the Grassmannian $Gr(2, n)$ of 2-dimensional subspaces of \mathbb{C}^n with the structure of a cluster algebra of types A, B, and D. Cluster algebras of type C are given a similar geometric realization in [5]. As finite type cluster algebras, these rings inherit linear bases of cluster monomials. In Sections 2 and 3 we use the geometric realizations of the types A and C cluster algebras presented in [5] to give representation theoretic proofs of multiplicity counting versions of Parts 1 and 2 of Theorem 1.1. In Section 4 we will use the geometric realization of the type D cluster algebra in [5] together

with some combinatorial reasoning to prove a multiplicity counting version of Part 3 of Theorem 1.1.

For the rest of the paper we will use the following notation related to symmetric functions, following the conventions of [13] and [12]. For $n \geq 0$ a *partition* λ of n is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ such that $\lambda_1 + \cdots + \lambda_k = n$. The number k is the *length* $\ell(\lambda)$ of λ and λ is said to have k *parts*. We write $\lambda \vdash n$ to mean that λ is a partition of n . For example, we have $(4, 2, 2) \vdash 8$ and $\ell((4, 2, 2)) = 3$. The *Ferrers diagram* of a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ is the figure consisting of k left-justified rows of dots with λ_i dots in row i for $1 \leq i \leq k$.

A λ -*tableau* T is an assignment of a positive integer to each dot in the Ferrers diagram of λ . The partition λ is the *shape* of the λ -tableau T . A tableau T is called *semistandard* if the entries in T increase weakly across rows and increase strictly down columns. A semistandard tableau T of shape $\lambda \vdash n$ is called *standard* if each of the letters $1, 2, \dots, n$ occur exactly once in T . The *content* $\text{cont}(T)$ of a λ -tableau T is the sequence $\text{cont}(T) = (\text{cont}(T)_1, \text{cont}(T)_2, \dots)$, where $\text{cont}(T)_i$ is the number of i 's in T for all i . Given a partition λ , the *Schur function* $s_\lambda(x_1, x_2, \dots, x_n)$ in n variables is the polynomial in the variable set $\{x_1, \dots, x_n\}$ defined by

$$s_\lambda(x_1, \dots, x_n) = \sum_T x_1^{\text{cont}(T)_1} \cdots x_n^{\text{cont}(T)_n},$$

where the sum ranges over all semistandard tableaux T of shape λ and entries bounded above by n . For $k \geq 0$, the *homogeneous symmetric function* $h_k(x_1, \dots, x_n)$ is given by $h_k(x_1, \dots, x_n) := \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$. We have that $h_k(x_1, \dots, x_n) = s_{(k)}(x_1, \dots, x_n)$. Given a partition $\lambda = (\lambda_1, \dots, \lambda_m)$, we extend the definition of the homogeneous symmetric functions by defining $h_\lambda(x_1, \dots, x_n)$ to be the product $h_{\lambda_1}(x_1, \dots, x_n) \cdots h_{\lambda_m}(x_1, \dots, x_n)$. Given a partition λ and $k \geq 0$, Pieri's Rule states that the product $h_k(x_1, \dots, x_n) s_\lambda(x_1, \dots, x_n)$ is equal to $\sum_\mu s_\mu(x_1, \dots, x_n)$, where μ ranges over the set of all partitions obtained by adding k dots to the Ferrers diagram of λ such that no two dots are added in the same column.

2. TYPE A

Our analog of Part 1 of Theorem 1.1 will involve an action on A -*multidissections* of \mathbb{P}_n which are, roughly speaking, noncrossing dissections of \mathbb{P}_n where boundary edges may be omitted and edges can occur with multiplicity. More formally, if $n > 2$ and $E_A = \binom{[n]}{2}$ is the set of edges in \mathbb{P}_n , an A -multidissection is a function $f : E_A \rightarrow \mathbb{N}$ such that whenever $e, e' \in E_A$ are crossing edges, we have that $f(e) = 0$ or $f(e') = 0$. An A -multidissection has k edges if $\sum_{e \in E_A} f(e) = k$. Figure 2.1 shows an A -multidissection of \mathbb{P}_9 with six edges. For k fixed, the set of A -multidissections of \mathbb{P}_n with k edges carries an action of rotation.

The simplicial complex Δ_n^A of A -multidissections of \mathbb{P}_n is closely related to the cluster complex $\Delta(\Phi_{A_{n-3}})$ of type A_{n-3} . In particular, if Δ is any simplicial complex on the ground set V , let $M(\Delta)$ be the associated multicomplex whose faces are multisets of the form $\{v_1^{a_1}, \dots, v_m^{a_m}\}$ where $a_1, \dots, a_m \geq 0$ and $\{v_1, \dots, v_m\} \subseteq V$ is a face of Δ . Using slightly nonstandard notation, denote by $2^{[n]}$ the $(n-1)$ -dimensional simplex of all subsets of $[n]$ (this is *not* the vertex set of the n -dimensional hypercube). Recall that the *join* $\Delta \star \Delta'$ of two simplicial complexes Δ and Δ' on the ground sets V and V' is the simplicial complex whose ground set is the disjoint union $V \uplus V'$ and whose faces are disjoint unions $F \uplus F'$ of faces $F \in \Delta$ and $F' \in \Delta'$. The fact that the n boundary edges of \mathbb{P}_n never occur in a crossing implies that the complex Δ_n^A decomposes as a join $\Delta_n^A \cong M(\Delta(\Phi_{A_{n-3}})) \star M(2^{[n]})$.

For $x = (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq 2}$ be an $n \times 2$ -matrix of variables and let $\mathbb{C}[x]$ be the polynomial ring in these variables over \mathbb{C} . Let \mathcal{A}_n be the subalgebra of $\mathbb{C}[x]$ generated by the 2×2 minors of the matrix x . For any edge $e = (i, j) \in E_A$ with $i < j$, let $z_e^A := \Delta_{ij} \in \mathcal{A}_n$ be the associated 2×2 matrix minor. Given an A -multidissection $f : E_A \rightarrow \mathbb{N}$, define $z_f^A \in \mathcal{A}_n$ by

$$z_f^A := \prod_{e \in E_A} (z_e^A)^{f(e)}.$$

For example, if f is the A -multidissection of \mathbb{P}_9 in Figure 2.1, then

$$z_f^A = \Delta_{15} \Delta_{19}^2 \Delta_{35} \Delta_{56} \Delta_{68}.$$

The ring \mathcal{A}_n is graded by polynomial degree. Since the generating minors of \mathcal{A}_n all have degree 2, we can write $\mathcal{A}_n \cong \bigoplus_{k \geq 0} V^A(n, k)$, where $V^A(n, k)$ is the subspace of \mathcal{A}_n with homogeneous polynomial degree $2k$.

