ENUMERATION OF CONNECTED CATALAN OBJECTS BY TYPE

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Abstract. Noncrossing set partitions, nonnesting set partitions, Dyck paths, and rooted plane trees are four classes of Catalan objects which carry a notion of type. There exists a product formula which enumerates these objects according to type. We define a notion of 'connectivity' for these objects and prove an analogous product formula which counts connected objects by type. Our proof of this product formula is combinatorial and bijective. We extend this to a product formula which counts objects with a fixed type and number of connected components. We relate our product formulas to symmetric functions arising from parking functions. We close by presenting an alternative proof of our product formulas communicated to us by Christian Krattenthaler [7] which uses generating functions and Lagrange inversion.

1. Introduction

The *Catalan numbers* $C_n = \frac{1}{n+1}\binom{2n}{n}$ are among the most important sequences of numbers in combinatorics. To name just a few examples (see [12] for many more), the number C_n counts 123-avoiding permutations in $\mathfrak{S}_n$, Dyck paths of length $2n$, standard Young tableaux of shape $2 \times n$, noncrossing or nonnesting set partitions of $[n]$, and rooted plane trees with $n+1$ vertices.

Certain families of Catalan objects come equipped with a natural notion of *type*. For example, the type of a noncrossing set partition of $[n]$ is the sequence $\mathbf{r} = (r_1, \ldots, r_n)$, where r_i is the number of blocks of size i. In the cases of noncrossing/nonnesting set partitions of $[n]$, Dyck paths of length $2n$, and plane trees on $n+1$ vertices, there exists a nice product formula (Theorem 1.1) which counts Catalan objects with fixed type $\mathbf{r}$. These four classes of Catalan objects also carry a notion of *connectivity*. In this paper we give a product formula which counts these objects with a fixed type and a fixed number of connected components.

The *bump diagram* of a set partition π of $[n]$ is obtained by drawing the numbers 1 through n in a line and drawing an arc between i and j with $i < j$ if i and j are blockmates in π and there does not exist k with $i < k < j$ such that i, k, and j are blockmates in π. The set partition π is *noncrossing* if the bump diagram of π has no crossing arcs or, equivalently, if there do not exist $a < b < c < d$ with a, c in a block of π and b, d in a different block of π. Similarly, the set partition π is *nonnesting* if the bump diagram of π contains no nesting arcs, that is, no pair of arcs of the form ad and bc with $a < b < c < d$. As above, the *type* of any set partition π of $[n]$ is the sequence $(r_1, \ldots, r_n)$, where r_i is the number of blocks in π of size i. The set partition π is called *connected* if there does not exist an index i with $1 \leq i \leq n-1$ such that there are no arcs connecting the intervals $[1, i]$ and $[i+1, n]$ in the bump diagram of π. The set partition π is said to have m *connected components* if there exist numbers $1 \leq i_1 < i_2 < \cdots < i_{m-1} \leq n$ such that the restriction of the bump diagram of π to each of the intervals $[1, i_1], [i_1 + 1, i_2], \ldots, [i_{m-1} + 1, n]$ is a connected set partition.

The bump diagram of the noncrossing partition $\{1, 8, 13/2, 5, 6, 7/3/4/9, 12/10, 11\}$ of $[13]$ with type $(2, 2, 1, 1, 0, \ldots, 0)$ is shown in the middle of Figure 1.1. The bump diagram of the nonnesting partition $\{1, 5, 7/2, 6, 8, 11/3/4/9, 12/10, 13\}$ of $[13]$ with type $(2, 2, 1, 1, 0, \ldots, 0)$ is shown in the top

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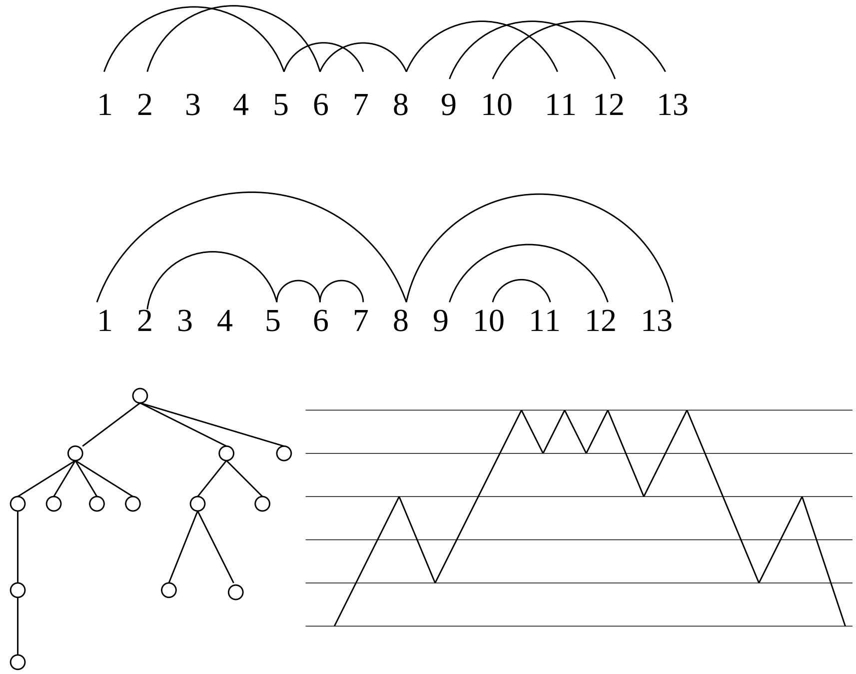

