HALL-LITTLEWOOD POLYNOMIALS AND FIXED POINT ENUMERATION

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ABSTRACT. We resolve affirmatively some conjectures of Reiner, Stanton, and White [12] regarding enumeration of various classes of matrices which are invariant under certain cyclic row and column rotations. Our results are phrased in terms of the bicyclic sieving phenomenon introduced by Barcelo, Reiner, and Stanton [1]. The proofs of our results use various tools from symmetric function theory such as the Stanton-White rim hook correspondence [18] and results concerning the specialization of Hall-Littlewood polynomials due to Lascoux, Leclerc, and Thibon [5] [6].

1. INTRODUCTION AND MAIN RESULTS

Let X be a finite set and $C \times C'$ be a direct product of two finite cyclic groups acting on X. Fix generators c and c' for C and C' and let $\zeta, \zeta' \in \mathbb{C}$ be two roots of unity having the same multiplicative orders as c, c'. Let $X(q, t) \in \mathbb{C}[q, t]$ be a polynomial in two variables. Following Barcelo, Reiner, and Stanton [1], we say that the triple $(X, C \times C', X(q, t))$ *exhibits the bicyclic sieving phenomenon* (biCSP) if for any integers $d, e \geq 0$ the cardinality of the fixed point set $X^{(c^d, c'^e)}$ is equal to the polynomial evaluation $X(\zeta^d, \zeta^e)$. The biCSP encapsulates several combinatorial phenomena: specializing to the case where one of the cyclic groups is trivial yields the *cyclic sieving phenomenon* of Reiner, Stanton, and White [13] and specializing further to the case where the nontrivial cyclic group has order two yields the $q = -1$ *phenomenon* of Stembridge [19]. Moreover, the fact that the identity element in any group action fixes everything implies that whenever $(X, C \times C', X(q, t))$ exhibits the biCSP, we must have that the $q = t = 1$ specialization $X(1, 1)$ is equal to the cardinality $|X|$ of the set X. In this paper we prove a pair of biCSPs conjectured by Reiner, Stanton, and White where the sets X are certain sets of matrices acted on by row and column rotation and the polynomials $X(q, t)$ are bivariate deformations of identities arising from the RSK insertion algorithm. Our proof, outlined in Section 2, relies on symmetric function theory and plethystic substitution. In Section 3 we outline an alternative argument due to Victor Reiner which proves these biCSPs 'up to modulus' using DeConcini-Procesi modules.

Given a partition $\lambda \vdash n$, recall that a *semistandard Young tableau (SSYT) of shape* λ is a filling of the Ferrers diagram of λ with positive numbers which increase strictly down columns and weakly across rows. For a SSYT T of shape λ, the *content* of T is the (weak) composition $\mu \models n$ given by letting μ_i equal the number of $i's$ in T. A SSYT T is called *standard* (SYT) if it has content 1^n. For a partition λ and a composition μ of n, the *Kostka number* $K_{\lambda,\mu}$ is equal to the number of SSYT of shape λ and content μ.

The *Kostka-Foulkes polynomials* $K_{\lambda,\mu}(q)$, indexed by a partition $\lambda \vdash n$ and a composition $\mu \models n$, arose originally as the entries of the transition matrix between the Schur function and Hall-Littlewood symmetric function bases of the ring of symmetric functions (with coefficients in $\mathbb{C}(q)$ where q is an indeterminate). A combinatorial proof of the positivity of their coefficients was given by Lascoux and Schützenberger [7] by identifying $K_{\lambda,\mu}(q)$ as the generating function for the statistic of *charge* on the set of semistandard tableaux of shape λ and content μ. We outline the definition of charge as the rank function of a cyclage poset.

Let $\mathcal{A}^*$ denote the free monoid of words $w_1 \ldots w_k$ of any length with letters drawn from $[n]$. Let $\equiv$ be the equivalence relation on $\mathcal{A}^*$ induced by

$$RkijR' \equiv RikjR', \quad RjikR \equiv RjkiR', \quad RjiiR' \equiv RijiR', \quad RjijR' \equiv RjjiR',$$

where $1 \leq i < j < k \leq n$ and R and R' are any words in the monoid $\mathcal{A}^*$. The *Robinson-Schensted-Knuth correspondence* yields an algorithmic bijection between words w in $\mathcal{A}^*$ and pairs $(P(w), Q(w))$ of tableaux, where P is a SSYT with entries $\leq n$ and Q is a SYT with the shape of $P(w)$ equal to the shape of $Q(w)$. For details on the RSK correspondence, see for example [14] or [17]. The RSK correspondence sets up an equivalence relation $\equiv'$ on words in $\mathcal{A}^*$ by setting $w \equiv' w'$ if and only if $Q(w) = Q(w')$. It is a result of Knuth [4] that the equivalence relations $\equiv$ and $\equiv'$ on $\mathcal{A}^*$ agree. That is, for any $w, w' \in \mathcal{A}^*$ we have $w \equiv w'$ if and only if $Q(w) = Q(w')$. Therefore, the quotient monoid $\mathcal{A}^*/ \equiv$ is in a natural bijective correspondence with the set of SSYT with entries $\leq n$. This quotient is called the *plactic monoid*.

