EXTENSIONS OF THE SHI/ISH DUALITY

MICHELLE BODNAR, CHRISTIAN GAETZ, NITIN PRASAD, BRENDON RHOADES, AND BJÖRN H. WEHLIN

1. Introduction

The *N-Ish arrangement*, introduced by Abe, Suyama and Tsujie [1], is given by

$$\mathrm{Ish}(N) := \mathrm{Cox}(n) \cup \{x_1 - x_j = i : 2 \leq j \leq n, i \in N_j\},$$

where $N = (N_2, \ldots, N_n)$ is a tuple of finite subsets N_j of some characteristic 0 field $\mathbb{K}$.

In the case where $\mathbb{K} = \mathbb{R}$ and N is such that $N_2 \subseteq N_3 \subseteq \cdots \subseteq N_n$ and $0 \notin N_n$, we say that $\mathrm{Ish}(N)$ is a *nested Ish arrangement*. We choose to limit ourselves to this case in what follows.

It is known from [1] that the characteristic polynomial of $\mathrm{Ish}(N)$ is given by

$$\chi_{\mathrm{Ish}(N)}(p) = p(p - \#N_n - 1)(p - \#N_{n-1} - 2) \cdots (p - \#N_2 - n + 1). \tag{1.1}$$

We will see that in a special case, $\chi_{\mathrm{Ish}(N)}$ is equivalent to $\chi_{\mathrm{Shi}^{(m)}(n)}$, where $\chi_{\mathrm{Shi}^{(m)}(n)}$ is the characteristic polynomial of the *extended Shi arrangement*, given by Stanley in [6]. This arrangement is defined by

$$\mathrm{Shi}^{(m)}(n) = \{x_i - x_j = -m+1, -m+2, \ldots, m : 1 \leq i < j \leq n\}$$

and has characteristic polynomial

$$\chi_{\mathrm{Shi}^{(m)}(n)}(p) = p(p - mn)^{n-1}, \tag{1.2}$$

as shown in [REFERENCE?].

Hence, if N is chosen such that $\#N_\ell = (m-1)n + \ell - 1$ for all $\ell = 2, \ldots, n$,

$$\chi_{\mathrm{Ish}(N)}(p) = p(p - mn)^{n-1}. \tag{1.3}$$

If Equation (1.3) holds, we will say that $\mathrm{Ish}(N)$ is *$\mathrm{Shi}^{(m)}(n)$-compatible*, or *Shi-compatible* for short. We further say that N is Shi-compatible if $\mathrm{Ish}(N)$ is Shi-compatible.

It is seen from Zaslavsky's Theorem [7] that if $\mathrm{Ish}(N)$ is Shi-compatible, $\mathrm{Shi}^{(m)}(n)$ and $\mathrm{Ish}(N)$ share

- the number of regions $r\left(\mathrm{Shi}^{(m)}(n)\right) = r\left(\mathrm{Ish}(N)\right)$, and
- the number of bounded regions $b\left(\mathrm{Shi}^{(m)}(n)\right) = b\left(\mathrm{Ish}(N)\right)$.

Our present goal will be to establish an explicit bijection between the regions of $\mathrm{Shi}^{(m)}(n)$ and an arbitrary Shi-compatible $\mathrm{Ish}(N)$ arrangement, a problem posed by [REFERENCE?].

We begin by giving some definitions.

$$\begin{tikzcd}
\mathcal{R}_{\mathrm{Ish}(\widetilde{N})} \arrow[d] & & \mathcal{R}_{\mathrm{Shi}^{(m)}(n)} \\
\left\{\begin{matrix}\text{Nested Ish} \\ \text{ceiling diagrams}\end{matrix}\right\} \arrow[d, "\rho"] & & \left\{\begin{matrix}\text{Nonnesting Shi} \\ \text{Ceiling Diagrams}\end{matrix}\right\} \arrow[u] \\
\left\{\begin{matrix}\text{Maximal rook} \\ \text{placements on } B_n^m\end{matrix}\right\} \arrow[rr, "\gamma"] & & \mathrm{Park}_n^m \arrow[u, "\sigma"']
\end{tikzcd}$$

FIGURE 1. The steps in our bijection.

