KAZHDAN-LUSZTIG IMMANANTS AND PRODUCTS OF MATRIX MINORS

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Abstract. We define a family of polynomials of the form $\sum f(\sigma)x_{1,\sigma(1)}\cdots x_{n,\sigma(n)}$ in terms of the Kazhdan-Lusztig basis $\{C'_w(1)\,|\,w\in S_n\}$ for the symmetric group algebra $\mathbb{C}[S_n]$. Using this family, we obtain nonnegativity properties of polynomials of the form $\sum c_{I,I'}\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$. In particular, we show that the application of certain of these polynomials to Jacobi-Trudi matrices yields symmetric functions which are equal to nonnegative linear combinations of Schur functions.

1. Introduction

Since its introduction in [20], the Kazhdan-Lusztig basis $\{C'_w(q)\,|\,w\in S_n\}$ of the Hecke algebra $H_n(q)$ has found many applications related to algebraic geometry, combinatorics, and Lie theory. One such application, due to Haiman [17], clarifies three nonnegativity properties of certain polynomials which arose in the representation theory of $H_n(q)$. Years later, two of these nonnegativity properties were observed in a family of polynomials which arose in the study of inequalities satisfied by minors of totally nonnegative matrices [8, 28]. Building upon the arguments of Haiman [17], we will show that this family posesses the third nonnegativity property as well.

The nonnegativity properties are as follows. Let $x=(x_{ij})$ be a generic square matrix. For each pair (I,I') of subsets of $[n]=\{1,\dots,n\}$, define $\Delta_{I,I'}(x)$ to be the (I,I') minor of x, i.e., the determinant of the submatrix of x corresponding to rows I and columns I'. A real matrix is called *totally nonnegative* (TNN) if each of its minors is nonnegative. A polynomial $p(x)=p(x_{1,1},\dots,x_{n,n})$ in n^2 variables is called totally nonnegative if for every TNN matrix A, the number

$$p(A)\underset{\mathrm{def}}{=}p(a_{1,1},\dots,a_{n,n})$$

is nonnegative. Much current work in total nonnegativity is motivated by problems in quantum Lie theory. (See e.g. [11, 24, 36].)

Other work in quantum Lie theory and the strong connection between total nonnegativity and symmetric functions lead to more nonnegativity properties. Somewhat

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analogous to TNN matrices are *Jacobi-Trudi* matrices $A = (h_{\lambda_i - \mu_j + j - i})_{i,j=1}^n$, whose entries are homogeneous symmetric functions. By convention we define $h_m = 0$ for $m < 0$. (See [27, 30] for information on Jacobi-Trudi matrices, and [13] for connections to total nonnegativity.) We will call the polynomial $p(x)$ *Schur nonnegative* (SNN) if for every $n \times n$ Jacobi-Trudi matrix A, the symmetric function $p(A)$ is equal to a nonnegative linear combination of Schur functions. We will also call such a symmetric function Schur nonnegative. Much current work in Schur nonnegativity is motivated by problems concerning the cohomology ring of the Grassmannian variety. (See e.g. [10].) In analogy to Schur nonnegativity, we will call $p(x)$ *monomial nonnegative* (MNN) if for every $n \times n$ Jacobi-Trudi matrix A, $p(A)$ is equal to a nonnegative linear combination of monomial symmetric functions. We will also call such a symmetric function monomial nonnegative. Since each Schur function is itself monomial nonnegative, any SNN polynomial must also be MNN.

Some nontrivial classes of polynomials possessing the TNN, SNN and MNN properties are contained in the complex span of the monomials $\{x_{1,w(1)} \cdots x_{n,w(n)} \,|\, w \in S_n\}$. We will call such polynomials *immanants.* In particular, for every function $f : S_n \to \mathbb{C}$ we define the *f-immanant* (as in [31, Sec. 3]) by

$$\mathrm{Imm}_f(x) \underset{\mathrm{def}}{=} \sum_{w \in S_n} f(w) x_{1,w(1)} \cdots x_{n,w(n)}.$$

Some familiar immanants are those of the form $\mathrm{Imm}_{\chi^\lambda}(x)$, where χ^λ is an irreducible character of S_n. Goulden and Jackson conjectured [15] and Greene proved [16] these immanants to be MNN. Stembridge then conjectured [34] these immanants to be TNN and SNN, and he [33] and Haiman [17] proved these two conjectures. (See [17, 18, 32, 33, 34] for related conjectures and results.) Other immanants of the form

$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \tag{1.1}$$

characterize the inequalities satisfied by products of two minors of TNN matrices. (Equivalently, these characterize the inequalities satisfied by products of two entries of the exterior power representation of TNN elements of $GL_n(\mathbb{C})$.) Fallat, Gekhtman and Johnson characterized [8] the TNN immanants of the form (1.1), in the principal minor case ($I = I'$, etc.) A characterization of the general case followed in [28], as did a proof that all such TNN immanants are MNN. More TNN, SNN and MNN immanants related to the Temperley-Lieb algebra and Bruhat order were studied in [6, 7, 26].

In Section 2 we define a family of immanants in terms of the Kazhdan-Lusztig basis of $\mathbb{C}[S_n]$ and discuss its nonnegativity properties. We then show in Sections 3-5 that the Kazhdan-Lusztig immanants unify *all* classes of TNN immanants mentioned in the previous paragraph. In particular, we prove that the immanants (1.1) are SNN, and apply this fact to problems concerning Schur functions in Section 5. In Sections 6-7 we consider determinant-like properties of the Kazhdan-Lusztig immanants and some open problems related to cones.