Cyclage is a monoid analogue of the group operation of conjugation introduced by Lascoux and Schützenberger [8]. Given $w, w' \in \mathcal{A}^*/ \equiv$, say that $w \prec w'$ if there exists $i \geq 2$ and $u \in \mathcal{A}^*/ \equiv$ so that $w = iu$ and $w' = ui$. For a fixed composition $\mu \models n$, the transitive closure of the relation $\prec$ induces a partial order on the subset of $\mathcal{A}^*/ \equiv$ consisting of words of content μ, and therefore also on the set of SSYT of content μ. For fixed μ, the rank generating function for this poset is called *cocharge* and is therefore a statistic on SSYT of content μ. The rank function of the order theoretic dual of this poset is called *charge*. Lascoux and Scützenberger [7] proved that for any partition $\lambda \vdash n$ and any composition $\mu \models n$, we have that

$$K_{\lambda,\mu}(q) = \sum_T q^{charge(T)},$$

where the sum ranges over all SSYT T of shape λ and content μ.

For $n \geq 0$, define $\epsilon_n(q, t) \in \mathbb{N}[q, t]$ to be $(qt)^{n/2}$ if n is even and 1 if n is odd. The type A specialization of Theorem 1.4 of Barcelo, Reiner, and Stanton [1] yields the following:

Theorem 1.1. *([1]) Let X be the set of $n \times n$ permutation matrices and $\mathbb{Z}_n \times \mathbb{Z}_n$ act on X by row and column rotation. The triple $(X, \mathbb{Z}_n \times \mathbb{Z}_n, X(q, t))$ exhibits the biCSP, where*

$$X(q, t) = \epsilon_n(q, t) \sum_{\lambda \vdash n} K_{\lambda, 1^n}(q) K_{\lambda, 1^n}(t).$$

Example 1.1. Let $n = 4$. We have that

$$X(q, t) = (qt)^2 \Big[(qt)^6 + (qt)^3(1 + q + q^2)(1 + t + t^2) + (qt)^2(1 + q^2)(1 + t^2)$$
$$+ (qt)(1 + q + q^2)(1 + t + t^2) + 1 \Big].$$

Consider the action of the diagonal subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_4$ on $X = S_4$. Let r be the generator of this subgroup, so that r acts on X by a simultaneous single row and column shift. We have that $X(i, i) = 4$, reflecting the fact that the fixed point set $X^r = \{1234, 2341, 3412, 4123\}$ has four elements. Also, $X(-1, -1) = 8$, whereas the fixed point set $X^{r^2} = \{1234, 2341, 3412, 4123, 1432, 2143, 3214, 4321\}$. Finally, we have that $X(i, -1) = 0$, reflecting the fact that no 4×4 permutation matrix is fixed by a simultaneous 1-fold row shift and 2-fold column shift.

The $q = t = 1$ specialization of $X(q, t)$ in the above result is implied by the RSK insertion algorithm on permutations. The following generalization of Theorem 1.1 to the case of words was known to Reiner and White but is unpublished. For any composition $\mu \models n$, let $\ell(\mu)$ denote the number of parts of μ and $|\mu| = n$ denote the sum of the parts of μ. A composition $\mu \models n$ is said to have cyclic symmetry of order a if one has $\mu_i = \mu_{i+a}$ always, where subscripts are interpreted modulo $\ell(\mu)$.

Theorem 1.2. *([12], [20]) Let $\mu \models n$ be a composition with cyclic symmetry of order $a | \ell(\mu)$. Let X be the set of length n words of content μ, thought of as 0,1-matrices in the standard way. The product of cyclic groups $\mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_n$ acts on X by a-fold row rotation and 1-fold column rotation.*

The triple $(X, \mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_n, X(q, t))$ exhibits the biCSP, where

$$X(q, t) = \epsilon_n(q, t) \sum_{\lambda \vdash n} K_{\lambda, \mu}(q) K_{\lambda, 1^n}(t).$$

Example 1.2. Let us give an example to show why the factor $\epsilon_n(q, t)$ is necessary in the definition of $X(q, t)$. Take $n = 2, \mu = (2)$, and $a = 1$. The set X is the singleton $\{11\}$ consisting of the word 11. One verifies that

$$K_{(1,1),(2)}(q) = 0, \quad K_{(2),(2)}(q) = 1, \quad K_{(1,1),(2)}(t) = 1, \quad K_{(2),(2)}(t) = t,$$

so that

$$X(q,t) = (qt)[0(1) + 1(t)] = qt^2.$$

We have the evaluation $X(1,-1) = 1$, which would have been negative if $X(q,t)$ did not contain the factor of $\epsilon_2(q,t) = qt$.

The $q = t = 1$ specialization of the identity in the above theorem arises from the application of RSK to the set of words with content μ. The following $\mathbb{N}$-matrix generalization of Theorem 1.2 was conjectured (unpublished) by Reiner and White in 2006.

Theorem 1.3. *Let $\mu, \nu \models n$ be two compositions having cyclic symmetries of orders $a|\ell(\mu)$ and $b|\ell(\nu)$, respectively. Let X be the set of $\ell(\mu) \times \ell(\nu)$ $\mathbb{N}$-matrices with row content μ and column content ν. The product of cyclic groups $\mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_{\ell(\nu)/b}$ acts on X by a-fold row rotation and b-fold column rotation.*

The triple $(X, \mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_{\ell(\nu)/b}, X(q,t))$ exhibits the biCSP, where

$$X(q,t) = \epsilon_n(q,t) \sum_{\lambda \vdash n} K_{\lambda,\mu}(q) K_{\lambda,\nu}(t).$$

As before, the $q = t = 1$ specialization of the above identity follows from applying RSK to the set X. The 'dual Cauchy' version of the previous result which follows was suggested by Dennis Stanton after the author's thesis defense.