Definition 1.1. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. We define $\mathcal{R}_{\mathcal{A}}$ to be the collection of regions of $\mathcal{A}$. That is, the elements of $\mathcal{R}_{\mathcal{A}}$ are the connected components of

$$\mathbb{R}^n \setminus \bigcup_{H\in\mathcal{A}} H.$$

Definition 1.2. An m-parking function on n letters is a sequence $f = (a_1 a_2 \cdots a_n)$ such that if $b_1 \le b_2 \le \cdots \le b_n$ is the unique rearrangement of the terms in f in increasing order, we have $b_i \le 1 + m(i-1)$ for all $i = 1, \ldots, n$. We denote the set of all m-parking functions on n letters by Park_n^m.

Definition 1.3. By the *simple nest* $\widetilde{N}_n^m$, we mean

$$\widetilde{N}_n^m := \{[(m-1)n+1], [(m-1)n+2], \ldots, [mn-1]\}.$$

If m and n are clear from the context, we simply write $\widetilde{N}$.

Remark. We note that $\widetilde{N}_n^m$ is Shi-compatible for any choice of m, n.

Our bijection will follow Figure 1.

2. Nested Ish Ceiling Diagrams

Let C denote the dominant cone $x_1 \ge x_2 \ge \cdots \ge x_n$ in $\mathbb{R}^n$. We will define *nested Ish ceiling diagrams* which label the regions of $\mathrm{Ish}(\widetilde{N})$ analogously to the construction in Section 4.2 of [2]. Since $\mathrm{Cox}(n) \subset \mathrm{Ish}(\widetilde{N})$, for a given region R of $\mathrm{Ish}(\widetilde{N})$, all vectors $v = (v_1, \ldots, v_n) \in R$ satisfy

$$v_{\pi(1)} > v_{\pi(2)} > \cdots > v_{\pi(n)}$$

for some permutation $\pi \in \mathfrak{S}_n$. Notice that the hyperplane $x_1 - x_j = i$ intersects the cone πC containing R if and only if $x_1 > x_j$ on πC. Thus the set of $\mathrm{Ish}(\widetilde{N})$ hyperplanes that intersect πC are exactly

$$\Psi^+(\widetilde{N}, \pi) := \{x_1 - x_j = i : i \in \widetilde{N}_j \text{ and } \pi^{-1}(1) < \pi^{-1}(j)\}.$$

We define a partial order on $\Psi^+(\widetilde{N}, \pi)$ by the convention that $(x_1 - x_j = i) < (x_1 - x_{j'} = i')$ whenever either $i < i'$ or $\pi^{-1}(j') < \pi^{-1}(j)$. We now have the following generalization of Theorem 4.3 from [2].

Theorem 2.1. *There is a bijection between regions of* $\mathrm{Ish}(\widetilde{N})$ *in the cone* πC *and order filters in the poset* $\Psi^+(\widetilde{N}, \pi)$. *This map sends a region* R *to the set of hyperplanes in* $\mathrm{Ish}(\widetilde{N})$ *that are "above"* R.

Proof. The proof of Theorem 4.3 in [2] carries over exactly to the general case. □

We are now ready to define nested Ish ceiling diagrams, which will provide a combinatorial characterization of the order filters.

Definition 2.2. Let $\pi \in \mathfrak{S}_n$. We call a pair (π, ε) a *nested Ish ceiling diagram* if the vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ satisfies

(1) $0 \le \varepsilon_i < (m-1)n + \pi(i)$,
(2) $\varepsilon_i = 0$ unless $\pi^{-1}(1) < \pi^{-1}(i)$, and
(3) the nonzero entries of ε strictly increase from left to right.

Remark. This definition is exactly analogous to Definition 4.2 in [2], except that condition 1 now allows for more dots above the entries in the permutation, corresponding to the fact that the families of parallel hyperplanes in $\mathrm{Ish}(\widetilde{N})$ are larger than those in the standard Ish arrangement, $\mathrm{Ish}(n)$.

3. Rook Placements

A board B is a finite subset of the two-dimensional integer lattice $\mathbb{Z} \times \mathbb{Z}$. A rook placement on B is a placement of non-attacking rooks (that is, no row or column of B contains more than one rook). A rook placement on B is called maximal if it is impossible to place any more rooks on B and maintain the non-attacking condition. We will focus on rook placements on a particular family of boards.