2. Kazhdan-Lusztig immanants and their nonnegativity properties

Let q be a formal parameter and define the *Hecke algebra* $H_n(q)$ to be the $\mathbb{C}[q^{1/2}, q^{-1/2}]$-algebra generated by elements $T_{s_1}, \ldots, T_{s_{n-1}}$, subject to the relations

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + q, && \text{for } i = 1, \ldots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j}, && \text{if } |i-j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i}, && \text{if } |i-j| \geq 2. \end{aligned}$$

For each permutation w we define the Hecke algebra element T_w by

$$T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}.$$

where $s_{i_1} \cdots s_{i_\ell}$ is any reduced expression for w. Specializing at $q = 1$ gives the symmetric group algebra $\mathbb{C}[S_n]$.

The elements $\{C'_v(q) \,|\, v \in S_n\}$ of the Kazhdan-Lusztig basis of $H_n(q)$ have the form

$$C'_v(q) = \sum_{u \leq v} P_{u,v}(q) q^{-\ell(v)/2} T_u, \tag{2.1}$$

where the comparison of permutations is in the Bruhat order, and

$$\{P_{u,v}(q) \,|\, u, v \in S_n\}$$

are certain polynomials in q, known as the *Kazhdan-Lusztig polynomials* [20]. Solving the equations (2.1) for T_v, we have

$$T_v = \sum_{u \leq v} (-1)^{\ell(v)-\ell(u)} P_{w_0 v, w_0 u}(q) q^{\ell(u)/2} C'_u(q), \tag{2.2}$$

where w_0 is the longest permutation in S_n [20, Thm. 3.1].

For each permutation v in S_n define the function $f_v : S_n \to \mathbb{C}$ by

$$f_v(w) = (-1)^{\ell(w)-\ell(v)} P_{w_0 w, w_0 v}(1).$$

Extending these functions linearly to $\mathbb{C}[S_n]$, we see that they are dual to the Kazhdan-Lusztig basis in the sense that

$$f_v(C'_w(1)) = \delta_{v,w}. \tag{2.3}$$

We will denote the f_v-immanant by

$$\mathrm{Imm}_v(x) \underset{\text{def}}{=} \sum_{w \geq v} f_v(w) x_{1,w(1)} \cdots x_{n,w(n)}, \tag{2.4}$$

and will call these immanants the *Kazhdan-Lusztig immanants*. In the case that v is the identity permutation, we obtain the determinant.

Results in [17, 33] imply that the Kazhdan-Lusztig immanants are TNN and SNN. To summarize these implications in Propositions 2.1-2.3, we shall consider the following elements of $H_n(q)$. Given indices $1 \leq i \leq j \leq n$, define $z_{[i,j]}$ to be the element of $H_n(q)$ which is the sum of elements T_w corresponding to permutations w in the parabolic subgroup of S_n generated by $s_i, \ldots, s_{j-1}$.

Proposition 2.1. *Let z be an element of $H_n(q)$ of the form*

$$z = z_{[i_1,j_1]} \cdots z_{[i_r,j_r]}. \tag{2.5}$$

Then we have

$$z = \sum_{w \in S_n} p_{z,w}(q) C'_w(q),$$

where the expressions $p_{z,w}(q)$ are Laurent polynomials in $q^{1/2}$ with nonnegative coefficients. In particular, an element of the form (2.5) in $\mathbb{C}[S_n]$ is equal to a nonnegative linear combination of the Kazhdan-Lusztig basis elements $\{C'_w(1) \,|\, w \in S_n\}$.

Proof. Let $s_{[i,j]}$ be the longest permutation in the subgroup generated by $s_i, \ldots s_{j-1}$. By [17, Prop. 3.1], we have

$$z_{[i,j]} = q^{\ell(s_{[i,j]})/2} C'_{s_{[i,j]}}(q).$$

A result of Springer [29] implies that for every pair (u, v) of permutations in S_n, we have

$$C'_u(q) C'_v(q) = \sum_{w \in S_n} f^w_{u,v}(q) C'_w(q),$$

where the expressions $f^w_{u,v}(q)$ are Laurent polynomials in $q^{1/2}$ with nonnegative coefficients. (See [17, Appendix].) $\square$

Proposition 2.2. *For each permutation w in S_n, the Kazhdan-Lusztig immanant* $\mathrm{Imm}_w(x)$ *is totally nonnegative.*

Proof. For any complex matrix A and any function $f : S_n \to \mathbb{C}$ we have

$$\mathrm{Imm}_f(A) = \sum_z c_z f(z),$$

where the sum is over elements z of $\mathbb{C}[S_n]$ of the form (2.5), and the coefficients c_z depend on A. If A is a totally nonnegative matrix, then these coefficients are real and nonnegative. (See, e.g., [26, Lem. 2.5], [33, Thm. 2.1].)