For any $n > 0$, let $\delta_n(q,t) \in \mathbb{C}[q,t]$ be a polynomial whose evaluations $\delta_n(\zeta, \zeta')$ at n^{th} roots of unity ζ, ζ' with multiplicative orders $|\zeta| = k$ and $|\zeta'| = \ell$ satisfy

$$\delta_n(\zeta, \zeta') = \begin{cases} 1 & \text{if } \frac{n}{k} \text{ and } \frac{n}{\ell} \text{ are even,} \\ 1 & \text{if } k \text{ and } \ell \text{ are odd,} \\ -1 & \text{if } k, \ell \text{ are even and } \frac{n}{k}, \frac{n}{\ell} \text{ are odd,} \\ -1 & \text{if exactly one of } \frac{n}{k}, \frac{n}{\ell} \text{ is even and both } k, \ell \text{ are even,} \\ 1 & \text{if exactly one of } \frac{n}{k}, \frac{n}{\ell} \text{ is even and exactly one of } k, \ell \text{ is even.} \end{cases}$$

An explicit formula for a choice of $\delta_n(q,t)$ can be found using Fourier analysis on the direct product $\mathbb{Z}_n \times \mathbb{Z}_n$ of cyclic groups, but the formula so obtained is somewhat messy. It should be noted that if n is odd, one can take $\delta_n(q,t) \equiv 1$.

Theorem 1.4. *Let $\mu, \nu \models n$ be two compositions having cyclic symmetries of orders $a|\ell(\mu)$ and $b|\ell(\nu)$, respectively. Let X be the set of $\ell(\mu) \times \ell(\nu)$ $0,1$-matrices with row content μ and column content ν. The product of cyclic groups $\mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_{\ell(\nu)/b}$ acts on X by a-fold row rotation and b-fold column rotation.*

The triple $(X, \mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_{\ell(\nu)/b}, X(q,t))$ *exhibits the biCSP, where* $X(q,t) \in \mathbb{C}[q,t]$ *is*

$$X(q,t) = \delta_n(q,t) \sum_{\lambda \vdash n} K_{\lambda',\mu}(q) K_{\lambda,\nu}(t).$$

Example 1.3. Let us give an example to show why the factor of $\delta_n(q,t)$ is necessary in the statement of Theorem 1.4. Take $n = 2$, $\mu = \nu = (1,1)$, and $a = b = 1$. The set X can be identified with the two permutation matrices in S_2. The polynomial $X(q,t)$ is given by $X(q,t) = \delta_2(q,t)(q+t)$ and the evaluation $X(-1,-1) = \delta_2(-1,-1)(-1 - 1) = (-1)(-2) = 2$ would have been negative without the factor $\delta_2(-1,-1)$.

The $q = t = 1$ specialization of Theorem 1.4 follows from applying the *dual* RSK algorithm to the set X (see [17]). By the definition of $\delta_n(q,t)$, we have that $\delta_n(q,t) \in \{1,-1\}$ whenever q and t are specialized to n^{th} roots of unity. Therefore, omitting the factor $\delta_n(q,t)$ in Theorem 1.4 gives a biCSP 'up to sign'.

Remark 1.4. Given a finite set X acted on by a finite product $C \times C'$ of cyclic groups, it is always possible to find some polynomial $X(q,t)$ such that the triple $(X, C \times C', X(q,t))$ exhibits the biCSP. The interest in a biCSP lies in giving a polynomial $X(q,t)$ with a particularly nice form, either as an explicit product/sum formula or as a generating function for some pair of natural combinatorial statistics on the set X. We observe that, apart from the factors of $\epsilon_n(q,t)$ and $\delta_n(q,t)$, our polynomials $X(q,t)$ are nice in this latter sense.

Indeed, one can represent any $\mathbb{N}$-matrix A whose entries sum to n as a $2 \times n$ matrix $\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \end{pmatrix}$, where the biletters $\begin{pmatrix} a_{1i} \\ a_{2i} \end{pmatrix}$ are in lexicographical order and the biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ occurs with multiplicity equal to the (i,j)-entry of A. The word $w_A := a_{21} a_{22} \ldots a_{2n}$ given by the bottom row of this matrix is mapped to a pair $(P(w_A), Q(w_A))$ under RSK insertion, where $P(w_A)$ is a semistandard tableau of content equal to the column content vector of the matrix A and $Q(w_A)$ is a standard tableau having the same shape as $P(w_A)$. Using the fact that matrix transposition corresponds under RSK to swapping tableau (see [17]), one sees that for any compositions $\mu, \nu \models n$,

$$\sum_{\lambda \vdash n} K_{\lambda,\mu}(q) K_{\lambda,\nu}(t) = \sum_A q^{charge(w_{A^T})} t^{charge(w_A)},$$

where the sum ranges over the set of all $\mathbb{N}$-matrices A with row content μ and column content ν and A^T is the transpose of A. Similarly, one has that

$$\sum_{\lambda \vdash n} K_{\lambda,\mu}(q) K_{\lambda',\nu}(t) = \sum_A q^{charge(w_{A^T})} t^{charge(w_A)},$$

where the sum ranges over all $0,1$-matrices A with row content μ and column content ν. Thus, apart from the factors $\epsilon_n(q,t)$ and $\delta_n(q,t)$, the polynomials $X(q,t)$ appearing in the biCSPs of Theorems 1.3 and 1.4 are the generating functions for the pair of statistics $A \mapsto (charge(A^T), charge(A))$ on the set X.