Theorem 2.1. *Let $n \geq 3$ and $k \geq 0$. The set $\{z_f^A\}$, where f ranges over all A -multidissections of \mathbb{P}_n with exactly k edges, is a \mathbb{C} -basis for the space $V^A(n, k)$.*

This result dates back to Kung and Rota [8]. Later work of Pylyavskyy [9, Theorem 24] on noncrossing tableaux implies this result, as well. Finally, this result was reproven by Fomin and Zelevinsky [5, Example 12.6] and vastly generalized in the context of cluster algebras. Explicitly, Fomin and Zelevinsky endow the ring \mathcal{A}_n with the structure of a type A_{n-3} cluster algebra such that the set $\{z_f^A\}$ of products of minors corresponding to A -multidissections are the cluster monomials. The general linear group $GL_n(\mathbb{C})$ acts on the matrix x of variables by left multiplication. This gives the graded ring \mathcal{A}_n the structure of a graded $GL_n(\mathbb{C})$ -module.

Theorem 2.1 can be used to determine the isomorphism type of the components $V^A(n, k)$ of the graded module \mathcal{A}_n . We will need the notion due to Pylyavskyy [9] of a *seminoncrossing tableau*. Two intervals $[a, b]$ and $[c, d]$ in \mathbb{N} with $a < b$ and $c < d$ are said to be *noncrossing* if neither $a < c < b < d$ nor $c < a < d < b$ hold. Call a rectangular tableau T with exactly two rows *seminoncrossing* if its entries

increase strictly down columns and its columns are pairwise noncrossing when viewed as intervals in \mathbb{N} . We consider two seminoncrossing tableaux to be the same if they differ only by a permutation of their columns.

Proposition 2.2. [9, Theorem 13] *For $n \geq 0$ and a rectangular partition λ with exactly two rows, there is a content preserving bijection between seminoncrossing tableaux of shape λ and entries $\leq n$ and semistandard tableaux of shape λ and entries $\leq n$.*

Pylyavskyy gives a more general definition of a seminoncrossing tableau which applies to general shapes with more than two rows under which this result still holds, but we will not need this here. In both this special case and the more general context, the proof of this result is combinatorial and uses the fact that every even length Yamanouchi word¹ on the letters 1 and 2 corresponds to a unique standard tableau and a unique seminoncrossing tableau of content $(1, \dots, 1)$. Since the condition for a tableau to be semistandard can be phrased as a nonnesting condition on its columns, Proposition 2.2 can be viewed as an instance of combinatorial ‘duality’ between noncrossing and nonnesting objects.

Lemma 2.3. *Let $n \geq 3$ and $k \geq 0$. The graded component $V^A(n, k)$ is isomorphic as a $GL_n(\mathbb{C})$ -module to the irreducible polynomial representation of $GL_n(\mathbb{C})$ of highest weight (k, k) .*

Proof. To prove the claimed module isomorphism, we compute the Weyl character of $V^A(n, k)$. Let $h = \text{diag}(y_1, \dots, y_n) \in GL_n(\mathbb{C})$ be an element of the Cartan subgroup of diagonal matrices. For any A-multidissection f of \mathbb{P}_n , the polynomial z_f^A is an eigenvector for the action of h on $V^A(n, k)$ with eigenvalue equal to

$$\prod_{ij \in \binom{[n]}{2}} (y_i y_j)^{f(ij)}.$$

Summing over A-multidissections f and applying Theorem 2.1, the trace of the action of h on $V^A(n, k)$ is equal to

$$\sum_f \prod_{ij \in \binom{[n]}{2}} (y_i y_j)^{f(ij)}.$$

There is an obvious bijection between the set of A-multidissections f of \mathbb{P}_n with k edges and seminoncrossing tableaux of shape (k, k) and entries bounded above by n obtained by letting the edge (i, j) with $i < j$ correspond to the length two column containing i above j . For example, the A-multidissection in Figure 2.1 is mapped to

¹A *Yamanouchi word* is a finite sequence $w_1 \dots w_n$ of positive integers such that for all $i > 0$ and all $1 \leq j \leq n$, the number of i 's in the prefix $w_1 \dots w_j$ is greater than or equal to the number of $(i+1)$'s in the prefix $w_1 \dots w_j$.

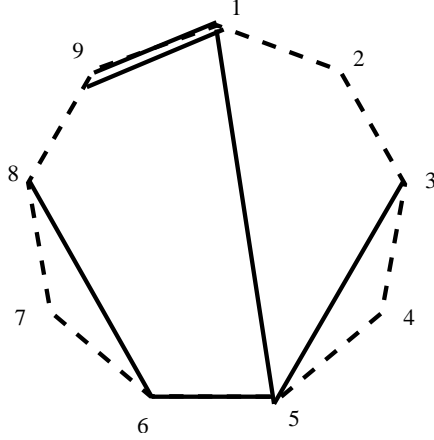


FIGURE 2.1. An A-multidissection of \mathbb{P}_9 with six edges

the seminoncrossing tableau: $\begin{matrix} 1 & 1 & 1 & 3 & 5 & 6 \\ 5 & 9 & 9 & 5 & 6 & 8 \end{matrix}$. (When drawing A-multidissections, edges are drawn with the multiplicity given by the multidissection and dashed boundary edges indicate boundary edges which are included with multiplicity zero.)

Since this bijection preserves weights, Proposition 2.2 implies that the above expression is equal to the Schur function

$$s_{(k,k)}(y_1, \dots, y_n),$$

which proves the desired module isomorphism. □

Define g^A to be the element of $GL_n(\mathbb{C})$ given by

$$g^A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Thus, g^A is the permutation matrix for the long cycle in \mathfrak{S}_n with the upper right 1 replaced with a -1 . The following observation is a simple computation.

Observation 2.4. *Let r denote the rotation operator and let f by any A-multidissection of \mathbb{P}_n . Then,*

$$g^A \cdot z_f^A = z_{r.f}^A.$$

Theorem 2.5. *Fix $n \geq 3$ and $k \geq 0$ and let X be the set of A-multidissections of \mathbb{P}_n with k edges. Let the cyclic group $C = \mathbb{Z}_n$ of order n act on X by rotation. The*

triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where

$$X(q) = q^{-k} s_{(k,k)}(1, q, \dots, q^{n-1}).$$

Proof. Fix $d \geq 0$ and let $\zeta = e^{\frac{2\pi i}{n}}$. Let $r : X \rightarrow X$ be the rotation operator. By Theorem 2.1, we know that the set $\{z_f^A \mid f \in X\}$ forms a basis for $V^A(n, k)$. Observation 2.4 implies that the fixed point set cardinality $|X^{r^d}|$ is equal to the trace of $(g^A)^d$ on $V^A(n, k)$.