Let B_n^m be the bottom-justified board with column heights from left to right given by

$$(n + m(n-1) + 1, n + m(n-1) + 2, \cdots, n + m(n-1) + n - 1).$$

We define the coordinates on B_n^m as follows.

$$B_n^m := \{(i, j) : 2 \le i \le n, 1 \le j \le n + m(n-1) + i - 1\}.$$

We will show a bijection between regions of $\mathrm{Ish}(\widetilde{N})$ and maximal rook placements on B_n^m.

Definition 3.1. Given a region R of $\mathrm{Ish}(\widetilde{N})$ with ceiling diagram (π, ε), we define a rook placement $\rho(R)$ onto B_n^m as follows. The rook on column i is placed on position

- $(i, \pi^{-1}(i))$ if $\varepsilon_{\pi^{-1}(i)} = 0$ (there are no dots atop i),
- $(i, n + \varepsilon_i)$ if $\varepsilon_{\pi^{-1}(i)} > 0$ (there are ε_i dots atop i).

It is easily seen that $\rho(R)$ is a maximal rook placement.

Definition 3.2. Let $w = w_1 w_2 \cdots w_n$ be a word with $w_i \in [mn]$ for all $i = 1, \ldots, n$ and such that each member of $[1, w_1]$ appears at least once as a letter in w. We then say w is an *m-rook word* and denote by Rook_n^m the set of all such words.

Lemma 3.3. *Let* g *be the action* $\mathbb{Z}_{mn+1}^n \curvearrowright [mn+1]^n$ *defined by*

$$1 \cdot (a_1, a_2, \ldots, a_n) \mapsto (a_1 + 1, a_2 + 1, \ldots, a_n + 1).$$

Then,

(1) *each orbit of g contains exactly one m-parking function,*
(2) *each orbit of g contains exactly one m-rook word.*

Proof.

(1) The proof will make use of the cycle lemma. Interpret a word $w = w_1...w_n$ in $[mn+1]^n$ as a parking preference function for n caravans of m cars seeking to park in mn spots, where w_i is the preferred spot of the i-th caravan. The caravans will park one at a time, with the cars in the i-th caravan parking in the first m available spots weakly after spot w_i. It is straightforward to show that the word w is an m-parking function if and only if all cars are able to find spots. To prove the lemma consider the lot arranged circularly, with an additional spot $mn+1$. The cars park in the same manner, but now all cars are able to park successfully. A word w is an m-parking function if and only if the single empty spot remaining at the end of the parking process is the $(mn+1)$-st spot. Since g acts by increasing spot preferences, each of the $mn+1$ words in the orbit of w will have a different empty spot at the end of the parking process, thus exactly one of them is an m-parking function.
(2) This proof follows easily from the proof of Lemma 10 in [4].

$\square$

Theorem 3.4. *There exists a bijection*

$$\gamma : \{\textit{Maximal rook placements on } B_n^m\} \to \mathrm{Park}_n^m\,.$$

Proof. We define a map γ from maximal rook placements on B_n^m to $m-$rook words. If P is a maximal rook placement on B_n^m, fire lasers rightward from every rook on P to the right side of the board. We then define a word $v_1 \dots v_n$ by the rule

$$v_i = \begin{cases} 1 & i = 1 \\ \# \left\{ \begin{matrix} \text{squares weakly below the rook in} \\ \text{Column } i \text{ that do not contain laser fire} \end{matrix} \right\} & 2 \le i \le n \end{cases} \tag{3.1}$$

Using Lemma 3.3, define $\gamma(P)$ to be the unique m-parking function in the $\mathbb{Z}_{mn+1}$-orbit of v. $\square$

4. Extended Shi Diagrams

Let R be a region of $\mathrm{Shi}^{(m)}(n)$. Then, for some multiset partition π of $M_n^m = \{1^m, \dots, n^m\}$, all points in R satisfy

$$x_{\pi(1)} + a(1) > x_{\pi(2)} + a(2) > \cdots > x_{\pi(mn)} + a(mn), \tag{4.1}$$

where $a(k) = m - 1 - \#\{k' : k' < k \text{ and } \pi(k') = \pi(k)\}$. Additionally, inequalities of the form $x_i - x_j < m$ or $x_i - x_j > m$, where $i < j$, may be required to specify R. Denote by $\mathfrak{R}_\pi$ the collection of regions of $\mathrm{Shi}^{(m)}(n)$ that satisfy (4.1). Let $\pi_k^{-1}(i)$ denote the k-th position where i occurs in π, counting right to left. Note that the $\pi_k^{-1}(i)$-th term in (4.1) is $x_i + (k-1)$ if (4.1) is written in order from greatest to least.