2. Proofs of Theorems 1.3 and 1.4

The proofs of all of the above biCSPs will be 'semi-combinatorial', relying on enumerative results arising from RSK and the Stanton-White rim hook correspondence [18] as well as algebraic results from symmetric function theory due to Lascoux, Leclerc, and Thibon [5] [6]. Interestingly, although the formulas for $X(q,t)$ involve many Kostka-Foulkes polynomials, we shall not explicitly need any facts about the charge statistic on tableaux. Let Λ denote the ring of symmetric functions in $x = (x_1, x_2, \dots)$ having coefficients in $\mathbb{C}(q)$, where q is a formal indeterminate. The *Hall inner product* $\langle \cdot, \cdot \rangle$ on Λ defined by declaring the basis $\{s_\lambda\}$ of Schur functions to be orthonormal. For any composition $\mu \models n$, the *Hall-Littlewood symmetric function* $Q_\mu(x_1, x_2, \dots; q)$ is defined by

$$Q_\mu(x; q) = \sum_{\lambda \vdash n} K_{\lambda,\mu}(q) s_\lambda(x).$$

Specializing to $q = 1$, we have that $Q_\mu(x; 1) = \sum_{\lambda \vdash n} K_{\lambda,\mu} s_\lambda(x) = h_\mu(x)$, where h_μ is the complete homogeneous symmetric function indexed by μ. Thus, the Hall-Littlewood symmetric functions may be regarded as q-deformations of the homogeneous symmetric functions.

For any $k \geq 0$, define a linear operator ψ^k on Λ by

$$\psi^k(F(x_1, x_2, \dots)) = p_k \circ F = F(x_1^k, x_2^k, \dots).$$

Here $p_k \circ F$ is plethystic substitution. Following Lascoux, Leclerc, and Thibon [5], let ϕ_k be the adjoint of ψ^k with respect to the Hall inner product. That is, ϕ_k is defined by the condition $\langle F, \phi_k(G) \rangle = \langle \psi^k(F), G \rangle$ for any symmetric functions F, G. For any composition $\mu \models n$ and any positive integer k so that $\mu_i | k$ for all i, define the composition $\frac{1}{k}\mu \models n/k$ by $(\frac{1}{k}\mu)_i = \frac{\mu_i}{k}$. In addition, for any composition $\mu \models n$ with all part multiplicities divisible by k, let $\mu^{1/k}$ be any composition of n/k obtained by dividing all part multiplicities in μ by k. In particular, if all of the part multiplicities in μ are divisible by k, the power sum symmetric function $p_{\mu^{1/k}}$, the elementary symmetric function $e_{\mu^{1/k}}$, and the complete homogeneous symmetric function $h_{\mu^{1/k}}$ are all well-defined. Finally, let $\omega : \Lambda \to \Lambda$ be the involution on the ring of symmetric functions which interchanges elementary and homogeneous symmetric functions: $\omega(e_n) = h_n$.

Lemma 2.1. *The operators ψ^k and ϕ_k are both ring homomorphisms. Moreover, we have the following equalities of operators on Λ for any $k, \ell \geq 0$.*

1. $\psi^k \psi^\ell = \psi^{k\ell}$
2. $\phi_k \phi_\ell = \phi_{k\ell}$
3. $\phi_k \psi^k \phi_\ell = \phi_\ell \phi_k \psi^k$
4. $\phi_k \psi^k \psi^\ell = \psi^\ell \phi_k \psi^k$.

If in addition k and ℓ are relatively prime, we also have

5. $\phi_k \psi^\ell = \psi^\ell \psi_k$.

Proof. Clearly ψ^k is a ring map. Using the fact that ϕ_k is the adjoint to ψ^k, it's easy to check that we have the following formula for ϕ_k evaluated on power sum symmetric functions p_μ for $\mu \models n$:

$$\phi_k(p_\mu) = k^{\ell(\mu)} p_{\mu/k}.$$

Here we interpret the right hand side to be 0 if k does not divide every part of μ. From this formula it follows that ϕ_k is a ring homomorphism. Now relations 1 through 5 can be routinely checked on the generating set $\{p_n\}$ of Λ given by power sums. $\qquad\square$

Remarkably, the operators ψ^k can be used to evaluate certain specialized Hall-Littlewood polynomials. The specializations involve application of the raising operators ψ^k to homogeneous symmetric functions. Recall that a composition $\mu \models n$ is *strict* if all of its parts are strictly positive.

Theorem 2.2. *(Lascoux-Leclerc-Thibon [5, Theorems 3.1, 3.2]) Let $\mu \models n$ be a strict composition and for $k|n$ let ζ be a primitive k^{th} root of unity. Assume that all the part multiplicities in μ are divisible by k. Then, we have*

$$Q_\mu(x; \zeta) = (-1)^{(k-1)\frac{n}{k}} \psi^k(h_{\mu^{1/k}}).$$

The sign appearing in the above theorem is the reason why we needed the factor of $\epsilon_n(q, t)$ in Theorem 1.3.