FIGURE 1.1. A connected nonnesting partition of [13], a connected noncrossing partition of [13], a plane tree with 14 vertices with a terminal rooted twig, and a Dyck path of length 26 with no returns

of Figure 1.1. Both of these set partitions are connected. The set partition $\{1, 4/2, 3/5/6, 7, 8\}$ is a noncrossing partition of [8] with 3 connected components and type $(1, 2, 1, 0, \ldots, 0)$.

A *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ starting at $(0, 0)$ and ending at $(2n, 0)$ which contains steps of the form $U = (1, 1)$ and $D = (1, -1)$ and never goes below the x-axis. An *ascent* in a Dyck path is a maximal sequence of U-steps. The *ascent type* of a Dyck path of length $2n$ is the sequence $(r_1, \ldots, r_n)$, where r_i is the number of ascents of length i. A *return* of a Dyck path of length $2n$ is a lattice point $(m, 0)$ with $0 < m < 2n$ which is contained in the Dyck path.

The ascent type of the Dyck path of length 26 shown on the lower right of Figure 1.1 is $(2, 2, 1, 1, 0, \ldots, 0)$. This Dyck path has no returns. The Dyck path $UUDDUDUUDUDD$ has length 12, ascent type $(2, 2, 0, \ldots, 0)$, and 2 returns.

A *(rooted) plane tree* is a graph T defined recursively as follows. A distinguished vertex is called the *root* of T and the vertices of T excluding the root are partitioned into an *ordered* list of k sets $T_1, \ldots, T_k$, each of which is a plane tree. Given a plane tree T on $n+1$ vertices, the *downdegree sequence* of T is the sequence $(r_0, r_1, \ldots, r_n)$, where r_i in the number of vertices $v \in T$ with i neighbors further from the root than v. If T is a plane tree with $n + 1$ vertices, there exists a labeling of the vertices of T with $[n + 1]$ called *preorder* (see [10] for the precise definition). The plane tree T with $n + 1$ vertices is said to have a *terminal rooted twig* if the vertex labeled $n + 1$ is attached to the root. A *plane forest* F is an *ordered* list of plane trees $F = (T_1, \ldots, T_k)$. The *downdegree sequence* of a plane forest F is the sum of the downdegree sequences of its constituent trees.

The downdegree sequence of the plane tree with 14 vertices shown on the lower left of Figure 1.1 is $(8, 2, 2, 1, 1, 0, \ldots, 0)$. This plane tree has a terminal rooted twig.

In order to avoid enforcing conventions such as $\frac{(-1)!}{(-1)!} = 1$ in the 'degenerate' cases of our product formulas, we adopt the following notation of Zeng [14]. Given any vectors $\mathbf{r} = (r_1, \dots, r_n), \mathbf{v} = (v_1, \dots, v_n) \in \mathbb{N}^n$, set $|\mathbf{r}| := r_1 + \cdots + r_n$, $\mathbf{r}! := r_1! r_2! \cdots r_n!$, and $\mathbf{r} \cdot \mathbf{v} := r_1 v_1 + \cdots + r_n v_n$. Let x be a variable and for any vectors $\mathbf{r}, \mathbf{v} \in \mathbb{N}^n$ let $A_{\mathbf{r}}(x; \mathbf{v}) \in \mathbb{R}[x]$ be the polynomial

$$A_{\mathbf{r}}(x; \mathbf{v}) = \frac{x}{x + \mathbf{r} \cdot \mathbf{v}} \frac{(x + \mathbf{r} \cdot \mathbf{v})_{|\mathbf{r}|}}{\mathbf{r}!}, \tag{1.1}$$

where $(y)_k = y(y-1)\cdots(y-k+1)$ is a falling factorial. Zeng used the polynomials $A_{\mathbf{r}}(x; \mathbf{v})$ to prove various convolution identities involving multinomial coefficients.