Let A be a TNN matrix. By Proposition 2.1 we have

$$\begin{aligned}\mathrm{Imm}_w(A) &= \sum_z c_z f_w(z)\\ &= \sum_z c_z \sum_v p_{z,v}(1) f_w(C'_v(1))\\ &= \sum_z c_z p_{z,w}(1)\\ &\geq 0.\end{aligned}$$

□

The following easy consequence of [17, Thm. 1.5] implies the Schur nonnegativity of the Kazhdan-Lusztig immanants. Following [17], we define a *generalized Jacobi-Trudi matrix* to be a finite matrix whose i, j entry is the homogeneous symmetric function $h_{\mu_i - \nu_j}$, where $\mu = (\mu_1, \dots, \mu_n)$ and $\nu = (\nu_1, \dots, \nu_n)$ are weakly decreasing nonnegative sequences, and by convention $h_m = 0$ if m is negative. Thus each generalized Jacobi-Trudi matrix is constructed from an ordinary Jacobi-Trudi matrix by repeating some rows and/or columns.

Proposition 2.3. *For each permutation w in S_n, and each $n \times n$ generalized Jacobi-Trudi matrix A, the symmetric function* $\mathrm{Imm}_w(A)$ *is Schur nonnegative.*

Proof. By [17, Thm. 1.5], we have

$$\sum_{v \in S_n} a_{1,v(1)} \cdots a_{n,v(n)} v = \sum_u g_{v,u}(A) C'_u(1),$$

where $g_{v,u}(A)$ is a Schur nonnegative symmetric function which depends upon A. Applying the function f_w to both sides of this equations, we have

$$\begin{aligned}\mathrm{Imm}_w(A) &= \sum_u g_{w,u}(A) f_w(C'_u(1))\\ &= g_{w,w}(A).\end{aligned}$$

□

3. Relation to Temperley-Lieb Immanants

The relationship of Kazhdan-Lusztig immanants to irreducible character immanants

$$\mathrm{Imm}_{\chi^\lambda}(x) = \sum_{w \in S_n} x_{1,w(1)} \cdots x_{n,w(n)} \tag{3.1}$$

was established as follows in [17, Lem. 1.1].

Proposition 3.1. *Each irreducible character immanant (3.1) is equal to a nonnegative linear combination of Kazhdan-Lusztig immanants.*

Thus the irreducible character immannats are TNN and SNN. In order to similarly prove the Schur nonnegativity of other immanants in Section 5, we will first relate the Kazhdan-Lusztig immanants to *Temperley-Lieb immanants* introduced in [26].

Given a formal parameter ξ, we define the *Temperley-Lieb algebra* $TL_n(\xi)$ to be the $\mathbb{C}[\xi]$-algebra generated by elements $t_1, \ldots, t_{n-1}$ subject to the relations

$$\begin{aligned} t_i^2 &= \xi t_i, && \text{for } i = 1, \ldots, n-1, \\ t_i t_j t_i &= t_i, && \text{if } |i-j| = 1, \\ t_i t_j &= t_j t_i, && \text{if } |i-j| \geq 2. \end{aligned}$$

The rank of $TL_n(\xi)$ as a $\mathbb{C}[\xi]$-module is well known to be $\frac{1}{n+1}\binom{2n}{n}$, and a natural basis is given by the elements of the form $t_{i_1} \cdots t_{i_\ell}$, where $i_1 \cdots i_\ell$ is a reduced word for a 321-avoiding permutation in S_n. (A permutation w is said to be 321-*avoiding* if there are no indices $i < j < k$ for which we have $w(i) > w(j) > w(k)$.) We shall call these elements the standard basis elements of $TL_n(\xi)$, or simply *the basis elements* of $TL_n(\xi)$.

The Temperley-Lieb algebra may be realized as a quotient of the Hecke algebra by

$$H_n(q)/(z_{[1,3]}) \cong TL_n(q^{1/2} + q^{-1/2}),$$

where the element $z_{[1,3]}$ of $H_n(q)$ is defined as before Proposition 2.1. We will let θ_q be the homomorphism

$$\begin{aligned} H_n(q) &\to TL_n(q^{1/2} + q^{-1/2}) \\ q^{-1/2}(T_{s_i} + 1) &\mapsto t_i. \end{aligned}$$

(See e.g. [9], [14, Sec. 2.1, Sec. 2.11], [35, Sec. 7].)

Temperley-Lieb immanants are defined in terms of the homomorphism θ_1 as follows. For each basis element τ of $TL_n(2)$, let $f_\tau : S_n \to \mathbb{R}$ be the function defined by

$$f_\tau(v) = \text{ coefficient of } \tau \text{ in } \theta_1(T_v),$$

and let

$$\mathrm{Imm}_\tau(x) = \sum_{w \in S_n} f_\tau(w) x_{1,w(1)} \cdots x_{n,w(n)}$$

be the corresponding immanant. By [26, Thm. 3.1], the Temperley-Lieb immanants are TNN. Furthermore, the following result shows that the Temperley-Lieb immanants are Kazhdan-Lusztig immanants. To prove this, we define for each 321-avoiding permutation w in S_n an element $D_w(q)$ of $H_n(q)$ as follows. For any reduced word $i_1 \cdots i_\ell$ for w, define

$$D_w(q) \underset{\text{def}}{=} q^{-1/\ell}(T_{s_{i_1}} + 1) \cdots (T_{s_{i_\ell}} + 1).$$

(This element does not depend upon the particular reduced word.) The element $D_w(q)$ satisfies

$$\theta_q(D_w(q)) = t_{i_1} \cdots t_{i_\ell},$$

and it follows that the set

$$\{\theta(D_w(q)) \mid w \text{ a 321-avoiding permutation }\}$$

is equal to the standard basis of $TL_n(q^{1/2} + q^{-1/2})$.