On the other hand, we have that $(g^A)^d$ is $GL_n(\mathbb{C})$ -conjugate to the diagonal matrix $\alpha^{-d} \text{diag}(1, \zeta^d, \dots, \zeta^{(n-1)d})$, where $\alpha = e^{\frac{\pi i}{n}}$. So, by Lemma 2.3, the trace of $(g^A)^d$ on $V^A(n, k)$ is the Schur function specialization $\alpha^{-2dk} s_{(k,k)}(1, \zeta^d, \dots, \zeta^{(n-1)d}) = X(\zeta^d)$, as desired. \square

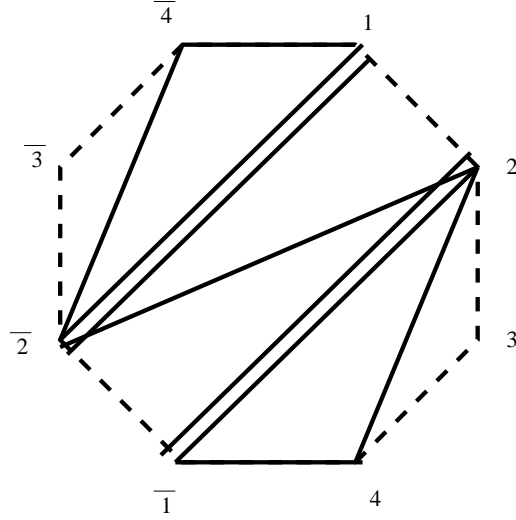
Fixing n and a divisor $d|n$, the set of A-multidissections of \mathbb{P}_n which are fixed by the d -th power of rotation is closely related to the set of classical dissections of \mathbb{P}_n which are fixed by the d -th power of rotation. An enumerative sieve argument implies that the collection of fixed point set cardinalities given by Part 1 of Theorem 1.1 is obtainable from the set of fixed point set cardinalities given in Theorem 2.5, and vice versa.

The polynomial $X(q)$ appearing in Theorem 2.5 has a nice product formula given by Stanley's q -hook content formula [13]. $X(q)$ is also the generating function for plane partitions inside a $2 \times k \times (n-2)$ -box. The polynomial $X(q)$ appears in a CSP involving the action of promotion on semistandard tableaux of shape (k, k) and entries bounded above by n [11]. The q -shift $q^k X(q) = [s_{(k,k)}(x_1, \dots, x_n)]_{x_i=q^{i-1}}$ is called the *principal specialization* of the Schur function $s_{(k,k)}(x_1, \dots, x_n)$. Finally, the polynomial $X(q)$ can be interpreted (up to q -shift) as the q -Weyl dimension formula for the irreducible polynomial representation of $GL_n(\mathbb{C})$ of highest weight (k, k) .

3. TYPE B/C

While the actions of the deformed Coxeter element τ on the cluster complexes of types B and C are identical, the geometric realizations of the cluster algebras of types B and C given in [5] are quite different. In proving our multiplicity counting analog of Part 2 of Theorem 1.1 we will use the geometric realization of the type C cluster algebra in [5].

For $n \geq 2$, a C -edge in \mathbb{P}_{2n} is either a pair of centrally symmetric nondiameter edges (which may be on the boundary of \mathbb{P}_{2n}) or a diameter of \mathbb{P}_{2n} . A C -multidissection of \mathbb{P}_{2n} is a function $f : E_C \rightarrow \mathbb{N}$, where E_C is the set of C-edges in \mathbb{P}_{2n} , such that for any pair e, e' of crossing C-edges we have $f(e) = 0$ or $f(e') = 0$. As in the type A case, the fact that the n pairs of centrally symmetric boundary edges of \mathbb{P}_{2n} never occur in a crossing implies that the simplicial complex Δ_n^C formed by the

FIGURE 3.1. A C-multidissection of \mathbb{P}_8 with five edges

C-multidissections of \mathbb{P}_{2n} is related to the type C_{n-1} cluster complex $\Delta(\Phi_{C_{n-1}})$ via $\Delta_n^C \cong M(\Delta(\Phi_{C_{n-1}})) \star M(2^{[n]})$. A C-multidissection f has k edges if $\sum_{e \in E_C} f(e) = k$. Figure 3.1 shows a C-multidissection of \mathbb{P}_8 with five edges. Rotation acts with order n on the set of C-multidissections of \mathbb{P}_{2n} with k edges.

We recall the geometric realization of the type C_{n-1} cluster algebra given in [5]. Fix $n \geq 2$, let $x = (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq 2}$ be an $n \times 2$ matrix of variables, and let $\mathbb{C}[x]$ be the polynomial ring in these variables over \mathbb{C} . The multiplicative group \mathbb{C}^\times of nonzero complex numbers acts on $\mathbb{C}[x]$ by $\alpha.x_{i1} = \alpha x_{i1}$ and $\alpha.x_{i2} = \alpha^{-1}x_{i2}$ for $1 \leq i \leq n$. Let $\mathbb{C}[x]^{\mathbb{C}^\times}$ be the invariant subalgebra for this action. Since no polynomial in $\mathbb{C}[x]$ containing a nonzero homogeneous component of odd degree is fixed by \mathbb{C}^\times , we have the grading $\mathbb{C}[x]^{\mathbb{C}^\times} \cong \bigoplus_{k \geq 0} V^C(n, k)$, where $V^C(n, k)$ is the subspace of $\mathbb{C}[x]^{\mathbb{C}^\times}$ spanned by polynomials of homogeneous degree $2k$.

We construct a graded action of $GL_n(\mathbb{C})$ on $\mathbb{C}[x]^{\mathbb{C}^\times}$ as follows. Let $\sigma : GL_n(\mathbb{C}) \rightarrow GL_n(\mathbb{C})$ be the involution defined by $\sigma(A) = (A^{-1})^T$. The group $GL_n(\mathbb{C})$ acts on the polynomial algebra $\mathbb{C}[x_{11}, \dots, x_{n1}]$ by considering this algebra as the symmetric algebra over the defining representation of $GL_n(\mathbb{C})$. In addition, the group $GL_n(\mathbb{C})$ acts on the algebra $\mathbb{C}[x_{12}, \dots, x_{n2}]$ via the above action precomposed with σ . These actions tensor together to give an action of $GL_n(\mathbb{C})$ on $\mathbb{C}[x] \cong \mathbb{C}[x_{11}, \dots, x_{n1}] \otimes_{\mathbb{C}} \mathbb{C}[x_{12}, \dots, x_{n2}]$. Restriction of this action to the center $Z(GL_n(\mathbb{C})) \cong \mathbb{C}^\times$ of nonzero multiples of the identity matrix yields the action of \mathbb{C}^\times in the above paragraph. Therefore, the invariant space $\mathbb{C}[x]^{\mathbb{C}^\times}$ is a graded $GL_n(\mathbb{C})$ -module. As a $GL_n(\mathbb{C})$ -module, we have that $\mathbb{C}[x] \cong \text{Sym}(\mathbb{C}^n) \otimes_{\mathbb{C}} \text{Sym}((\mathbb{C}^n)^*)$, where \mathbb{C}^n carries the defining representation of $GL_n(\mathbb{C})$ and $(\mathbb{C}^n)^*$ carries the dual of the defining representation of

$GL_n(\mathbb{C})$. The invariant subalgebra $\mathbb{C}[x]^{\mathbb{C}^\times}$ is the Serge subalgebra of $\mathbb{C}[x]$ consisting of polynomials which are bihomogeneous with degrees of the form (k, k) . Therefore, we have the decomposition of $GL_n(\mathbb{C})$ -modules:

$$\mathbb{C}[x]^{\mathbb{C}^\times} = \bigoplus_{k \geq 0} \text{Sym}^k(\mathbb{C}^n) \otimes_{\mathbb{C}} \text{Sym}^k((\mathbb{C}^n)^*).$$

The next result follows from looking at the k -th graded piece of the above direct sum.