Theorem 1.1. *Let $n \geq 1$, let $\mathbf{v} = (1, 2, \dots, n)$, and suppose $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{N}^n$ satisfies $\mathbf{r} \cdot \mathbf{v} = n$.*

The polynomial evaluation $A_{\mathbf{r}}(1; \mathbf{v}) = [A_{\mathbf{r}}(x; \mathbf{v})]_{x=1}$ is equal to [1] *the cardinality of:*
1. *the set of noncrossing partitions of $[n]$ of type $\mathbf{r}$;*
2. *the set of nonnesting partitions of $[n]$ of type $\mathbf{r}$;*
3. *the set of Dyck paths of length $2n$ with ascent type $\mathbf{r}$;*
4. *the set of plane trees with $n+1$ vertices and with downdegree sequence $(n - |\mathbf{r}| + 1, r_1, \dots, r_n)$.*

Part 1 of Theorem 1.1 is due to Kreweras [8, Theorem 4]. A type-preserving bijection showing the equivalence of Parts 1 and 2 was discovered by Athanasiadis [3, Theorem 3.1]. A similar bijection showing the equivalence of Parts 1 and 3 was proven by Dershowitz and Zaks [4]. Armstrong and Eu [1, Lemma 3.2] give an example of a bijection proving the equivalence of Parts 1 and 4.

The rest of this paper is organized as follows. In Section 2 we prove an analogous product formula (Theorem 2.2) which counts connected objects according to type. The proof of Theorem 2.2 is bijective and relies on certain properties of words in monoids. We extend this result to another product formula (Theorem 2.3) which counts objects which have a fixed number of connected components according to type. These product formulas have found a geometric application in [2] where they are used to count regions of hyperplane arrangements related to the Shi arrangement according to 'ideal dimension' in the sense of Zaslavsky [13]. We then apply our product formulas to the theory of symmetric functions, refining a formula of Stanley [11]. In Section 3 we present an alternative proof of Theorem 2.3 communicated to us by Christian Krattenthaler [7] which uses generating functions and Lagrange inversion.

2. Main Results

The proofs of our product formulas will rest on a lemma about words in monoids which can be viewed as a 'connected analog' of the 'cycle lemma' due to Dvoretzky and Motzkin [6] (see also [5]). For a more leisurely introduction to this material, see [12].

Let $\mathcal{A}$ denote the infinite alphabet $\{x_0, x_1, x_2, \dots\}$ and let $\mathcal{A}^*$ denote the free (noncommutative) monoid generated by $\mathcal{A}$. Denote the empty word by $e \in \mathcal{A}^*$. The *weight function* is the monoid homomorphism $\omega : \mathcal{A}^* \to (\mathbb{Z}, +)$ induced by $\omega(x_i) = i - 1$ for all i. We define a subset $\mathcal{B} \subset \mathcal{A}^*$ by

$$\mathcal{B} = \{w = w_1 \dots w_n \in \mathcal{A}^* \,|\, \omega(w) = 1, \omega(w_1 w_2 \dots w_j) > 0 \text{ for } 1 \leq j \leq n\}.$$

That is, a word $w \in \mathcal{A}^*$ is contained in $\mathcal{B}$ if and only if it has weight 1 and all of its nonempty prefixes have positive weight. In particular, we have that $e \notin \mathcal{B}$.

Given any word $w = w_1 \dots w_n \in \mathcal{A}^*$, a *conjugate* of w is an element of $\mathcal{A}^*$ of the form $w_i w_{i+1} \dots w_n w_1 w_2 \dots w_{i-1}$ for some $1 \leq i \leq n$ (this is the monoid-theoretic analog of conjugation in groups). We have the following result concerning conjugacy classes of elements of $\mathcal{B}$. It is our 'connected analog of' [12, Lemma 5.3.7] and is an analog of the 'cycle lemma' in tree enumeration.

[1] This polynomial evaluation can also be expressed as $\frac{n!}{(n-|\mathbf{r}|+1)!\mathbf{r}!}$.

Lemma 2.1. *A word $w \in \mathcal{A}^*$ is conjugate to an element of $\mathcal{B}$ if and only if $\omega(w) = 1$, in which case w is conjugate to a unique element of $\mathcal{B}$ and the conjugacy class of w has size equal to the length of w.*

Proof. Let $w \in \mathcal{B}$ have length n and suppose the conjugacy class of w has size $k|n$. Then we can write $w = v^{n/k}$ for some $v \in \mathcal{A}^*$. Since v is a nonempty prefix of w, we have $\omega(v) > 0$ and the fact that $1 = \omega(w) = \frac{n}{k}\omega(v)$ forces $k = n$.