Proposition 3.2. *Let w be a 321-avoiding permutation and define $\tau = \theta_1(D_w(1))$. Then the Temperley-Lieb immanant $\mathrm{Imm}_\tau(x)$ is equal to the Kazhdan-Lusztig immanant $\mathrm{Imm}_w(x)$.*

Proof. Let v be any permutation in S_n. Then we have

$$v = \sum_{u \leq v} (-1)^{\ell(v)-\ell(u)} P_{w_0 v, w_0 u}(1) C'_u(1).$$

The coefficient of $x_{1,v(1)} \cdots x_{n,v(n)}$ in $\mathrm{Imm}_\tau(x)$ is equal to $f_\tau(v)$, which is the coefficient of τ in

$$\theta_1(v) = \sum_{u \leq v} (-1)^{\ell(v)-\ell(u)} P_{w_0 v, w_0 u}(1) \theta_1(C'_u(1)). \tag{3.2}$$

A result of Fan and Green [9, Thm. 3.8.2] implies that we have

$$\theta_q(C'_w(q)) = \begin{cases} \theta_q(D_w(q)) & \text{if } w \text{ is 321-avoiding,} \\ 0 & \text{otherwise.} \end{cases}$$

(See also [3, Thm. 4].) We may therefore assume that each permutation u appearing in (3.2) is 321-avoiding, and we may rewrite the sum as

$$\theta_1(v) = \sum_{u \leq v} (-1)^{\ell(v)-\ell(u)} P_{w_0 v, w_0 u}(1) \theta_1(D_u(1)).$$

The coefficient of $\tau = \theta_1(D_w(1))$ in this expression is $f_w(v)$, as desired. □

Thus the Temperley-Lieb immanants are precisely the Kazhdan-Lusztig immanants corresponding to 321-avoiding permutations.

4. Relation to the Bruhat order

The Bruhat order on S_n may be defined by setting $u \leq v$ whenever some (equivalently, each) reduced expression for v contains a subexpression which is a reduced expression for u. (See references of [6, 7] for other definitions.) Three more definitions concern nonnegativity properties of immanants [6, Thm. 2], [7, Thm. 2].

Theorem 4.1. *The following conditions on two permutations in S_n are equivalent:*

(1) $u \leq v$ *in the Bruhat order.*
(2) $x_{1,u(1)} \cdots x_{n,u(n)} - x_{1,v(1)} \cdots x_{n,v(n)}$ *is MNN.*
(3) $x_{1,u(1)} \cdots x_{n,u(n)} - x_{1,v(1)} \cdots x_{n,v(n)}$ *is SNN.*
(4) $x_{1,u(1)} \cdots x_{n,u(n)} - x_{1,v(1)} \cdots x_{n,v(n)}$ *is TNN.*

To relate these definitions to the Kazhdan-Lusztig immanants, we offer one more.

Theorem 4.2. *We have* $u \leq v$ *in the Bruhat order if and only if the immanant*

$$x_{1,u(1)} \cdots x_{n,u(n)} - x_{1,v(1)} \cdots x_{n,v(n)} \tag{4.1}$$

is equal to a nonnegative linear combination of Kazhdan-Lusztig immanants.

Proof. Solving the equations (2.4) for the monomials, we have

$$x_{1,v(1)} \cdots x_{n,v(n)} = \sum_{w \geq v} P_{v,w}(1)\mathrm{Imm}_w(x).$$

Thus the coefficient of $\mathrm{Imm}_w(x)$ in (4.1) is $P_{v,w}(1) - P_{u,w}(1)$. This is clearly nonnegative whenever $u \not\leq w$, so assume that $u \leq w$.

If $v \not\leq u$, then we have $P_{v,u}(1) - P_{u,u}(1) = -1$. Suppose therefore that $v \leq u \leq w$. By a result of Braden and Macpherson [4, Cor. 3.7], the polynomial $P_{v,w}(q) - P_{u,w}(q)$ has nonnegative integer coefficients. Thus, $P_{v,w}(1) - P_{u,w}(1)$ is nonnegative. □

5. Applications to Products of Matrix Minors

Studying inequalities satisfied by products of principal minors of TNN matrices, Fallat, Gekhtman and Johnson [8, Thm. 4.6] characterized all TNN immanants of the form

$$\Delta_{J,J}(x)\Delta_{\overline{J},\overline{J}}(x) - \Delta_{I,I}(x)\Delta_{\overline{I},\overline{I}}(x),$$

(where $\overline{I} = [n] \smallsetminus I$, $\overline{J} = [n] \smallsetminus J$) and more generally, all TNN polynomials of the form

$$\Delta_{J,J}(x)\Delta_{L,L}(x) - \Delta_{I,I}(x)\Delta_{K,K}(x),$$

where the index sets need not be complementary. This result was generalized further in [28, Thm. 3.2] to apply to polynomials of the form

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x), \tag{5.1}$$

in which the minors need not be principal, i.e. I need not be equal to I'. We will show in Theorem 5.2 that conditions on the sets $I, \ldots, L$, $I', \ldots, L'$ which are equivalent to the total nonnegativity of (5.1) are sufficient to imply the Schur nonnegativity of (5.1). One characterization of TNN polynomials of this form is the following [28, Thm. 4.2]. (See also [26, Thm. 5.2, Cor. 5.5].)