Proof. (of Theorem 1.3) Without loss of generality we may assume that the compositions μ and ν are strict. Let ζ and ζ' be roots of unity of multiplicative orders k and ℓ, where each part of μ has multiplicity divisible by k and each part of ν has multiplicity divisible by ℓ. Temporarily ignoring the factor $\epsilon_n(q, t)$, we are interested in expressions like

$$\sum_{\lambda \vdash n} K_{\lambda,\mu}(\zeta) K_{\lambda,\nu}(\zeta').$$

This sum is equal to the Hall inner product

$$\langle Q_\mu(x; \zeta), Q_\nu(x; \zeta') \rangle$$

of specialized Hall-Littlewood functions.

By Theorem 2.2, the above inner product up to sign is equal to

$$\langle \psi^k(h_{\mu^{1/k}}), \psi^\ell(h_{\nu^{1/\ell}})\rangle.$$

Let m be the greatest common divisor of k and ℓ. Applying the operator calculus in Lemma 2.1, we see that the previous inner product is equal to

$$\langle \psi^m \phi_{\ell/m}(h_{\mu^{1/k}}), \psi^m \phi_{k/m}(h_{\nu^{1/\ell}})\rangle.$$

For $N \geq 0$, recall that the a-core of the partition (N) with a single part is empty if and only if $a|N$, in which case the a-quotient of (N) is the sequence $((\frac{N}{a}), \emptyset, \ldots, \emptyset)$ and the a-sign of (N) is 1. By a result of Littlewood [9] (See Formula 13 of [6]), the evaluation $\phi_a(h_N)$ is equal to $h_{N/a}$ if $a|N$ and 0 otherwise. From this and the fact that the ϕ operators are ring homomorphisms we get that the last inner product is equal to

$$\langle \psi^m(h_{\frac{m}{\ell}\mu^{1/k}}), \psi^m(h_{\frac{m}{k}\nu^{1/\ell}})\rangle,$$

where we interpret $h_{\frac{1}{a}\alpha}$ to be equal to zero if every part size in α is not divisible by a.

Formula 17 in [6] implies that

$$\psi^m(h_\alpha) = \sum_T \epsilon_m(T) s_{sh(T)},$$

where the sum ranges over all semistandard m-ribbon tableaux T having content α, $\epsilon_m(T)$ is the m-sign of the ribbon tableau T, and $\mathrm{sh}(T)$ is the shape of T. By the orthonormality of the Schur function basis, this implies that the inner product of interest

$$\langle \psi^m(h_{\frac{m}{\ell}\mu^{1/k}}), \psi^m(h_{\frac{m}{k}\nu^{1/\ell}})\rangle,$$

is equal to the number of ordered pairs (P, Q) of semistandard m-ribbon tableaux of the same shape where P has content $\frac{m}{\ell}\mu^{1/k}$ and Q has content $\frac{m}{k}\nu^{1/\ell}$. By the Stanton-White rim hook correspondence [18], this latter number is equal to the number of pairs (P, Q), where $P = (P_1, \ldots, P_m)$ and $Q = (Q_1, \ldots, Q_m)$ are m-tuples of semistandard tableaux with P_i having the same shape as Q_i for all i and such that P and Q have contents $\frac{m}{\ell}\mu^{1/k}$ and $\frac{m}{k}\nu^{1/\ell}$. By RSK insertion, this enumeration is again equal to the number of sequences $(A_1, \ldots, A_m)$ of $\frac{\ell(\mu)m}{\ell} \times \frac{\ell(\nu)m}{k}$ $\mathbb{N}$-matrices with row vectors summing to $\frac{m}{\ell}\mu^{1/k}$ and column vectors summing to $\frac{m}{k}\nu^{1/\ell}$. An analysis of fundamental domains under the action of row and column shifts shows that sequences of matrices as above are in bijection with $\ell(\mu) \times \ell(\nu)$ matrices A with row vector μ and column vector ν which are fixed under $\ell(\mu)/k$-fold row rotation and $\ell(\nu)/\ell$-fold column rotation. Up to sign, this proves Theorem 1.3.

To make sure the sign in Theorem 1.3 is correct, we need to show that the expression

$$\epsilon_n(\zeta, \zeta') \sum_{\lambda \vdash n} K_{\lambda,\mu}(\zeta) K_{\lambda,\nu}(\zeta')$$

is nonnegative. By Theorem 2.2 we need only check that

$$\epsilon_n(\zeta, \zeta') = (-1)^{((k-1)\frac{n}{k} + (\ell-1)\frac{n}{\ell})}.$$

This is a routine exercise.

$\square$

In order to prove Theorem 1.4 we will need a pair of commutativity results regarding the raising and lowering operators and the involution ω.

Lemma 2.3. *1. If k is odd, we have that $\omega\phi_k = \phi_k\omega$ and $\omega\psi^k = \psi^k\omega$.*
2. If $\ell > k$, we have the relation $\phi_{2^\ell}\omega\psi^{2^k} = \phi_{2^k}\omega\psi^{2^k}\phi_{2^{\ell-k}}$.
3. For any $\ell > 0$ and any composition μ such that $2^\ell || \mu|$, we have that $\phi_{2^\ell}\omega(h_\mu) = (-1)^{\frac{|\mu|}{2^\ell}}\omega\phi_{2^\ell}(h_\mu)$.