Lemma 3.1. *Let $n \geq 2$ and $k \geq 0$. Let \mathbb{C}^n carry the defining representation of $GL_n(\mathbb{C})$ and let $(\mathbb{C}^n)^*$ be the dual of this representation. The $GL_n(\mathbb{C})$ -module $V^C(n, k)$ is isomorphic to the rational $GL_n(\mathbb{C})$ -module $\text{Sym}^k(\mathbb{C}^n) \otimes_{\mathbb{C}} \text{Sym}^k((\mathbb{C}^n)^*)$.*

As in [5, Example 12.12], \mathbb{C} -multidissections can be used to build \mathbb{C} -bases for the spaces $V^C(n, k)$. Label the vertices of \mathbb{P}_{2n} clockwise with $1, 2, \dots, n, \bar{1}, \bar{2}, \dots, \bar{n}$. For any \mathbb{C} -edge $e \in E_C$ we associate a polynomial $z_e^C \in V^C(n, 1)$ as follows. If e is a diameter of the form $a\bar{a}$ for $1 \leq a \leq n$, define $z_e^C = x_{a1}x_{a2}$. If e is a pair of ‘integrated’ centrally symmetric nondiameter edges of the form $a\bar{b}, \bar{a}b$ for $1 \leq a < b \leq n$, define $z_e^C = \frac{x_{a1}x_{b2} + x_{a2}x_{b1}}{2}$. Finally, if e is a pair of ‘segregated’ centrally symmetric nondiameter edges of the form $ab, \bar{a}\bar{b}$ for $1 \leq a < b \leq n$, define $z_e^C = \frac{x_{a1}x_{b2} - x_{a2}x_{b1}}{2i}$. If $f : E_C \rightarrow \mathbb{N}$ is any \mathbb{C} -multidissection of \mathbb{P}_{2n} , define z_f^C by

$$z_f^C := \prod_{e \in E_C} (z_e^C)^{f(e)}.$$

For example, if f is the \mathbb{C} -multidissection of \mathbb{P}_8 shown in Figure 3.1, then

$$z_f^C = \left(\frac{x_{11}x_{22} + x_{12}x_{21}}{2}\right)^2 \left(\frac{x_{11}x_{24} + x_{14}x_{21}}{2}\right) \left(\frac{x_{12}x_{24} - x_{14}x_{22}}{2i}\right) (x_{21}x_{22}) \in V^C(4, 5).$$

The type \mathbb{C} analog of Theorem 2.1 is as follows.

Theorem 3.2. [5, Proposition 12.13] *Let $n \geq 2$ and $k \geq 0$. Then, the set $\{z_f^C\}$, where f ranges over all \mathbb{C} -multidissections of \mathbb{P}_{2n} with exactly k edges, forms a \mathbb{C} -basis for $V^C(n, k)$.*

Let g^C be the element of $GL_n(\mathbb{C})$ given by

$$g^C = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & i \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Thus, g^C is the permutation matrix for the long cycle in \mathfrak{S}_n , except that the upper right hand entry is i instead of 1. The following observation is a direct computation involving the action of g^C on z_e^C in the cases where e is a diameter or a pair of

centrally symmetric nondiameter edges. This latter case breaks up into two subcases depending on whether the vertex n is involved in the edges in the C-edge e .

Observation 3.3. *Let r denote the rotation operator and let f be any C-multidissection of \mathbb{P}_{2n} . Then,*

$$g^C \cdot z_f^C = z_{r \cdot f}^C.$$

Theorem 3.4. *Let $n \geq 2$ and $k \geq 0$. Let X be the set of C-multidissections of \mathbb{P}_{2n} with k edges and let the cyclic group $C = \mathbb{Z}_n$ act on X by rotation. The triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where*

$$X(q) = h_{(k,k)}(1, q, \dots, q^{n-1}).$$

Proof. Let $r : X \rightarrow X$ be the rotation operator and fix $d \geq 0$. By Theorem 3.2 and Observation 3.3, the cardinality of the fixed point set X^{r^d} is equal to the trace of the linear operator $(g^C)^d$ on the $GL_n(\mathbb{C})$ -module $V^C(n, k)$. We have that $(g^C)^d$ is $GL_n(\mathbb{C})$ -conjugate to the diagonal matrix $\alpha^d \text{diag}(1, \zeta^d, \dots, \zeta^{(n-1)d})$, where $\zeta = e^{\frac{2\pi i}{n}}$ and $\alpha = e^{\frac{\pi i}{2n}}$. Lemma 3.1 implies that the trace in question is equal to

$$\alpha^d h_k(1, \zeta^d, \dots, \zeta^{(n-1)d}) \alpha^{-d} h_k(1, \zeta^{-d}, \dots, \zeta^{(1-n)d}) = h_{(k,k)}(1, \zeta^d, \dots, \zeta^{(n-1)d}) = X(\zeta^d),$$

as desired. \square

As with Theorem 2.5, an enumerative sieve can be used to relate the fixed point set sizes predicted by Theorem 3.4 to those predicted by Part 2 of Theorem 1.1.

The polynomial $h_k(1, q, \dots, q^{n-1})$ is the q -analog of a multinomial coefficient, and therefore the polynomial $X(q)$ in Theorem 3.4 has a nice product formula. To our knowledge this is the first occurrence of the polynomial $h_{(k,k)}(1, q, \dots, q^{n-1})$ in a CSP.

4. TYPE D

For $n \geq 2$, a D -edge of \mathbb{P}_{2n} is either a pair of centrally symmetric nondiameter edges of \mathbb{P}_{2n} (which may be boundary edges) or a diameter of \mathbb{P}_{2n} colored one of two colors, solid/blue or dotted/red. As in Section 1, two D -edges are said to *cross* if they cross in the classical sense, except that distinct diameters of the same color and identical diameters of different colors do not cross. A D -multidissection of \mathbb{P}_{2n} is a function $f : E_D \rightarrow \mathbb{N}$, where E_D is the set of D -edges in \mathbb{P}_{2n} and whenever e and e' are crossing D -edges, we have $f(e) = 0$ or $f(e') = 0$. Also as in Section 1, D -rotation acts on the set of D -multidissections by standard rotation, except that D -rotation switches the colors of the diameters.

When counting the edges in a D -multidissection, we count diameters as one edge and pairs of centrally symmetric nondiameter edges as two edges. More formally, if $E_D(CS)$ denotes the set of centrally symmetric pairs of nondiameters in \mathbb{P}_{2n} and if $E_D(D)$ denotes the set of colored diameters in \mathbb{P}_{2n} , we say that a D -multidissection

f of \mathbb{P}_{2n} has k edges if $\sum_{e \in E_D(CS)} 2f(e) + \sum_{e \in E_D(D)} f(e) = k$. For example, the D-multidissection of \mathbb{P}_8 on the right in Figure 4.1 has nine edges. This counting convention *differs* from that of Part 3 of Theorem 1.1, where a pair of centrally symmetric nondiameters counted as a single edge. Although this counting convention is less natural from the point of view of cluster complexes in which a pair of centrally symmetric nondiameters corresponds to a single almost positive root, it is more natural from the standpoint of edge enumeration in polygon dissections.