Proposition 5.1. *Let I, J, K, L be subsets of $[n]$ and let I', J', K', L' be subsets of $[n']$, and define the subsets I'', J'', K'', L'' of $[n+n']$ by*

$$\begin{aligned} I'' &= I \cup \{n+n'+1-i \,|\, i \in K'\}, \\ J'' &= J \cup \{n+n'+1-i \,|\, i \in L'\}, \\ K'' &= K \cup \{n+n'+1-i \,|\, i \in I'\}, \\ L'' &= L \cup \{n+n'+1-i \,|\, i \in J'\}. \end{aligned} \tag{5.2}$$

Then the polynomial

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x) \tag{5.3}$$

is totally nonnegative if and only if the sets $I,\dots,L$, $I',\dots,L'$ satisfy

$$\begin{aligned} I \cup K = J \cup L, &\qquad I' \cup K' = J' \cup L', \\ I \cap K = J \cap L, &\qquad I' \cap K' = J' \cap L', \end{aligned} \tag{5.4}$$

and for each subinterval B of $[n+n']$ the sets $I'',\dots,L''$ satisfy

$$\max\{|B \cap J''|, |B \cap L''|\} \leq \max\{|B \cap I''|, |B \cap K''|\}. \tag{5.5}$$

The proof in [28] shows that these polynomials are MNN as well. (See [26, Cor. 6.1].) The characterization of these polynomials [28, Cor. 5.5] replaces the equalities (5.2) and the inequalities (5.5) with conditions stated in terms of $TL_n(2)$. This alternative characterization plays a crucial role in the proof of the following result.

Theorem 5.2. *Let I, J, K, L be subsets of $[n]$, let I', J', K', L' be subsets of $[n']$, and suppose that these satisfy the conditions of Proposition 5.1. Then the polynomial*

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x) \tag{5.6}$$

is Schur nonnegative.

Proof. Define $r = |I|+|K|$, and let $k_1 \leq \cdots \leq k_r$ be the nondecreasing rearrangement of the elements of I and K, including repeated elements. Define $k'_1,\dots,k'_r$ analogously, and let y be the $r\times r$ matrix whose i,j entry is the variable x_{k_i,k'_j}. Thus y is the matrix obtained from x by duplicating rows whose indices belong to $I\cap K$ and columns whose indices belong to $I' \cap K'$.

By Proposition 5.1, the polynomial (5.6) is TNN, and by [26, Cor. 5.5] we have

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x) = \sum_{\tau} \mathrm{Imm}_\tau(y),$$

where the sum is over a subset of basis elements of $TL_r(2)$. By Proposition 3.2 this is a sum of Kazhdan-Lustig immanants,

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x) = \sum_{w} \mathrm{Imm}_w(y), \tag{5.7}$$

where the sum is over an appropriate set of 321-avoiding permutations w in S_r.

Now let A be an arbitrary $n \times n'$ Jacobi-Trudi matrix, and let B be the generalized Jacobi-Trudi matrix whose i, j entry is a_{k_i,k'_j}. Then the evaluation of the left-hand side of (5.7) at $x = A$ is equal to the evaluation of the right-hand side at $y = B$. By Proposition 2.3, the resulting symmetric function on the right-hand side is SNN. Thus the polynomial $\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x)$ is SNN. □

Of course it is also true that any linear combination of products of matrix minors which can be expressed as

$$\sum_i \Delta_{J_i,J'_i}(x)\Delta_{L_i,L'_i}(x) - \Delta_{I_i,I'_i}(x)\Delta_{K_i,K'_i}(x) = \sum_w d_w \mathrm{Imm}_w(y),$$

where y is obtained from x as in the preceding proof, is SNN if the coeffients d_w are all nonnegative. Theorem 5.2 is a special case of this. On the other hand, while the conditions of Proposition 5.1 are sufficient to ensure the Schur nonnegativity of the polynomial (5.3), it is not clear that they are necessary.

Question 5.3. *Is Proposition 5.1 a characterization of the Schur nonnegative differences of products of matrix minors?*

Theorem 5.2 provides new machinery for proving that certain symmetric functions are SNN. In particular, various special cases of the following question have appeared in the literature.

Question 5.4. *What conditions on the integer partitions α, β, γ, δ, κ, λ, μ, ν imply the Schur nonnegativity of the symmetric function $s_{\alpha/\kappa}s_{\beta/\lambda} - s_{\gamma/\mu}s_{\delta/\nu}$?*

We will provide some simple examples.

Proposition 5.5. *Given an integer n, define the partitions ρ, ρ' by*

$$\rho = (\lceil \tfrac{n}{2} \rceil - 1, \lceil \tfrac{n}{2} \rceil - 2, \dots, 1)$$
$$\rho' = (\lfloor \tfrac{n}{2} \rfloor - 1, \lfloor \tfrac{n}{2} \rfloor - 2, \dots, 1).$$

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition. Then for any k in $[n]$ the symmetric function

$$s_{(\lambda_1,\lambda_3,\dots)/\rho}s_{(\lambda_2,\lambda_4,\dots)/\rho'} - s_{(\lambda_1,\dots,\lambda_k)}s_{(\lambda_{k+1},\dots,\lambda_n)}$$

is Schur nonnegative.