Lemma 4.1. *Let $\Phi^m(\pi)$ denote the set of hyperplanes that intersect the $\mathfrak{R}_\pi$. Then, $\Phi^m(\pi)$ is precisely the set of hyperplanes $x_i - x_j = m$ for which $\pi_1^{-1}(i) < \pi_m^{-1}(j)$ and where all $\pi(k)$ satisfying $\pi_1^{-1}(i) < k < \pi_m^{-1}(j)$ are distinct (that is, π has no repeated entries between positions $\pi_1^{-1}(i)$ and $\pi_m^{-1}(j)$).*

Proof. Any hyperplane of the form $x_i - x_j = a$ for $a < m$ determines π. Such hyperplanes form the boundaries of the $\mathfrak{R}_\pi$ and thus cannot intersect them. If $\pi_a^{-1}(j) > \pi_1^{-1}(i)$ for some $1 \leq a \leq m$, then $x_i - x_j < a - 1 \leq m - 1$, implying that $x_i - x_j = m$ does not intersect $\mathfrak{R}_\pi$. If there are distinct $\pi(k)$ satisfying $\pi_1^{-1}(i) < k < \pi_m^{-1}(j)$, we have for $1 \leq a \leq m-2$ that $x_i > x_k + a > x_k + a - 1 > x_j + (m-1)$, and it follows that $x_i - x_j > m$. □

We impose a partial order on $\Phi^m(\pi)$ by declaring that $x_{i'} - x_{j'} = m$ is less than or equal to $x_i - x_j = m$ if $i \leq i' < j' \leq j$. This ensures that if a region R lies below the first hyperplane, it also lies below the second, so that the collection of hyperplanes of the form $x_i - x_j = m$ above R form a down-closed set.

Theorem 4.2. *There is a bijection between the regions of* $\mathrm{Shi}^{(m)}(n)$ *in* $\mathfrak{R}_\pi$ *and the order ideals in* $\Phi^m(\pi)$*. The maximal elements of this ideal are the ceilings of the* $R \in \mathfrak{R}_\pi$ *of the form* $x_i - x_j = m$*.*

Proof. Let $R \in \mathfrak{R}_\pi$. We established above that the collection of hyperplanes above R form an order ideal in $\Phi^m(\pi)$. The map is injective since the hyperplanes above a region uniquely determine R in any particular R_π (note that π and the ceilings are all possible information about a particular region). The ceilings of R are the elements of the ideal which may be removed to obtain another ideal, which are by definition maximal. □

Corollary 4.3. *There is a bijection between the regions of* $\mathrm{Shi}^{(m)}(n)$ *in* $\mathfrak{R}_\pi$ *and the order filters in* $\Phi^m(n)$*. The minimal elements of this ideal are the floors of the* $R \in \mathfrak{R}_\pi$ *of the form* $x_i - x_j = m$*.*

Proof. The proof is identical to that of Theorem 4.2 with floors instead of ceilings and filters instead of ideals. □

Definition 4.4. Let $R \in \mathfrak{R}_\pi$. We associate with R a pair (π, C), where C is an order ideal in the poset $\Phi^m(\pi)$. The *extended Shi diagram* of R is obtained using the following procedure.

(1) Write π in one-line notation,
(2) For each $k \in [n]$, draw arcs from positions $\pi_a^{-1}(k)$ to $\pi_{a+1}^{-1}(k)$, for $a \in [m-1]$,
(3) For maximal $x_i - x_j = m$ in C, draw an arc from $\pi_1^{-1}(i)$ to $\pi_m^{-1}(j)$.