The proof of our CSP involving D-multidissections will use both algebraic and combinatorial methods. Recall the definition of the algebra \mathcal{A}_n from Section 2. We define the algebra $V^D(n)$ to be the quotient of \mathcal{A}_{n+2} by the ideal generated by the minor $\Delta_{n+1, n+2}$. For $1 \leq i \leq n+2$ and $1 \leq j \leq 2$, the *D-degree* of the variable $x_{ij} \in \mathcal{A}_{n+2}$ is defined to be 1 if $1 \leq i \leq n$ and 0 otherwise. The D-degree of a 2×2 -minor Δ_{ij} remains well-defined in the quotient ring $V^D(n)$ and induces a grading $V^D(n) \cong \bigoplus_{k \geq 0} V^D(n, k)$, where $V^D(n, k)$ is the subspace of $V^D(n)$ with homogeneous D-degree k . We can use A-multidissections to write down a basis for $V^D(n, k)$. Viewing A-multidissections as multidissections of edges, any A-multidissection of \mathbb{P}_{n+2} gives rise to a multiset of endvertices. If we label the vertices of \mathbb{P}_{n+2} clockwise with $1, 2, \dots, n+2$, it makes sense to count how many of these vertices lie in $[n]$ with multiplicity. For example, if $n = 7$, we have that 9 vertices of the A-multidissection of \mathbb{P}_9 shown in Figure 2.1 lie in $[7]$ counting multiplicity.

Lemma 4.1. *Let $n \geq 2$ and $k \geq 0$. Abusing notation, identify polynomials in \mathcal{A}_{n+2} with their images in $V^D(n)$. We have that the set $\{z_f^A\}$, where f ranges over all A-multidissections of \mathbb{P}_{n+2} with $f(n+1, n+2) = 0$ and k vertices of f counting multiplicity are in $[n]$, is a \mathbb{C} -basis for $V^D(n, k)$.*

Proof. Let $J \subset \mathcal{A}_{n+2}$ be the ideal generated by the minor $\Delta_{n+1, n+2}$. By Theorem 2.1 it is enough to show that J is spanned over \mathbb{C} by all polynomials of the form z_f^A , where f is an A-multidissection of \mathbb{P}_{n+2} satisfying $f(n+1, n+2) > 0$. Clearly this span is contained in J . The reverse containment follows from the fact that the boundary edge $(n+1, n+2)$ crosses none of the edges in \mathbb{P}_{n+2} , and can therefore be added to any A-multidissection without creating any crossings. \square

As in Section 2, the space \mathcal{A}_{n+2} is a $GL_{n+2}(\mathbb{C})$ -module. Considering the inclusion $GL_n(\mathbb{C}) \times GL_2(\mathbb{C}) \subset GL_{n+2}(\mathbb{C})$ via $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, the space \mathcal{A}_{n+2} is also a $GL_n(\mathbb{C}) \times GL_2(\mathbb{C})$ -module by restriction. This latter action descends to the quotient $V^D(n)$ and respects D-degree, giving $V^D(n) \cong \bigoplus_{k \geq 0} V^D(n, k)$ the structure of a graded $GL_n(\mathbb{C}) \times GL_2(\mathbb{C})$ -module. The isomorphism type of $V^D(n, k)$ can be determined using Lemma 4.1.

Lemma 4.2. *Let $n \geq 2$ and $k \geq 0$. For any partition λ , let V_λ be the irreducible polynomial representation of $GL_n(\mathbb{C})$ of highest weight λ . The $GL_n(\mathbb{C}) \times GL_2(\mathbb{C})$ -module $V^D(n, k)$ is isomorphic to the representation of $GL_n(\mathbb{C}) \times GL_2(\mathbb{C})$ given by $\bigoplus_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} V_{(k-\ell, \ell)} \otimes_{\mathbb{C}} \text{Sym}^{k-2\ell}(\mathbb{C}^2)$, where \mathbb{C}^2 carries the defining representation of $GL_2(\mathbb{C})$.*

Proof. Let $\mathcal{A}_{n+2}(-z_1 z_2)$ denote the $GL_n(\mathbb{C}) \times GL_2(\mathbb{C})$ -module whose underlying vector space is the same as \mathcal{A}_{n+2} and whose module structure is obtained by $(g, h) \cdot v := \det(h)(g, h) \cdot v$, where the raised dot denotes action on $\mathcal{A}_{n+2}(-z_1 z_2)$ and the lowered dot denotes action on \mathcal{A}_{n+2} . Lemma 4.1 implies that we have the following short exact sequence of $GL_n(\mathbb{C}) \times GL_2(\mathbb{C})$ -modules.

$$0 \rightarrow \mathcal{A}_{n+2}(-z_1 z_2) \xrightarrow{\cdot \Delta_{n+1, n+2}} \mathcal{A}_{n+2} \rightarrow V^D(n) \rightarrow 0,$$

where the left hand map is multiplication by $\Delta_{n+1, n+2}$ and the right hand map is the canonical projection. This short exact sequence implies that the trace of the action of $(\text{diag}(y_1, \dots, y_n), \text{diag}(z_1, z_2)) \in GL_n(\mathbb{C}) \times GL_2(\mathbb{C})$ on $V^D(n)$ is equal to

$$(1 - z_1 z_2) \sum_{m \geq 0} s_{(m, m)}(y_1, \dots, y_n, z_1, z_2) = \sum_T (yz)^T,$$

where the latter sum ranges over all semistandard tableaux T of shape (m, m) and entries $y_1 < \dots < y_n < z_1 < z_2$ having no z 's in the last column. Restricting such a tableau T to its y and z entries gives a semistandard tableau T_1 of some shape (m, ℓ) in the y entries and another semistandard tableau T_2 of shape $(m - \ell)$ in the z entries. Such a tableau T therefore contributes a term to the Weyl character of $V^D(n, k)$, where $k = m + \ell$. We have that $m - \ell = k - 2\ell$, so that T contributes a typical term of $s_{(k-\ell, \ell)}(y_1, \dots, y_n) h_{k-2\ell}(z_1, z_2)$. It follows that the Weyl character of $V^D(n, k)$ agrees with the Weyl character of the module in the statement of the lemma. \square

We remark that a combinatorial proof of Lemma 4.2 can be obtained using Lemma 4.1 and an explicit weight-preserving bijection between A-multidissections and semi-noncrossing tableaux which computes the Weyl character of $V^D(n, k)$. A more general notion (due to Pylyavskyy [9]) of seminoncrossing tableaux for two-row shapes which are not rectangular is needed in this context.

The module isomorphism in Lemma 4.2 can be combined with a geometric realization of the type D cluster algebras in [5] to give an enumeration of the k -edge D-multidissections of \mathbb{P}_{2n} in terms of symmetric function evaluations. In what follows, if $f(x_1, \dots, x_n)$ is a symmetric function in n variables, abbreviate by $f(1^n)$ the evaluation $f(1, \dots, 1)$. Also, adopt the convention that the digon \mathbb{P}_2 has two types of D-edges: a 'diameter' which can be colored solid/blue or dotted/red. In accordance with our earlier conventions regarding crossings in D-multidissections, no multiset of

D-edges in \mathbb{P}_2 has a crossing. We count each diameter in \mathbb{P}_2 as a *single* edge, so that the number of k -edge D-multidissections of \mathbb{P}_2 is equal to $k + 1$ for all k .