Proof. Let J be the set of odd integers in $[n]$, and let $I = [k]$. By Proposition 5.1, the polynomial

$$\Delta_{J,J}(x)\Delta_{\overline{J},\overline{J}}(x) - \Delta_{I,I}(x)\Delta_{\overline{I},\overline{I}}(x)$$

is SNN, and its evaluation at the Jacobi-Trudi matrix $(h_{\lambda_i+j-i})_{i,j=1}^n$ gives the symmetric function (5.5). □

For instance, we may choose $n = 7$, $k = 4$, $\lambda = 5444333$ to prove the Schur nonnegativity of the symmetric function

$$s_{8643/432}s_{653/32} - s_{5444}s_{333}.$$

Proposition 5.6. *Let $\theta = (\theta_1, \dots, \theta_{2k})$ and $\gamma = (\gamma_1, \dots, \gamma_{2k})$ be partitions and define*

$$\begin{aligned}
\kappa &= (\theta_1 - \gamma_k, \dots, \theta_k - \gamma_k),\\
\lambda &= (\theta_{k+1}, \dots, \theta_{2k}),\\
\mu &= (\theta_1 + k, \dots, \theta_k + k),\\
\nu &= (\theta_{k+1} - \gamma_k - k, \dots, \theta_{2k} - \gamma_k - k),\\
\alpha &= (\gamma_1 - \gamma_k, \dots, \gamma_k - \gamma_k),\\
\beta &= (\gamma_{k+1}, \dots, \gamma_{2k}).
\end{aligned}$$

Then the symmetric function

$$s_{\kappa/\alpha}s_{\lambda/\beta} - s_{\mu/\beta}s_{\nu/\alpha} \tag{5.8}$$

is Schur nonnegative.

Proof. Let $n = 2k$ and let $J = [k]$. By Proposition 5.1, the polynomial

$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{J,\overline{J}}(x)\Delta_{\overline{J},J}(x)$$

is SNN, and its evaluation at the Jacobi-Trudi matrix $H_{\lambda/\mu}$ gives the symmetric function (5.8). $\square$

For instance, we may choose $k = 3$, $\theta = (13, 11, 8, 8, 7, 5)$, $\gamma = (3, 2, 1, 1, 1)$ and apply the above proposition to the Jacobi-Trudi matrix $(h_{\theta_i-\gamma_j+j-i})_{i,j=1}^6$ to prove the Schur nonnegativity of the symmetric function

$$s_{(12,10,7)/(2,1)}s_{(8,7,5)/(1,1)} - s_{(16,14,11)/(1,1)}s_{(4,3,1)/(2,1)}.$$

More answers to Question5.4 have recently been provided by Lam, Postnikov and Pylyavskyy [21]. In particular they have applied Theorem 5.2 to prove conjectures of their own [22], of Lascoux, Leclerc and Thibon [23, Conj. 6.4], of Okounkov [25, p. 269], of Fomin, Fulton, Li and Poon [10, Conj. 2.8], and of Bergeron and McNamara [2, Conj. 5.2] It would be interesting to use these methods to settle [10, Conj. 5.1] and the stronger [1, Conj. 2.9].

6. Determinant-like properties of Kazhdan-Lusztig immanants

In the following propositions, we use $<$ to denote the Bruhat order on S_n, $\ell(w)$ to denote the length of a reduced expression for $w \in S_n$, and $\mu(u, v)$ to denote the nonnegative integer which is the coefficient of $q^{(\ell(v)-\ell(u)-1)/2}$ in $P_{u,v}(q)$. (See [19] for more information.)

Lemma 6.1. *Let u, v be permutations in S_n. Then we have*

$$P_{u,v}(q) = P_{u^{-1},v^{-1}}(q) = P_{w_0uw_0,w_0vw_0}(q), \tag{6.1}$$

$$\mu(u,v) = \mu(u^{-1},v^{-1}) = \mu(w_0uw_0,w_0vw_0). \tag{6.2}$$

Proof. Kazhdan and Lusztig's *R-polynomials* $\{R_{u,v}(q) \,|\, u,v \in S_n\}$, introduced in [20], satisfy

$$R_{u,v}(q) = R_{u^{-1},v^{-1}}(q) = R_{w_0uw_0,w_0vw_0}(q)$$

by [20, Sec. 2] and [19, Sec. 7.6].

Applying these facts to the recursive definition of the Kazhdan-Lusztig polynomials in [20, Eq. (2.2.b)] and using induction on $\ell(v) - \ell(u)$ we obtain (6.1). The equations (6.2) follow immediately. □

Proposition 6.2. *For any permutation w in S_n we have*

$$\mathrm{Imm}_w(x^T) = \mathrm{Imm}_{w^{-1}}(x),$$

where $x^T_{i,j} = x_{j,i}$.