Proof. The operator relations 1 and 2 can both be checked on the power sum functions $\{p_n\}$ using the identity $\omega(p_n) = (-1)^{n-1}p_n$ together with the fact that ω, the raising operators, and the lowering operators are all ring maps.

For 3, we again appeal to Formula 13 of [6] to get that the evaluation $\phi_a(e_N)$ of the lowering operator ϕ_a on the elementary symmetric function e_N is equal to $(-1)^{\frac{N}{a}(a-1)}e_{\frac{N}{a}}$ if $a|N$ and 0 otherwise for any $a, N \geq 0$. Here we have used that the a-core of the partition (1^N) is empty if and only if $a|N$, in which case the a-quotient of (1^N) is $((1^{\frac{N}{a}}), \emptyset, \ldots, \emptyset)$ and the a-sign of (1^N) is $(-1)^{\frac{N}{a}}$. Using this evaluation, the desired identity can be proven using the fact that ϕ_{2^ℓ} and ω are ring maps. $\square$

Proof. (of Theorem 1.4) Without loss of generality, we may again assume that μ and ν are strict. Fix divisors $k|\frac{\ell(\mu)}{a}$ and $\ell|\frac{\ell(\nu)}{b}$, where each part of μ has multiplicity divisible by k and each part of ν has multiplicity divisible by ℓ. Let ζ and ζ' be roots of unity of multiplicative orders k and ℓ. Recalling that $\omega(s_\lambda) = s_{\lambda'}$, up to sign we were interested in expressions like

$$\sum_{\lambda \vdash n} K_{\lambda',\mu}(\zeta)K_{\lambda,\nu}(\zeta') = \langle \omega(Q_\mu(x;\zeta)), Q_\nu(x;\zeta') \rangle.$$

Applying Theorem 2.2 we see that, up to the sign $(-1)^{\frac{n}{k}(k-1) + \frac{n}{\ell}(\ell-1)}$, the above expression is equal to

$$\langle \omega\psi^k(h_{\mu^{1/k}}), \psi^\ell(h_{\nu^{1/\ell}}) \rangle.$$

Let m be the greatest common divisor of k and ℓ. We consider several cases depending on the parities of k and ℓ.

If k and ℓ are both odd, we can use Part 1 of Lemma 2.3 together with Lemma 2.1 to derive the identity

$$\langle \omega\psi^k(h_{\mu^{1/k}}), \psi^\ell(h_{\nu^{1/\ell}}) \rangle = \langle \omega\psi^m(h_{\frac{m}{\ell}\mu^{1/k}}), \psi^m(h_{\frac{m}{k}\nu^{1/\ell}}) \rangle.$$

If at least one of k and ℓ are even, since ω is involutive and an isometry with respect to the Hall inner product, we can assume that $\frac{k}{m}$ is odd. If both k and ℓ are even, we can use Parts 1 and 2 of Lemma 2.3 together with Lemma 2.1 to again show that

$$\langle \omega\psi^k(h_{\mu^{1/k}}), \psi^\ell(h_{\nu^{1/\ell}}) \rangle = \langle \omega\psi^m(h_{\frac{m}{\ell}\mu^{1/k}}), \psi^m(h_{\frac{m}{k}\nu^{1/\ell}}) \rangle.$$

However, if k is odd and ℓ is even, assuming as before the $\frac{k}{m}$ is odd, we use Parts 1 and 3 of Lemma 2.3 together with Lemma 2.1 to show that

$$\langle \omega\psi^k(h_{\mu^{1/k}}), \psi^\ell(h_{\nu^{1/\ell}}) \rangle = (-1)^{\frac{n}{\ell}} \langle \omega\psi^m(h_{\frac{m}{\ell}\mu^{1/k}}), \psi^m(h_{\frac{m}{k}\nu^{1/\ell}}) \rangle.$$

Regardless of the parities of k and ℓ, consider the Hall inner product

$$\langle \omega\psi^m(h_{\frac{m}{\ell}\mu^{1/k}}), \psi^m(h_{\frac{m}{k}\nu^{1/\ell}}) \rangle.$$

Formula 17 of [6] again allows us to perform the raising operator evaluations

$$\psi^m(h_\alpha) = \sum_T \epsilon_m(T) s_{sh(T)},$$

and we have that $\omega(s_\lambda) = s_{\lambda'}$ for any partition λ. In addition, given any partition λ with empty m-core, we have that the m-signs of λ and λ' are related by

$$\epsilon_m(\lambda) = (-1)^{(m-1)\frac{|\lambda|}{m}} \epsilon_m(\lambda').$$

Therefore, the Hall inner product of interest is equal to $(-1)^{(m-1)\frac{mn}{k\ell}}$ times the number of pairs (P, Q) of m-tuples $P = (P_1, \ldots, P_m)$ and $Q = (Q_1, \ldots, Q_m)$ of SSYT such that P has content $\frac{m}{\ell}\mu^{1/k}$, Q has content $\frac{m}{k}\nu^{1/\ell}$, and the shape of P_i is the *conjugate* of the shape of Q_i for all i. By the dual RSK algorithm, this is the number of m-tuples $(A_1, \ldots, A_m)$ of $\frac{\ell(\mu)m}{\ell} \times \frac{\ell(\nu)m}{k}$ $0, 1$-matrices with row vectors summing to $\frac{m}{\ell}\mu^{1/k}$ and column vectors summing to $\frac{m}{k}\nu^{1/\ell}$. Again, an elementary analysis of fundamental domains implies that such m-tuples are in bijective correspondence with the fixed point set of interest.