Corollary 4.3. *Let $n \geq 1$ and $k \geq 0$. The number of k -edge D-multidissections of \mathbb{P}_{2n} is equal to $2h_{(k)}(1^n) + 2h_{(k-1,1)}(1^n) + \cdots + 2h_{(\frac{k+1}{2}, \frac{k-1}{2})}(1^n)$ if k is odd, and $2h_{(k)}(1^n) + 2h_{(k-1,1)}(1^n) + \cdots + 2h_{(\frac{k}{2}+1, \frac{k}{2}-1)}(1^n) + h_{(\frac{k}{2}, \frac{k}{2})}(1^n)$ if k is even.*

Proof. If $n = 1$ this enumeration of D-multidissections of the digon can be checked directly. For $n > 1$, the geometric realization of the type D cluster algebra given in [5, Section 12.4] gives a cluster monomial basis for $V^D(n)$ where cluster variables are indexed by D-edges, clusters monomials are indexed by D-multidissections, the cluster variables corresponding to colored diameters have D-degree 1, and the cluster variables corresponding to pairs of centrally symmetric nondiameters have D-degree 2. Therefore, the number of k -edge D-multidissections of \mathbb{P}_{2n} is equal to the dimension of the space $V^D(n, k)$. This dimension is equal to the Weyl character in the proof of Lemma 4.2 evaluated at the identity matrix $y_1 = \cdots = y_n = z_1 = z_2 = 1$. For all k and ℓ we have that $h_{k-2\ell}(1^2) = k - 2\ell + 1$. So, the number of k -edge D-multidissections of \mathbb{P}_{2n} is equal to

$$\sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} (k - 2\ell + 1) s_{(k-\ell, \ell)}(1^n).$$

The equality of this expression and the expression in the statement of the corollary is a consequence of Pieri's Rule. \square

The following lemma will be useful in the proof of our type D CSP in Theorem 4.6. Roughly, it states that D-multidissections of \mathbb{P}_{2n} which are invariant under a fixed even power of D-rotation are in bijection with D-multidissections of a smaller polygon.

Lemma 4.4. (*Folding Lemma*) *Let $k \geq 0$, $n \geq 2$, and let d be an even divisor of $2n$. Let r be the D-rotation operator on D-multidissections of \mathbb{P}_{2n} .*

1. *If d divides n , the set of k -edge r^d -invariant D-multidissections of \mathbb{P}_{2n} is in bijective correspondence with the set of $\frac{kd}{n}$ -edge D-multidissections of \mathbb{P}_{2d} .*

2. *If d does not divide n , the set of k -edge r^d -invariant D-multidissections of \mathbb{P}_{2n} is in bijective correspondence with the set of $\frac{kd}{2n}$ -edge D-multidissections of \mathbb{P}_d .*

We interpret the set of m -edge D-multidissections of a polygon to be empty if m is not an integer.

Lemma 4.4 does not hold if d is odd. In this case, the operator r^d swaps diameter colors and therefore no r^d -invariant D-multidissection of \mathbb{P}_{2n} contains a diameter. When d is odd, there is a natural bijection between k -edge r^d -invariant D-multidissections of \mathbb{P}_{2n} and $\frac{k}{2}$ -edge C-multidissections of \mathbb{P}_{2n} which are invariant under the d -th power of rotation.

Proof. Let $C = \langle r^d \rangle$ be the cyclic group generated by r^d . Since d is even, observe that r^d preserves diameter colors and is therefore the d -th power of the classical rotation operator. The idea behind the bijective correspondences in Parts 1 and 2 is to map each C -orbit \mathcal{O} of mutually noncrossing D -edges in \mathbb{P}_{2n} to a one- or two-element set $\psi(\mathcal{O})$ of noncrossing D -edges in \mathbb{P}_{2d} (in Part 1) or \mathbb{P}_d (in Part 2). Roughly speaking, $\psi(\mathcal{O})$ will be a pair of diameters of different colors when \mathcal{O} is an inscribed polygon and $\psi(\mathcal{O})$ will be a singleton consisting of a D -edge obtained by ‘folding’ \mathbb{P}_{2n} otherwise. This assignment $\mathcal{O} \mapsto \psi(\mathcal{O})$ will induce the desired bijection Ψ of D -multidissections.

For Part 1, label the vertices of \mathbb{P}_{2n} clockwise with the ordered pairs

$$(1, 1), (2, 1), \dots, (d, 1), (\bar{1}, 1), (\bar{2}, 1), \dots, (\bar{d}, 1), \dots, (1, \frac{n}{d}), \dots, (d, \frac{n}{d}), (\bar{1}, \frac{n}{d}), \dots, (\bar{d}, \frac{n}{d}).$$

There are four types of C -orbits of mutually noncrossing D -edges in \mathbb{P}_{2n} :

(1) monochromatic sets of diameters

$$\{((i, j), (\bar{i}, j + \lfloor \frac{n}{2d} \rfloor)) \mid 1 \leq j \leq \frac{n}{d}\}$$

of \mathbb{P}_{2n} , where the second indices of ordered pairs are interpreted modulo $\frac{n}{d}$ and $1 \leq i \leq d$,

(2) sets of ‘segregated’ nondiameter edges of the form

$$\{((i, k), (j, k)), ((\bar{i}, k), (\bar{j}, k)) \mid 1 \leq k \leq \frac{n}{d}\},$$

where $1 \leq i < j \leq d$,

(3) sets of ‘integrated’ nondiameter edges of the form

$$\{((j, k), (\bar{i}, k)), ((\bar{j}, k), (i, k + 1)) \mid 1 \leq k \leq \frac{n}{d}\},$$

where the second indices of the ordered pairs are interpreted modulo $\frac{n}{d}$ and $1 \leq i < j \leq d$, and

(4) sets of ‘integrated’ nondiameter edges of the form

$$\{((i, k), (\bar{i}, k)), ((\bar{i}, k), (i, k + 1)) \mid 1 \leq k \leq \frac{n}{d}\},$$

where the second indices of the ordered pairs are interpreted modulo $\frac{n}{d}$ and $1 \leq i \leq d$.

Observe that these edges form an inscribed $\frac{2n}{d}$ -gon in \mathbb{P}_{2n} .

To every C -orbit \mathcal{O} of mutually noncrossing D -edges in \mathbb{P}_{2n} we associate a one- or two-element set $\psi(\mathcal{O})$ of mutually noncrossing D -edges in \mathbb{P}_{2d} as follows. Label the vertices of \mathbb{P}_{2d} clockwise with $1, 2, \dots, d, \bar{1}, \bar{2}, \dots, \bar{d}$.

(1) If \mathcal{O} is a monochromatic set of diameters

$$\{((i, j), (\bar{i}, j + \lfloor \frac{n}{2d} \rfloor)) \mid 1 \leq j \leq \frac{n}{d}\},$$

let $\psi(\mathcal{O})$ be the singleton consisting of the diameter $i\bar{i}$ in \mathbb{P}_{2d} which has the same color as the diameters in \mathcal{O} .