Remark. The resulting diagram is non-nesting. First, no nests are introduced in Step (2) of the construction, because if $\mathfrak{R}_\pi$ is to contain any regions, π must avoid the pattern *a_b_b_a*. Next, no arc drawn in Step (3) can induce a nest with any other arc in the same step because each hyperplane in $\Phi^m(\pi)$ intersects $\mathfrak{R}_\pi$. Also, no arc in Step (3) can induce a nest with any arc in a previous step as only maximal arcs are drawn.

Lemma 4.5. *Let* $R \in \mathcal{R}_{\mathrm{Shi}^{(m)}(n)}$*. The extended Shi ceiling diagram* (π, C) *for* R *has* d *connected components if and only if* R *has* d *degrees of freedom.*

Proof. Consider $v = (v_1, \ldots, v_n) \in \mathrm{Rec}(R)$. By definition, we have $x_{\pi(1)} + a(1) > \cdots > x_{\pi(mn)} + a(mn)$ and so, v must satisfy $v_{\pi(1)} \geq \cdots \geq v_{\pi(mn)}$. Any ideal $x_i - x_j = m$ in C forces $v_i = v_j$. Since these are exactly the constraints on v, we conclude that $\mathrm{Rec}(R)$ consists of all vectors of the form $(a_1, \ldots, a_n)$ where $a_i = a_j$ if i and j are in the same connected component of (π, C). The dimension of $\mathrm{Rec}(R)$ is therefore d. □

Definition 4.6. Define $\nu : \mathcal{R}_{\mathrm{Shi}^{(m)}(n)} \to \mathcal{R}_{\mathrm{Shi}^{(m)}(n)}$. In general, ν sends the region R in $\mathfrak{R}_\pi$ to R' in $\mathfrak{R}_\pi$ with the property that set of floors of R of the form $x_i - x_j = m$ is the set of ceilings of R' of the form $x_i - x_j = m$.

Remark. This procedure is well-defined since in any $\mathfrak{R}_\pi$, a region is determined by the associated ideal C. The ceilings or $R \in \mathfrak{R}_\pi$ are then the maximal elements of C, which uniquely determine C. Then, by Corollary 4.3, there is some region within the same $\mathfrak{R}_\pi$ determined by the order filter generated by these maximal elements.

Lemma 4.7. *The set of extended Shi ceiling diagrams is equivalent to the set of extended Shi floor diagrams given in Athanasiadis and Linusson* [3].

Proof. By construction, the extended Shi ceiling diagram of a region R is the extended floor diagram of $\nu(R)$. Since ν is a bijection, the result follows. □

5. A Bijection Preserving Degrees of Freedom

We will define a map $\sigma : \{\text{diagrams } D \text{ of } \mathrm{Shi}^{(m)}(n) \text{ regions}\} \to \mathrm{Park}_n^m$ which is very similar to a map defined by Athanasiadis and Linusson [3]. For each $1 \le i \le n$, let $1 \le w_i \le mn$ be the position of the leftmost number in the chain of arcs in D that contains i. Note that by our construction of the diagram D, all m copies of i do in fact appear in the same chain of arcs. We define $\sigma(D) = w_1 \cdots w_n$.

Proposition 5.1. *The words $\sigma(D)$ are m-parking functions, and σ is a bijection.*

Proof. By Lemma 4.7, the sets of extended Shi ceiling diagrams and extended Shi floor diagrams are equal. Hence, Theorem 3.5 in [3] holds for extended Shi ceiling diagrams. □

Proposition 5.2. *Let D be an extended Shi ceiling diagram with d connected components. Then, the non-decreasing rearrangement $a_1 \cdots a_n$ of $\sigma(D)$ satisfies $a_i = 1 + m(i-1)$ for exactly d values of i.*

Proof. Since the size of connected components in D must be divisible by m, suppose that a death ray fired between positions $(i-1)m$ and $(i-1)m+1$ separates connected components in D. The number in position $(i-1)m+1$ is contained in some arc, and so must be the left endpoint of a chain of arcs. Since the arcs ending in positions $1, 2, ..., (i-1)m$ determine $a_1, ..., a_{i-1}$, we see that $a_i = (i-1)m+1$. Since there must exist $d-1$ such death rays, and since $a_1 = 1 = (1-1)m+1$, we have that there are at least d values of i for which the desired equality holds. Conversely, this equality implies that only arcs beginning in positions $1, 2, ...(i-1)m$ could end in these positions, so that a death ray separating position $(i-1)m$ and $(i-1)m+1$ exists. □

6. Degrees of Freedom

In this section, we will develop enumerative results on the degrees of freedom statistics for the generalized Shi arrangement and the nested Ish arrangement. We begin with the Shi case.