To check that the sign in Theorem 1.4 is correct, we check that the expression

$$\delta_n(\zeta, \zeta') \sum_{\lambda \vdash n} K_{\lambda', \mu}(\zeta) K_{\lambda, \nu}(\zeta')$$

is nonnegative. This is a routine case by case check depending on the parities of the numbers $k, \ell, \frac{n}{k}$, and $\frac{n}{\ell}$. $\square$

3. Proofs of Theorems 1.3 and 1.4 using Representation Theory

In this section we use results about the graded characters of DeConcini-Procesi modules [2] to sketch a representation theoretic proof of Theorems 1.3 and 1.4 up to modulus. The author is grateful to Victor Reiner for outlining this argument.

Given any integer $n > 0$, let X_n denote the variety of complete flags $0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ in $\mathbb{C}^n$ with $\dim V_i = i$. For any composition $\mu \models n$, let $u \in GL_n(\mathbb{C})$ be an $n \times n$ unipotent complex matrix with Jordan block decomposition given by μ. The subset $X_\mu \subseteq X_n$ of flags stabilized by the action of u is a subvariety of X_n and Springer [16] showed that the cohomology ring $H^*(X_\mu)$ carries a natural graded representation of S_n. It turns out that $H^i(X_\mu) = 0$ for odd i, so one defines a graded S_n-module $R_\mu := \bigoplus_{d \geq 0} R_\mu^d$, with $R_\mu^d := H^{2d}(X_\mu)$. The graded character $\mathrm{char}_q R_\mu$ is the symmetric function $\mathrm{char}_q R_\mu = \sum_{d \geq 0} q^d \mathrm{char} R_\mu^d$, where $\mathrm{char} R_\mu^d$ is the Frobenius character of the S_n-module R_μ^d.

Define the *modified Kostka-Foulkes polynomial* $\widetilde{K}_{\lambda,\mu}(q) \in \mathbb{N}[q]$ to be the generating function for the cocharge statistic on SSYT of shape λ and content μ:

$$\widetilde{K}_{\lambda,\mu}(q) := \sum_T q^{cocharge(T)}.$$

For μ a partition, the modified Kostka-Foulkes polynomials are related to the ordinary Kostka-Foulkes polynomials by $K_{\lambda,\mu}(q) = q^{n(\mu)} \widetilde{K}_{\lambda,\mu}(\frac{1}{q})$, where $n(\mu) = \sum_i (i-1)\mu_i$. The *modified Hall-Littlewood polynomials* $\widetilde{Q}_\mu(x; q)$ for $\mu \models n$ a composition are given by

$$\widetilde{Q}_\mu(x; q) := \sum_{\lambda \vdash n} \widetilde{K}_{\lambda,\mu}(q) s_\lambda(x).$$

Garsia and Procesi [3] proved that that the graded character of the module R_μ is equal to the modified Hall-Littlewood polynomial: $\mathrm{char}_q R_\mu = \widetilde{Q}_\mu(x; q)$. For any number $\ell > 0$, we can regard R_μ as a graded $S_n \times \mathbb{Z}_\ell$-module by letting the cyclic group $\mathbb{Z}_\ell$ act on the graded component R_μ^d by scaling by a factor of $e^{\frac{2\pi i d}{\ell}}$.

Suppose now that the composition $\mu \models n$ has cyclic symmetry of order $a | \ell(\mu)$. Let Y_μ be the set of all words $(w_1, \ldots, w_n)$ of length n and content μ. Then Y_μ is naturally a $S_n \times \mathbb{Z}_{\ell(\mu)/a}$-set, where the symmetric group S_n acts on the indices and the cyclic group $\mathbb{Z}_{\ell(\mu)/a}$ acts on the letter values, sending i to $i + a \bmod \ell(\mu)$. The vector space $\mathbb{C}[Y_\mu]$ is therefore a module over $S_n \times \mathbb{Z}_{\ell(\mu)/a}$ by linear extension. The following module isomorphism is a remarkable result of Morita and Nakajima.

Theorem 3.1. [11, Theorem 13] *Let $\mu \models n$ by a composition with cyclic symmetry $a | \ell(\mu)$. We have an isomorphism of $S_n \times \mathbb{Z}_{\ell(\mu)/a}$-modules*

$$R_\mu \cong \mathbb{C}[Y_\mu].$$

Morita and Nakajima proved this result by comparing the characters of the modules in question. Morita [10, Theorem 4] gave another character theoretic proof using the plethystic operators ψ^k and ϕ_k in Section 3 of this paper. Shoji [15] proved a generalization of this result to other types in which one replaces the variety X_μ with

the variety of Borel subgroups containing a unipotent element u of a simple algebraic group G over $\mathbb{C}$.