Since conjugation does not affect weight, every element w of the conjugacy class of an element of $\mathcal{B}$ satisfies $\omega(w) = 1$.

Suppose that $w \in \mathcal{A}^*$ satisfies $\omega(w) = 1$. We show that w is conjugate to an element of $\mathcal{B}$. The proof of this fact breaks up into three cases depending on the letters which occur in w.

Case 1: w contains no occurrences of x_0. Since $\omega(w) = 1$, in this case w must be of the form $x_1 \dots x_1 x_2 x_1 \dots x_1$ and w is conjugate to $x_2 x_1 \dots x_1 \in \mathcal{B}$.

Case 2: w contains at most one occurrence of a letter other than x_0. In this case, the condition $\omega(w) = 1$ forces a conjugate of w to be of the form $x_s x_0^{s-2} \in \mathcal{B}$ for some $s > 1$.

Case 3: w at least one occurrence of x_0 and at least two occurrences of letters other than x_0. We claim that there exists a conjugate w' of w of the form $w' = x_{s+1} x_0^s v$ for some $s \geq 0$. If this were not the case, consider the word w written around a circle. Every maximal contiguous string of x_0's in w of length ℓ must be preceded by a letter of the form x_s for some $s > \ell + 1$. The weight of any such contiguous string taken together with its preceding letter is $\omega(x_s x_0^{\ell}) = s - 1 - \ell > 0$. Since $\omega(w) = 1$, it follows that w has a conjugate of the form $x_s x_0^{s-2}$, which contradicts our assumption that w has at least two occurrences of a letter other than x_0. Let w' be a conjugate of w of the form $w' = x_{s+1} x_0^s v$ for some $v \in \mathcal{A}^*$. Since $1 = \omega(w) = \omega(w') = s - s + \omega(v)$, by induction on length we can assume that a conjugate of v is contained in $\mathcal{B}$. Say that $v = yz$ such that $zy \in \mathcal{B}$ with $z \neq e$. Then $z x_{s+1} x_0^s y$ is a conjugate of $w' = x_{s+1} x_0^s yz$ satisfying $z x_{s+1} x_0^s y \in \mathcal{B}$. □

Let $\mathcal{B}^*$ denote the submonoid of $\mathcal{A}^*$ generated by $\mathcal{B}$. In view of [12, Lemma 5.3.7], it is tempting to guess that any element $w \in \mathcal{A}^*$ obtained by permuting the letters of an element of $\mathcal{B}^*$ is itself conjugate to an element of $\mathcal{B}^*$. However, this is false. For example, the element $x_3 x_0 x_2 = (x_3 x_0)(x_2) \in \mathcal{A}^*$ is contained in $\mathcal{B}^*$ but $x_3 x_2 x_0$ has no conjugate in $\mathcal{B}^*$. (However, the analog of [12, Lemma 5.3.6] does hold in this context - the monoid $\mathcal{B}^*$ is very pure.) Lemma 2.1 is the key tool we will use in proving our connected analog of Theorem 1.1.

Theorem 2.2. *Let $n \geq 1$, let $\mathbf{v} = (1, 2, \dots, n)$, and suppose $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{N}^n$ satisfies $\mathbf{r} \cdot \mathbf{v} = n$.*

The polynomial evaluation $-A_{\mathbf{r}}(-1; \mathbf{v}) = [-A_{\mathbf{r}}(x; \mathbf{v})]_{x=-1}$ is equal to [2] the cardinality of:
1. the set of connected noncrossing partitions of $[n]$ of type $\mathbf{r}$;
2. the set of connected nonnesting partitions of $[n]$ of type $\mathbf{r}$;
3. the set of Dyck paths of length $2n$ with no returns and ascent type $\mathbf{r}$;
4. the set of plane trees with a terminal rooted twig and $n+1$ vertices with downdegree sequence $(n - |\mathbf{r}| + 1, r_1, \dots, r_n)$.

Proof. The line of reasoning which we follow here should be compared to that in [12, Chapter 5].

Observe first that when $\mathbf{r} = (n, 0, \dots, 0)$ we have that

$$-A_{\mathbf{r}}(-1; \mathbf{v}) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{2.1}$$

[2]In the case where $n > 1$ and $\mathbf{r} \neq (n, 0, \dots, 0)$, this can also be expressed as $\frac{(n-2)!}{(n-|\mathbf{r}|-1)!\mathbf{r}!}$.