(2) If \mathcal{O} is a set of ‘segregated’ nondiameter edges of the form

$$\{((i, k), (j, k)), ((\bar{i}, k), (\bar{j}, k)) \mid 1 \leq k \leq \frac{n}{d}\},$$

let $\psi(\mathcal{O})$ be the singleton consisting of the D-edge in \mathbb{P}_{2d} which is the pair $i\bar{j}, \bar{i}j$ of centrally symmetric nondiameters.

(3) If \mathcal{O} is a set of ‘integrated’ nondiameter edges of the form

$$\{((j, k), (\bar{i}, k)), ((\bar{j}, k), (i, k+1)) \mid 1 \leq k \leq \frac{n}{d}\},$$

let $\psi(\mathcal{O})$ be the singleton consisting of the D-edge in \mathbb{P}_{2d} which is the pair $i\bar{j}, j\bar{i}$ of centrally symmetric nondiameters.

(4) If \mathcal{O} is an inscribed polygon of the form

$$\{((i, k), (\bar{i}, k)), ((\bar{i}, k), (i, k+1)) \mid 1 \leq k \leq \frac{n}{d}\},$$

let $\psi(\mathcal{O})$ be the two element set of D-edges in \mathbb{P}_{2d} consisting of a solid/blue and dotted/red copy of the diameter $i\bar{i}$.

Given any C -orbit \mathcal{O} of mutually noncrossing D-edges, let $\chi_{\mathcal{O}}$ be the D-multidissection of \mathbb{P}_{2n} corresponding to \mathcal{O} and let $\chi_{\psi(\mathcal{O})}$ be the D-multidissection of \mathbb{P}_{2d} corresponding to $\psi(\mathcal{O})$. Any C -invariant D-multidissection f of \mathbb{P}_{2n} can be written uniquely as a sum $f = \sum_{\mathcal{O}} c_{\mathcal{O}} \chi_{\mathcal{O}}$, where the sum is over all C -orbits \mathcal{O} of mutually noncrossing D-edges and the $c_{\mathcal{O}}$ are nonnegative integers. It is easy to verify that if the D-edges in two C -orbits \mathcal{O} and \mathcal{O}' are mutually nonintersecting, then the D-edges in the sets $\psi(\mathcal{O})$ and $\psi(\mathcal{O}')$ are mutually nonintersecting, as well. Thus, the map $\Psi(f) := \sum_{\mathcal{O}} c_{\mathcal{O}} \chi_{\psi(\mathcal{O})}$ from the D-edges in \mathbb{P}_{2d} to \mathbb{N} is a D-multidissection of \mathbb{P}_{2d} .

One checks that Ψ maps k -edge D-multidissections to $\frac{kd}{n}$ -edge D-multidissections and that Ψ gives the bijective correspondence in Part 1. Given a D-multidissection g of \mathbb{P}_{2d} , to construct $\Psi^{-1}(g)$ one first checks if g contains diameters of both colors. If not, for every D-edge e occurring in g , one includes the edges in the unique C -orbit \mathcal{O} with $\psi(\mathcal{O}) = \{e\}$ in $\Psi^{-1}(g)$ with the appropriate multiplicity. If g contains diameters of both colors, one includes the the appropriate inscribed $\frac{2d}{n}$ -gon in $\Psi^{-1}(g)$ with multiplicity equal to the minimum of the multiplicities of the solid/blue and dotted/red diameters in g and then includes the C -orbits corresponding to the remaining D-edges in g in $\Psi^{-1}(g)$ with the appropriate multiplicities.

An example of the map Ψ for $n = 8$ and $d = 4$ is shown in Figure 4.1. Repeated edges correspond to edges counted with multiplicity and dashed edges are boundary sides which are counted with multiplicity zero in the D-multidissection. Observe that the inscribed 4-gon in \mathbb{P}_{16} maps to a pair of solid/blue and dotted/red diameters in \mathbb{P}_8 . The C -orbit of solid/blue diameters in \mathbb{P}_{16} maps to an additional solid/blue diameter in \mathbb{P}_8 .

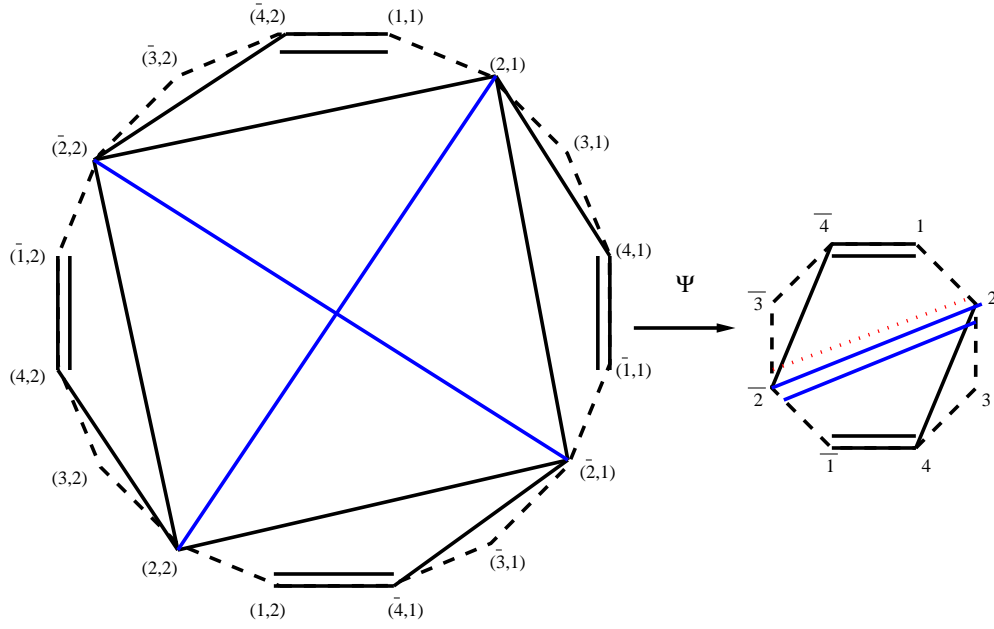


FIGURE 4.1. The action of the map Ψ on a D -multidissection with $n = 8$ and $d = 4$

The proof of Part 2 mimics the proof of Part 1 and is left to the reader. The reason why the polygon \mathbb{P}_d appears in Part 2 instead of \mathbb{P}_{2d} is that the antipodal image of a nondiameter edge (i, j) in \mathbb{P}_{2n} is not contained in the orbit of (i, j) under d -fold rotation if d does not divide n . In the case $d = 2$ one must apply our conventions regarding D -multidissections of the digon \mathbb{P}_2 . \square

We will also need a result on the specialization of homogeneous symmetric functions at roots of unity which is implicit in a CSP of Reiner, Stanton, and White.