Definition 6.1. [2, Section 2.1] Let π be a partition of $[n]$. We say the *type* of π is $(r_1, r_2, \dots, r_n)$, where r_i is the number of blocks of π with size i.

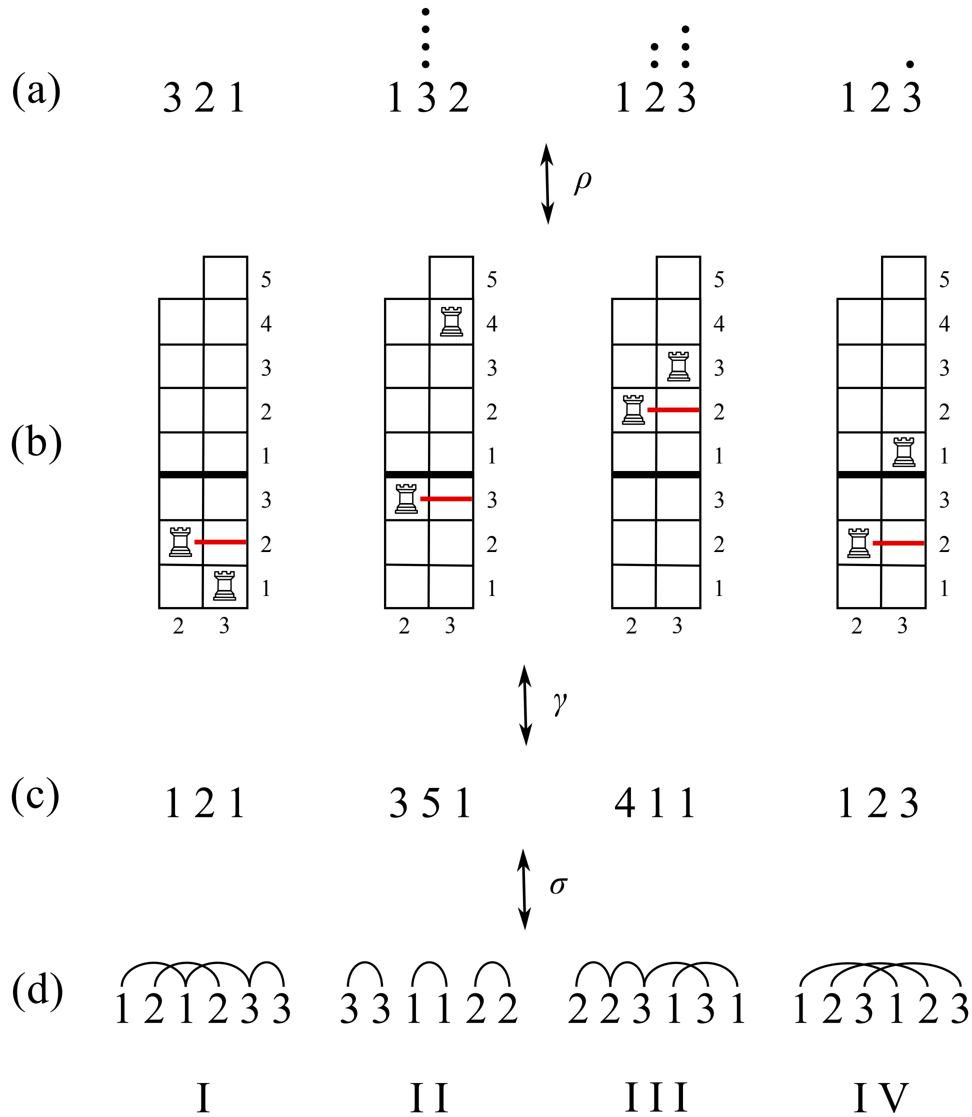

FIGURE 2. Four examples of our bijection from $\mathcal{R}_{\mathrm{Ish}(\tilde{N})}$ to $\mathcal{R}_{\mathrm{Shi}^{(m)}(n)}$ with $m = 2$ and $n = 3$. Here, *(a)* are Ish diagrams, *(b)* are maximal rook placements, *(c)* are m-parking functions, and *(d)* are Shi diagrams, all corresponding to the regions I, II, III, and IV in Figure 3.