Proof. By Lemma 6.1 we have

$$\begin{aligned} f_{w^{-1}}(v^{-1}) &= (-1)^{\ell(v^{-1})-\ell(w^{-1})} P_{w_0v^{-1},w_0w^{-1}}(1) \\ &= (-1)^{\ell(v)-\ell(w)} P_{w_0v,w_0w}(1) \\ &= f_w(v). \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{Imm}_w(x^T) &= \sum_{v \in S_n} f_w(v) x_{v(1),1} \cdots x_{v(n),n} \\ &= \sum_{v \in S_n} f_{w^{-1}}(v^{-1}) x_{v(1),1} \cdots x_{v(n),n} \\ &= \sum_{v \in S_n} f_{w^{-1}}(v) x_{1,v(1)} \cdots x_{n,v(n)} \\ &= \mathrm{Imm}_{w^{-1}}(x). \end{aligned}$$

□

Let P be the $n \times n$ permutation matrix corresponding to the adjacent transposition s_i in S_n, so that the matrices A and PA differ by a transposition of their ith and $(i+1)$st rows. Recalling that the determinant satisfies

$$\det(PA) = -\det(A),$$

we will prove similar properties of the Kazhdan-Lusztig immanants.

Proposition 6.3. *Let A be an $n \times n$ matrix and let P be the permutation matrix corresponding to the adjacent transposition s_i of S_n. Then we have*

$$\mathrm{Imm}_w(PA) = \begin{cases} -\mathrm{Imm}_w(A), & \text{if } s_i w > w, \\ \mathrm{Imm}_w(A) + \mathrm{Imm}_{s_i w}(A) + \displaystyle\sum_{s_i z > z} \mu(w,z)\,\mathrm{Imm}_z(A) & \text{if } s_i w < w. \end{cases}$$

$$\mathrm{Imm}_w(AP) = \begin{cases} -\mathrm{Imm}_w(A), & \text{if } w s_i > w, \\ \mathrm{Imm}_w(A) + \mathrm{Imm}_{w s_i}(A) + \displaystyle\sum_{z s_i > z} \mu(w,z)\,\mathrm{Imm}_z(A) & \text{if } w s_i < w. \end{cases}$$

Proof. By the duality of Kazhdan-Lusztig immanants and Kazhdan-Lusztig basis elements (2.3), we have

$$\sum_{w \in S_n} a_{1,w(1)} \cdots a_{n,w(n)} w = \sum_{w \in S_n} \mathrm{Imm}_w(A) C'_w(1).$$

Thus $\mathrm{Imm}_w(PA)$ is equal to the coefficient of $C'_w(1)$ in

$$\begin{aligned} \sum_{v \in S_n} a_{1,s_i v(1)} \cdots a_{n,s_i v(n)} v &= s_i \sum_{v \in S_n} a_{1,v(1)} \cdots a_{n,v(n)} v \\ &= \sum_{v \in S_n} \mathrm{Imm}_v(A) s_i C'_v(1). \end{aligned}$$

One can show (e.g. mimicking the proof of [12, Thm 2.1]) that the Kazhdan-Lusztig basis elements satisfy

$$s_i C'_v(1) = \begin{cases} C'_v(1) & \text{if } s_i v < v, \\ C'_{s_i v}(1) - C'_v(1) + \displaystyle\sum_{s_i y < y} \mu(y,v) C'_y(1) & \text{if } s_i v > v. \end{cases}$$

Thus the coefficient in question is that of $C'_w(1)$ in

$$\sum_{s_i v < v} \mathrm{Imm}_v(A) C'_v(1) + \sum_{s_i v > v} \mathrm{Imm}_v(A) \left(C'_{s_i v}(1) - C'_v(1) + \sum_{s_i y < y} \mu(y,v) C'_y(1) \right).$$

If $s_i w > w$, this coefficient is $-\mathrm{Imm}_w(A)$; if $s_i w < w$, it is

$$\mathrm{Imm}_w(A) + \mathrm{Imm}_{s_i w}(A) + \sum_{s_i z > z} \mu(w,z)\,\mathrm{Imm}_z(A),$$

as desired.

Applying Proposition 6.2 to this result and using (6.2), we obtain the stated expression for $\mathrm{Imm}_w(AP)$.

□

Corollary 6.4. *Let A be an $n \times n$ matrix in which rows i and $i+1$ are equal, and let w be a permutation in S_n. If some reduced expression for w begins with s_i, then we have*

$$\mathrm{Imm}_w(A) = 0.$$

If some reduced expression for w ends with s_i, then we have

$$\mathrm{Imm}_w(A^T) = 0.$$

To generalize the identity

$$\det \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \det(B)\det(D)$$

concerning block-upper-triangular matrices, we introduce the following operation on permutations. Given permutations

$$w_1 = s_{i_1} \cdots s_{i_k} \in S_n,$$
$$w_2 = s_{j_1} \cdots s_{j_\ell} \in S_m,$$

define the permutation $w_1 \oplus w_2$ in S_{n+m} by

$$w_1 \oplus w_2 = s_{i_1} \cdots s_{i_k} s_{n+j_1} \cdots s_{n+j_\ell}.$$

It is clear that a permutation $w \in S_{n+m}$ decomposes as $w_1 \oplus w_2$ with $w_1 \in S_n$, $w_2 \in S_m$ if and only if no reduced expression for w contains the transposition s_n.