Now suppose that we are given two compositions $\mu, \nu \models n$. Elements of the product $Y_\mu \times Y_\nu$ can be thought of as $2 \times n$ matrices $\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \end{pmatrix}$ of letters such that the content of the word $a_{11}a_{12}\ldots a_{1n}$ is equal to μ and the content of the word $a_{21}a_{22}\ldots a_{2n}$ is equal to ν. The product $S_n \times S_n$ of symmetric groups acts on these matrices by independent permutation of the indices in the top and bottom rows. If in addition the compositions μ and ν have cyclic symmetries of orders $a | \ell(\mu)$ and $b | \ell(\nu)$, then the product set $Y_\mu \times Y_\nu$ carries an action of $S_n \times \mathbb{Z}_{\ell(\mu)/a} \times S_n \times \mathbb{Z}_{\ell(\nu)/b}$, where the cyclic groups act by modular addition on the letter values.

As a direct consequence of Theorem 3.1 we have that

$$R_\mu \otimes_{\mathbb{C}} R_\nu \cong \mathbb{C}[Y_\mu \times Y_\nu]$$

as modules over the group $S_n \times \mathbb{Z}_{\ell(\mu)/a} \times S_n \times \mathbb{Z}_{\ell(\nu)/b}$, where the module on the left hand side is bigraded. Considering the diagonal embedding $S_n \hookrightarrow S_n \times S_n$ given by $w \mapsto (w, w)$, restricting the above isomorphism yields an isomorphism

$$R_\mu \otimes_{\mathbb{C}} R_\nu \cong \mathbb{C}[Y_\mu \times Y_\nu]$$

of $S_n \times \mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_{\ell(\nu)/b}$-modules. Viewing elements of $Y_\mu \times Y_\nu$ as $2 \times n$ matrices, the action of S_n on the right hand side is induced by its natural action on matrix columns. Finally, if ϵ is any irreducible character of S_n, we may restrict the above isomorphism to its ϵ-isotypic component to get an isomorphism

$$[R_\mu \otimes_{\mathbb{C}} R_\nu]^\epsilon \cong \mathbb{C}[Y_\mu \times Y_\nu]^\epsilon$$

of modules over $\mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_{\ell(\nu)/b}$, where the exponential notation denotes taking iso-typic components and the left hand side is bigraded with the cyclic groups acting by scaling by a root of unity in each grade. At least up to modulus, Theorems 1.3 and 1.4 can be deduced from specializing ϵ to the trivial and sign characters of S_n, respectively.

Suppose first that $\epsilon = \mathrm{triv}$ is the trivial character of S_n. Then the isotypic component $\mathbb{C}[Y_\mu \times Y_\nu]^{\mathrm{triv}} = \mathbb{C}[Y_\mu \times Y_\nu]^{S_n}$ has a natural basis given by sums over orbits of the action of S_n on $Y_\mu \times Y_\nu$. Each of these orbits has a unique representative of the form $\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \end{pmatrix}$, where the biletters $\begin{pmatrix} a_{1i} \\ a_{2i} \end{pmatrix}$ are in lexicographical order. Such orbit representatives are in natural bijection with $\mathbb{N}$-matrices with row content μ and column content ν. It is easy to see that the action of the cyclic group product $\mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_{\ell(\nu)/b}$ is given by a-fold row and b-fold column rotation. Therefore, the number of fixed points of a group element $g \in \mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_{\ell(\nu)/b}$ in the action of Theorem 1.3 is equal to the trace of g on $\mathbb{C}[Y_\mu \times Y_\nu]^{\mathrm{triv}}$ and therefore is also equal to the trace of g on $[R_\mu \otimes_{\mathbb{C}} R_\nu]^{\mathrm{triv}}$. This latter trace can be identified with a polynomial evaluation

at roots of unity by considering the bigraded Hilbert series of $[R_\mu \otimes_{\mathbb{C}} R_\nu]^{\mathrm{triv}}$, proving Theorem 1.3 up to modulus.

To prove Theorem 1.4, we instead focus on the *sign* character $\epsilon = \mathrm{sgn}$ of the symmetric group S_n. The isotypic component $\mathbb{C}[Y_\mu \times Y_\nu]^{\mathrm{sgn}}$ has as basis the set of S_n-antisymmetrized sums over the element of the set $Y_\mu \times Y_\nu$. Representing elements of $Y_\mu \times Y_\nu$ as $2 \times n$ matrices, since antisymmetrization kills any matrix with repeated biletters, these basis elements are in natural bijection with $0, 1-$matrices of row content μ and column content ν. The cyclic group product $\mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_{\ell(\nu)/b}$ acts on this basis by a-fold row and b-fold column rotation, up to a plus or minus sign which arises from antisymmetrization and sorting biletters into lexicographical order. It is fairly easy to see that up to sign the number of fixed points of a group element $g \in \mathbb{Z}_{\ell(\mu)/a} \times \mathbb{Z}_{\ell(\nu)/b}$ is the absolute value of the trace of g on $\mathbb{C}[Y_\mu \times Y_\nu]^{\mathrm{sgn}}$. This latter number is also the absolute value of the trace of g on $[R_\mu \otimes_{\mathbb{C}} R_\nu]^{\mathrm{sgn}}$. This trace can be identified with a polynomial evaluation by considering bigraded Hilbert series as in the case of the trivial isotypic component. Up to modulus, this verifies Theorem 1.4.

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