This is in agreement with the relevant set cardinalities, so from now on we assume that $n > 1$ and $\mathbf{r} \neq (n, 0, \ldots, 0)$. Let $\mathcal{B}(\mathbf{r})$ denote the set of length $n-1$ words $w \in \mathcal{B}$ with $n - |\mathbf{r}| - 1$ x_0's, r_1 x_1's, $\ldots$, and $r_n x_n$'s. By Lemma 2.1, we have that

$$|\mathcal{B}(\mathbf{r})| = \frac{1}{n-1}\binom{n-1}{n-|\mathbf{r}|-1, r_1, \ldots, r_n} = -A_{\mathbf{r}}(-1; \mathbf{v}), \tag{2.2}$$

where the second equality follows from the definition of $A_{\mathbf{r}}(x; \mathbf{v})$. Therefore, it suffices to biject each of the sets in Parts 1-4 with the set $\mathcal{B}(\mathbf{r})$. We present a bijection in each case.

1. Let $NC(\mathbf{r})$ be the set of noncrossing partitions we wish to enumerate. Given any partition π of $[n]$, define a word $\psi(\pi) = w_1 w_2 \ldots w_{n-1} \in \mathcal{A}^*$ as follows. For $1 \leq i \leq n-1$, if i is the minimal element of a block of π, let $w_i = x_j$ where j is the size of the block containing i. Otherwise, let $w_i = x_0$. For example, if π is the connected nonnesting partition of [13] shown on the top of Figure 1.1, we have that $\psi(\pi) = x_3x_4x_1x_1x_0x_0x_0x_0x_2x_2x_0x_0$. It is easy to see that the mapping $\pi \mapsto \psi(\pi)$ sets up a bijection between set partitions in $NC(\mathbf{r})$ and words in $\mathcal{B}(\mathbf{r})$.

2. Let $NN(\mathbf{r})$ be the set of nonnesting partitions we wish to enumerate. It is easy to verify (see [12, Solution to Exercise 5.44]) that the map ψ from the proof of Part 1 restricts to a bijection between $NN(\mathbf{r})$ and $\mathcal{B}(\mathbf{r})$.

3. Let $\mathbb{D}$ be a Dyck path with no returns of length $2n$ and ascent type $\mathbf{r}$. Define a length $n-1$ word $\delta(\mathbb{D}) \in \mathcal{A}^*$ as follows. Let $w_1 w_2 \ldots w_n \in \mathcal{A}^*$ be the word obtained by reading $\mathbb{D}$ from left to right, replacing every ascent of length i with x_i and replacing every maximal contiguous sequence of downsteps of length ℓ with $x_0^{\ell-1}$. Set $\delta(\mathbb{D}) := w_1 w_2 \ldots w_{n-1}$. For example, if $\mathbb{D}$ is the Dyck path shown in Figure 1.1 we have that $\delta(\mathbb{D}) = x_3x_0x_4x_1x_1x_0x_2x_0^3x_2x_0$. It is easy to verify that $\delta(\mathbb{D}) \in \mathcal{B}(\mathbf{r})$ and that the map $\mathbb{D} \mapsto \delta(\mathbb{D})$ sets up a bijection between Dyck paths with no returns of length $2n$ and ascent type $\mathbf{r}$ to $\mathcal{B}(\mathbf{r})$.

4. For T be a plane tree on $n+1$ vertices with a terminal rooted twig, let $w_1 w_2 \ldots w_{n+1} \in \mathcal{A}^*$ be the word obtained by setting $w_i = x_j$, where j is the downdegree of the i^{th} vertex of T in preorder. Since T has a terminal rooted twig, we have $w_n = w_{n+1} = x_0$. Set $\chi(T) := w_1 w_2 \ldots w_{n-1} \in \mathcal{A}^*$. For example, if T is the tree shown in Figure 1.1, we have that $\chi(T) = x_3x_4x_1x_1x_0x_0x_0x_0x_2x_2x_0x_0$. The mapping $T \mapsto \chi(T)$ sets up a bijection between the set of trees of interest and $\mathcal{B}(\mathbf{r})$. □

An alternative proof of Parts 1 and 2 of Theorem 2.2 which relies on a product formula enumerating noncrossing partitions by 'reduced type' due to Armstrong and Eu [1] (which in turn relies on the original enumeration of noncrossing partitions by type due to Kreweras) can be found in [2].