Proposition 6.5. *Let v be an element of S_{n+m} and let A be an $(n+m) \times (n+m)$ block-upper-triangular matrix of the form*

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

with B an $n \times n$ matrix and D an $m \times m$ matrix. Then we have

$$\mathrm{Imm}_v(A) = \begin{cases} \mathrm{Imm}_{v_1}(B)\mathrm{Imm}_{v_2}(D) & \text{if } v = v_1 \oplus v_2 \text{ for some } v_1 \in S_n, v_2 \in S_m, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

Proof. The block-upper-triangular form of A implies that

$$a_{1,w(1)} \cdots a_{n,w(n)} = 0$$

whenever w does not decompose as $w = w_1 \oplus w_2$ with $w_1 \in S_n$, $w_2 \in S_m$. Thus we have

$$\mathrm{Imm}_v(A) = \sum_{w_1 \oplus w_2 \geq v} f_v(w_1 \oplus w_2) b_{1,w_1(1)} \cdots b_{n,w_1(n)} d_{1,w_2(1)} \cdots d_{m,w_2(m)}.$$

If some reduced expression for v contains the transposition s_n, then the above sum is empty and the immanant is equal to zero. Suppose therefore that v decomposes as $v = v_1 \oplus v_2$. Then we have

$$\mathrm{Imm}_v(A) = \sum_{w_1 \geq v_1} \sum_{w_2 \geq v_2} f_{v_1 \oplus v_2}(w_1 \oplus w_2) b_{1,w_1(1)} \cdots b_{n,w_1(n)} d_{1,w_2(1)} \cdots d_{m,w_2(m)}.$$

Let w_0' and w_0'' be the longest elements of S_n and S_m respectively. A result of Brenti [5, Thm. 4.4] concerning the factorization of Kazhdan-Lusztig polynomials implies that we have

$$P_{w_0(w_1 \oplus w_2), w_0(v_1 \oplus v_2)}(1) = P_{w_0' w_1, w_0'' v_1}(1) P_{w_0' w_2, w_0'' v_2}(1).$$

Thus we have

$$f_{v_1 \oplus v_2}(w_1 \oplus w_2) = f_{v_1}(w_1) f_{v_2}(w_2)$$

and our result follows. □

7. Cones of immanants

Work on immanants related to representations of S_n has led to the study of certain elements of $\mathbb{C}[S_n]$ associated to total nonnegativity. Following Stembridge [33], we define the *cone of total nonnegativity* to be the smallest cone in $\mathbb{C}[S_n]$ containing the set

$$\{ \sum_w a_{1,w(1)} \cdots a_{n,w(n)} w \mid A \text{ TNN } \}.$$

We shall denote this cone by $\mathcal{C}_{TNN}$. (We omit the number n from this notation, although the cone obviously depends upon n.) Dual to $\mathcal{C}_{TNN}$ is the cone of TNN immanants, which we shall denote by $\check{\mathcal{C}}_{TNN}$,

$$\check{\mathcal{C}}_{TNN} = \{ \mathrm{Imm}_f(x) \mid f(z) \geq 0 \text{ for all } z \in \mathcal{C}_{TNN} \}.$$

No simple description of the extremal rays of these cones is known. However, Stembridge showed [33, Thm. 2.1] that $\mathcal{C}_{TNN}$ is contained in the cone whose extremal rays are elements of $\mathbb{C}[S_n]$ of the form (2.5). We shall denote this third cone by $\mathcal{C}_{INT}$. Furthermore, Stembridge showed that this containment $\mathcal{C}_{TNN} \subset \mathcal{C}_{INT}$ is proper for $n \geq 4$.

Define C_{KL} to be the cone whose extremal rays are the Kazhdan-Lusztig basis elements $\{ C_w'(1) \mid w \in S_n \}$. By Proposition 2.1 ([17, Prop. 3.1]), $\mathcal{C}_{INT}$ is contained in $\mathcal{C}_{KL}$. It is not difficult to show that this containment is proper for $n \geq 4$. Thus we have the proper containment of the dual cones

$$\check{\mathcal{C}}_{KL} \subset \check{\mathcal{C}}_{INT} \subset \check{\mathcal{C}}_{TNN}.$$

For small n, many of the Kazhdan-Lusztig immanants seem to be extremal rays in $\check{\mathcal{C}}_{TNN}$.

An interesting related fact concerns TNN immanants in the variables $x_{1,1}, \ldots, x_{4,4}$. Writing such an immanant as

$$\mathrm{Imm}_f(x) = \sum_{w \in S_4} d_w \mathrm{Imm}_w(x),$$

It is straightforward to show that d_w must be nonnegative if $w \notin \{3412, 4231\}$. This suggests the following question.

Question 7.1. *Let* $\mathrm{Imm}_f(x_{1,1} \ldots, x_{n,n})$ *be a totally nonnegative immanant and write*

$$\mathrm{Imm}_f(x) = \sum_{w \in S_n} d_w \mathrm{Imm}_w(x).$$

Must d_w *be nonnegative when the Schubert variety* Γ_w *is smooth? (i.e. when* w *avoids the patterns* $3412, 4231$*?)*

In analogy to the dual cone of nonnegativity, one may define cones $\check{\mathcal{C}}_{SNN}$ and $\check{\mathcal{C}}_{MNN}$ of Schur nonnegative immanants and monomial nonnegative immanants. While these cones are not known to differ from one another, or from $\check{C}_{TNN}$, we do have the containments

$$\check{\mathcal{C}}_{KL} \subset \check{\mathcal{C}}_{INT} \subseteq \check{\mathcal{C}}_{SNN} \subseteq \check{\mathcal{C}}_{MNN}.$$

This suggests the following problem.

Problem 7.2. *Describe the precise containment relationships between the cones* $\check{C}_{MNN}$*,* $\check{C}_{SNN}$*, and* $\check{C}_{TNN}$*.*

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