Lemma 4.5. [10, Theorem 1.1, Part a] *Let ζ be a root of unity of order d . If $d|n$ we have the polynomial evaluation*

$$h_k(1, \zeta, \zeta^2, \dots, \zeta^{n-1}) = \begin{cases} h_{\frac{k}{d}}(1^{\frac{n}{d}}), & \text{if } d|k \\ 0, & \text{otherwise.} \end{cases}$$

Our CSP for D -multidissections is as follows.

Theorem 4.6. *Let $n \geq 2$ and $k \geq 0$. Let X be the set of D -multidissections of \mathbb{P}_{2n} with k edges and let the cyclic group $C = \mathbb{Z}_{2n}$ act on X by D -rotation. The triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where*

$$X(q) = \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} s_{(k-\ell, \ell)}(1, q^2, \dots, q^{2(n-1)}) h_{k-2\ell}(1, q^n).$$

Proof. Let r be the D-rotation operator and let ζ be a root of unity of order $2n$. We equate the fixed point sets corresponding to powers r^d of r with $d|2n$ with the appropriate polynomial evaluations. Our proof breaks up into several cases depending on the parity of d and whether d divides n .

If d is even and $d|n$, we have that $h_{k-2\ell}(1, (\zeta^d)^n) = h_{k-2\ell}(1^2) = k - 2\ell + 1$ for all ℓ . Pieri's Rule, Corollary 4.3, and Lemma 4.5 imply that the polynomial evaluation $X(\zeta^d)$ is equal to the number of D-multidissections of \mathbb{P}_{2d} with $\frac{kd}{2n}$ edges. By Part 1 of Lemma 4.4, this polynomial evaluation is the fixed point set cardinality $|X^{r^d}|$. The case of d even with d not dividing n is similar, but uses Part 2 of Lemma 4.4.

If d is odd, for $\ell \leq \lfloor \frac{k}{2} \rfloor$ we have that $h_{k-2\ell}(1, (\zeta^d)^n) = h_{k-2\ell}(1, -1)$, which is 0 if k is odd and 1 if k is even. An easy exercise using Pieri's Rule implies that $X(\zeta^d)$ is equal to $h_{(\frac{k}{2}, \frac{k}{2})}(1, \zeta^{2d}, \dots, \zeta^{2(n-1)d})$ if k is even and $X(\zeta^d) = 0$ if k is odd. On the other hand, since r^d reverses diameter colors, no r^d -invariant D-multidissection of \mathbb{P}_{2n} can contain a diameter. It follows that r^d -invariant D-multidissections of \mathbb{P}_{2n} are in natural bijection with C-multidissections of \mathbb{P}_{2n} which are invariant under d powers of rotation. The equality of the fixed point set cardinality $|X^{r^d}|$ and the polynomial evaluation $X(\zeta^d)$ is a consequence of Theorem 3.4. \square

Our proof of Theorem 4.6 relied both on the algebraic result in Lemma 4.2 and the combinatorial result in Lemma 4.4. A purely algebraic approach is possible modulo the following conjecture regarding a possible basis for the space $V^D(n, k)$. For $n \geq 2$, recall that we label the vertices of \mathbb{P}_{2n} clockwise with $1, 2, \dots, n, \bar{1}, \bar{2}, \dots, \bar{n}$. To any D-edge e in \mathbb{P}_{2n} , we associate an element $z_e^D \in V^D(n)$ as follows. Abusing notation, identify polynomials in \mathcal{A}_{n+2} with their images in the quotient $V^D(n)$. If e is a solid/blue diameter of the form $i\bar{i}$ for $1 \leq i \leq n$, let $z_e^D = \Delta_{i, n+1}$. If e is a dotted/red diameter of the form $i\bar{i}$ for $1 \leq i \leq n$, let $z_e^D = \Delta_{i, n+2}$. If e is a pair of centrally symmetric nondiameters of the form $ij, \bar{i}\bar{j}$ for $1 \leq i < j \leq n$, let $z_e^D = \Delta_{i, n+1}\Delta_{j, n+2} + \Delta_{ij}$. Finally, if e is a pair of centrally symmetric nondiameters of the form $i\bar{j}, \bar{i}j$ for $1 \leq i < j \leq n$, let $z_e^D = \Delta_{i, n+1}\Delta_{j, n+2} - \Delta_{ij}$. Observe that $z_e^D \in V^D(n, 1)$ if e is a diameter of either color and $z_e^D \in V^D(n, 2)$ if e is a pair of centrally symmetric nondiameters. Given a D-multidissection f , define $z_f^D \in V^D(n)$ by

$$z_f^D := \prod_{e \in E_D} (z_e^D)^{f(e)}.$$

For example, if f is the D-multidissection of \mathbb{P}_8 on the right of Figure 4.1, then

$$z_f^D = (\Delta_{25}\Delta_{46} + \Delta_{24})(\Delta_{15}\Delta_{46} - \Delta_{14})^2(\Delta_{25})(\Delta_{26})^2 \in V^D(4, 9).$$

Conjecture 4.7. *The set $\{z_f^D\}$, where f ranges over all D-multidissections of \mathbb{P}_{2n} , is a \mathbb{C} -basis for $V^D(n)$.*

The geometric realization of the type D cluster algebras in [5] implies that one need only show that the set of Conjecture 4.7 spans $V^D(n)$ or is linearly independent. The polynomials which we have attached to D-edges do not satisfy the type D exchange relations and are not related to the cluster monomial basis presented in [5] via a unitriangular transition matrix. Assuming Conjecture 4.7 is true, we can give the following alternative proof of Theorem 4.6.

Proof. (of Theorem 4.6, assuming Conjecture 4.7) Embed the direct product $\mathfrak{S}_n \times \mathfrak{S}_2$ of symmetric groups into $GL_n(\mathbb{C}) \times GL_2(\mathbb{C})$ via permutation matrices. Writing permutations in cycle notation, let g^D be the image of $(1, 2, \dots, n) \times (1, 2)$ under this embedding. It is routine to verify that g^D maps the module element z_e^D to the module element $z_{r.e}^D$ for all D-edges e , where r is the D-rotation operator. The homogeneity of the z_e^D combined with Conjecture 4.7 implies that the set $\{z_f^D\}$, where f ranges over all D-multidissections of \mathbb{P}_{2n} with k edges, forms a \mathbb{C} -basis for the space $V^D(n, k)$. The desired CSP follows from using the Weyl character evaluation in Lemma 4.2 to calculate the traces of powers of the operator g^D . \square

We close by noting that the acting groups in our representation theoretic proofs in types A and C and our conjectural representation theoretic proof in type D are not equal to the associated Lie group outside of type A. In the type C case, the action involved can be reformulated as an action of the intersection symplectic group $Sp_{2n}(\mathbb{C}) = \{A \in GL_{2n}(\mathbb{C}) \mid A \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A^T = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\}$ with the Levi subgroup $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ of block diagonal matrices in $GL_{2n}(\mathbb{C})$. In type D, we can see no obvious relation between the acting group $GL_n(\mathbb{C}) \times GL_2(\mathbb{C})$ and the even special orthogonal groups. This is perhaps not surprising given that the geometric realizations of finite type cluster algebras typically have little obvious connection with the Lie group of the same Cartan-Killing type.

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E-mail address: `brhoades@math.mit.edu`

BRENDON RHOADES, DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA, 02139