It is natural to ask if the formula in Theorem 2.2 can be generalized to the case of multiple connected components. The answer is 'yes', and to avoid enforcing nonstandard conventions in degenerate cases we will again state the relevant product formula in terms of a polynomial specialization. Suppose that $\mathbf{r}, \mathbf{v} \in \mathbb{N}^n$ and $1 \leq m \leq |\mathbf{r}|$. We define the polynomial $A_{\mathbf{r}}^{(m)}(x; \mathbf{v}) \in \mathbb{R}[x]$ by

$$A_{\mathbf{r}}^{(m)}(x; \mathbf{v}) = \frac{(|\mathbf{r}|-1)!}{(|\mathbf{r}|-m)!} \frac{x}{x + \mathbf{r} \cdot \mathbf{v}} \frac{(x + \mathbf{r} \cdot \mathbf{v})_{|\mathbf{r}|-m+1}}{\mathbf{r}!}. \tag{2.3}$$

Observe that in the case $m = 1$ we have $A_{\mathbf{r}}^{(1)}(x; \mathbf{v}) = A_{\mathbf{r}}(x; \mathbf{v})$.

Theorem 2.3. *Let $n \geq m \geq 1$ and let $\mathbf{v} = (1, 2, \ldots, n) \in \mathbb{N}^n$. Suppose that $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{N}^n$ satisfies $\mathbf{r} \cdot \mathbf{v} = n$ and $m \leq |\mathbf{r}|$.*

The polynomial evaluation $-A_{\mathbf{r}}^{(m)}(-m; \mathbf{v}) = [-A_{\mathbf{r}}^{(m)}(x; \mathbf{v})]_{x=-m}$ is equal to [3] the cardinality of:
1. the set of noncrossing partitions of $[n]$ with exactly m connected components of type $\mathbf{r}$;

[3] In the case where $n > m$ and $\mathbf{r} \neq (n, 0, \ldots, 0)$, this can also be expressed as $\frac{m(n-m-1)!(|\mathbf{r}|-1)!}{(n-|\mathbf{r}|-1)!(|\mathbf{r}|-m)!\mathbf{r}!}$.

2. the set of nonnesting partitions of $[n]$ *with exactly* m *connected components of type* $\mathbf{r}$*;*
3. the set of Dyck paths of length $2n$ *with exactly* $m-1$ *returns of ascent type* $\mathbf{r}$*;*
4. the set of plane forests with $n+m$ *vertices and exactly* m *trees with downdegree sequence* $(n-|\mathbf{r}|+m, r_1, \dots, r_n)$ *such that every tree has a terminal rooted twig.*

Proof. In light of Theorem 2.2, it suffices to prove Part 1. The polynomial $A_{\mathbf{r}}(mx;\mathbf{v})$ can be obtained via the following convolution-type identity which follows from a result of Raney [9, Theorems 2.2, 2.3] and induction:

$$\sum_{\mathbf{r^{(1)}}+\cdots+\mathbf{r^{(m)}}=\mathbf{r}} A_{\mathbf{r^{(1)}}}(x;\mathbf{v})\cdots A_{\mathbf{r^{(m)}}}(x;\mathbf{v}) = A_{\mathbf{r}}(mx;\mathbf{v}), \tag{2.4}$$

where $\mathbf{r^{(i)}} \in \mathbb{N}^n$ for all i. Let $\mathbf{0} \in \mathbb{N}^n$ be the zero vector. By Theorem 2.2 and the fact that $A_{\mathbf{0}}(x;\mathbf{v}) = 1$, we can set $x = -1$ to obtain

$$\sum_{k=1}^{m} (-1)^k \binom{m}{k} C(n,k,\mathbf{r}) = A_{\mathbf{r}}(-m;\mathbf{v}), \tag{2.5}$$

where $C(n,k,\mathbf{r})$ denotes the number of noncrossing partitions of $[n]$ with exactly k connected components and type $\mathbf{r}$. By the Principle of Inclusion-Exclusion (see [10]), it follows that

$$C(n,m,\mathbf{r}) = \sum_{k=1}^{m} (-1)^k \binom{m}{k} A_{\mathbf{r}}(-k;\mathbf{v}). \tag{2.6}$$

Therefore, it suffices to show that the right hand side of Equation 2.6 is equal to $-A_{\mathbf{r}}^{(m)}(-m;\mathbf{v})$. We sketch this verification here for the case $m, |\mathbf{r}| < n$; the other degenerate cases are left to the reader.

We start with the following binomial coefficient identity:

$$\sum_{k=1}^{m} (-1)^{k+1} k \binom{m}{k} \binom{n-k-1}{|\mathbf{r}|-1} = m \binom{n-m-1}{n-|\mathbf{r}|-1}. \tag{2.7}$$

This identity can be obtained by comparing like powers of x on both sides of the equation $r(1+x)^{r+s-1} = (1+x)^s \frac{d}{dx}(1+x)^r = (1+x)^s(\binom{r}{1} + 2\binom{r}{2}x + 3\binom{r}{3}x^2 + \cdots)$. Multiplying both sides of Equation 2.7 by $\frac{(|\mathbf{r}|-1)!}{\mathbf{r}!}$ and using the definition of $A_{\mathbf{r}}(x;\mathbf{v})$ we obtain

$$\sum_{k=1}^{m} (-1)^k \binom{m}{k} A_{\mathbf{r}}(-k;\mathbf{v}) = \frac{m(n-m-1)!(|\mathbf{r}|-1)!}{(n-|\mathbf{r}|-1)!(|\mathbf{r}|-m)!\mathbf{r}!}. \tag{2.8}$$

The right hand side of Equation 2.8 is equal to $-A_{\mathbf{r}}^{(m)}(-m;\mathbf{v})$. □

We close this section by relating the product formulas in this paper to Frobenius characters arising from the theory of parking functions. For $n \geq 0$ a *parking function of length* n is a sequence $(a_1, \dots, a_n)$ of positive integers whose nondecreasing rearrangement $(b_1, \dots, b_n)$ satisfies $b_i \leq i$ for all i. A nondecreasing parking function is called *primitive* and primitive parking functions of length n are in an obvious bijective correspondence (see [1]) with Dyck paths of length $2n$. The *type* of a parking function is the ascent type of its nondecreasing rearrangement. A parking function $(a_1, \dots, a_n)$ will be said to have m *returns* if its nondecreasing rearrangement has m returns when viewed as a Dyck path.

The symmetric group $\mathfrak{S}_n$ acts on the set of parking functions of length n. Stanley [11] computed the Frobenius character of this action with respect to the standard bases (monomial, homogeneous, elementary, power sum, and Schur) of the ring of symmetric functions. To compute this character in the basis $\{h_\lambda\}$ of complete homogeneous symmetric functions, he observed that every orbit $\mathcal{O}$ of this action contains a unique primitive parking function $(b_1, \ldots, b_n)$ and that the Frobenius character of the action of $\mathfrak{S}_n$ on $\mathcal{O}$ is h_λ, where $\lambda = (1^{r_1}2^{r_2} \ldots n^{r_n})$ and $(r_1, \ldots, r_n)$ is the type of $(b_1, \ldots, b_n)$. By applying the formula in Theorem 1.1, one immediately gets the expansion

$$\mathrm{Frob}(P_n) = \sum_{\lambda \vdash n} \frac{n!}{(n - |\mathbf{r}(\lambda)| + 1)!\mathbf{r}(\lambda)!} h_\lambda, \tag{2.9}$$

where Frob is the Frobenius characteristic map, P_n is the permutation module for the action of $\mathfrak{S}_n$ on parking functions of length n, $r_i(\lambda)$ is the multiplicity of i in λ for $1 \leq i \leq n$, and $\mathbf{r}(\lambda) = (r_1(\lambda), \ldots, r_n(\lambda))$. (See [1] for a nonhomogeneous generalization of this formula.)

Using the same reasoning as in [11] we can compute the Frobenius characters of other modules related to parking functions. In particular, for $m > 0$, the symmetric group $\mathfrak{S}_n$ acts on the set of parking functions of length n with $m-1$ returns. Let $P_n^{(m)}$ be the permutation module corresponding to this action, so that $P_n \cong \bigoplus_{m=0}^{n-1} P_n^{(m)}$. Applying Theorem 2.3, we have that the Frobenius character of this module is

$$\mathrm{Frob}(P_n^{(m)}) = \sum_{\lambda \vdash n} -A^{(m)}_{\mathbf{r}(\lambda)}(-m; \mathbf{v}) h_\lambda, \tag{2.10}$$

where $\mathbf{v} = (1, 2, \ldots, n) \in \mathbb{N}^n$.

3. Proof of Theorem 2.3 using Lagrange Inversion

In this section we outline an alternative proof of Theorem 2.3 using generating functions and Lagrange inversion which was pointed out to the author by Christian Krattenthaler [7]. This method has the advantage of immediately proving Theorem 2.3 without first proving the single connected component case of Theorem 2.2. We only handle the case of noncrossing partitions.

Let $y = \{y_1, y_2, \ldots, \}$ and z be commuting variables. If π is a noncrossing partition of $[n]$ for $n \geq 0$, the *weight* of π is the monomial

$$\mathrm{wt}(\pi) = z^n \prod_{i \geq 1} y_i^{r_i(\pi)}, \tag{3.1}$$

where $r_i(\pi)$ is the number of blocks in π of size i. (The unique partition of $[0]$ has weight 1.) We define $P(z) \in \mathbb{R}(y_1, y_2, \ldots)[[z]]$ by grouping these monomials together in a generating function. That is,

$$P(z) = \sum_{\pi} \mathrm{wt}(\pi), \tag{3.2}$$

where the sum is over all noncrossing partitions π.

Given any noncrossing partition π of $[n]$ with $n > 1$, if the block of π containing 1 has size k, drawing π on a circle one obtains k (possibly empty) noncrossing partitions 'between' each successive pair of elements in this k element block. This combinatorial observation yields the following formula:

$$P(z) = 1 + \sum_{k=1}^{\infty} y_k z^k P(z)^k. \tag{3.3}$$

Rearranging this expression, we get that

$$\frac{zP(z)}{1 + \sum_{k=1}^{\infty} y_k z^k P(z)^k} = z, \tag{3.4}$$

and therefore $zP(z)$ is the compositional inverse of $\frac{z}{X(z)}$, where $X(z) = 1 + \sum_{k=1}^{\infty} y_k z^k$. This implies that

$$P\left(\frac{z}{X(z)}\right) = X(z). \tag{3.5}$$

In order to prove Theorem 2.3, we need to keep track of the number of connected components of a noncrossing partition. To do this, let $C(z) \in \mathbb{R}(y_1, y_2, \dots)[[z]]$ the generating function

$$C(z) = \sum_{\pi} \mathrm{wt}(\pi), \tag{3.6}$$

where the sum ranges over all *connected* noncrossing partitions of $[n]$ where $n \geq 1$. It is immediate that the generating functions $P(z)$ and $C(z)$ are related by

$$P(z) = \frac{1}{1 - C(z)} \tag{3.7}$$

or equivalently,

$$C(z) = \frac{P(z) - 1}{P(z)}. \tag{3.8}$$

As in the first proof of Theorem 2.3, let $C(n, m, \mathbf{r})$ denote the number of noncrossing partitions of $[n]$ with exactly m connected components and type $\mathbf{r}$. It is evident that

$$C(z)^m = \left(\frac{P(z) - 1}{P(z)}\right)^m = \sum_{n \geq 0} \sum_{\mathbf{r} \geq \mathbf{0}} C(n, m, \mathbf{r}) y^{\mathbf{r}} z^n, \tag{3.9}$$

where the inequality in the inner summation is componentwise and $y^{\mathbf{r}} = y_1^{r_1} y_2^{r_2} \cdots$ if $\mathbf{r} = (r_1, r_2, \dots)$.

To find an expression for $C(n, m, \mathbf{r})$ it is enough to extract the coefficient of $z^n y^{\mathbf{r}}$ from the generating function in Equation 3.9. We use Lagrange inversion to do this. Set $F(z) := \frac{z}{X(z)}$, so that the compositional inverse of $F(z)$ is $F^{\langle -1 \rangle}(z) = zP(z)$. Also set $H(z) := \left(\frac{X(z)-1}{X(z)}\right)^m$. In light of Equation 3.5 we have the identity $\left(\frac{P(z)-1}{P(z)}\right)^m = H(F^{\langle -1 \rangle}(z))$. Let $\langle - \rangle$ denote taking a coefficient in a Laurent series. Applying Lagrange inversion as in [12, Corollary 5.4.3] we get that

$$\begin{aligned} C(n, m, \mathbf{r}) &= \langle z^n y^{\mathbf{r}} \rangle H(F^{\langle -1 \rangle}(z)) \\ &= \frac{1}{n} \langle z^{n-1} y^{\mathbf{r}} \rangle H'(z) \left(\frac{z}{F(z)}\right)^n \\ &= \frac{1}{n} \langle z^{n-1} y^{\mathbf{r}} \rangle m X(z)^{n-m-1} (X(z) - 1)^{m-1} X'(z) \\ &= \frac{m}{n} \langle z^{n-1} y^{\mathbf{r}} \rangle (X(z) - 1)^{m-1} \sum_{\ell \geq 0} \binom{n-m-1}{\ell} (X(z) - 1)^{\ell} X'(z) \\ &= \frac{m}{n} \langle z^{n-1} y^{\mathbf{r}} \rangle (X(z) - 1)^{m-1} \sum_{\ell \geq 0} \frac{1}{m+\ell} \binom{n-m-1}{\ell} \left((X(z) - 1)^{m+\ell}\right)', \end{aligned}$$

where all derivatives are partial derivatives with respect to z. Suppose $\mathbf{r} = (r_1, r_2, \dots)$. Taking the coefficient in the bottom line yields the equality

$$C(n, m, \mathbf{r}) = \frac{m}{|\mathbf{r}|} \binom{n-m-1}{|\mathbf{r}| - m} \binom{|\mathbf{r}|}{r_1, r_2, \dots}, \tag{3.10}$$

which is equivalent to Part 1 of Theorem 2.3.

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