For example, the partition $\{13/24/5678/9\}$ of $[9]$ has type $(1, 2, 0, 1, 0, 0, 0, 0, 0)$.

Since our construction of the extended Shi ceiling diagrams implies that the nonnesting multiset partition corresponding to an extended Shi region has all occurrences of the same number in the same block, we may define a corresponding type for these multisets. For a partition π^m of the multiset $M_n^m = \{1^m, \dots, n^m\}$,

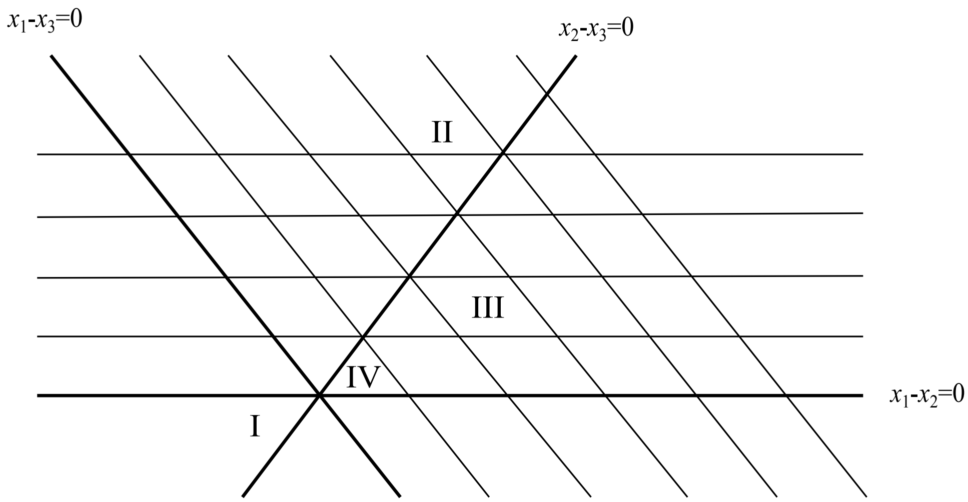

FIGURE 3. $\mathrm{Shi}^2(3)$-compatible standardly nested Ish arrangement with some representative regions marked.

we let π be the partition of $[n]$ such that i and j appear in the same block of π if the same is true for π^m. We say π is the *copartition* of π^m.

For instance, the multiset partition $\pi^2 = \{11/2244/33\}$ of M_3^2 has copartition $\pi = \{1/24/3\}$ of $[3]$, and hence type $\mathbf{r}_\pi = (2,1,0)$.

Theorem 6.2. *The number of regions of* $\mathrm{Shi}^{(m)}(n)$ *with* d *degrees of freedom is*

$$\sum_{k=d}^{n} \left\{ {n \atop k} \right\} \frac{d(mn-d-1)!(k-1)!}{(mn-k-1)!(k-d)!}, \tag{6.1}$$

where $\left\{ {n \atop k} \right\}$ *is the Stirling number of the second kind.*

Proof. Let $\Pi_{\mathbf{r}}$ be the collection of nonnesting set partitions of M_n^m of type $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ and take d to be fixed.

The number of nonnesting partitions of $[nm]$ with r_i $(i = 1, \ldots, n)$ blocks of size mi and d connected components is (see [5, Theorem 2.3, Part 2])

$$\frac{d(mn-d-1)!\,(\sum_{i=1}^n r_i - 1)!}{r_1!r_2!\cdots r_n!\,(mn - \sum_{i=1}^n r_i - 1)!\,(\sum_{i=1}^n r_i - d)!}. \tag{6.2}$$

There are $r_1!r_2!\cdots r_n!$ ways to fill in a copartition π of $[n]$ with type $\mathbf{r}$. Hence, if $\sum r_i = k$,

$$\#\Pi_{\mathbf{r}} = \frac{d(mn-d-1)!\,(k-1)!}{(mn-k-1)!\,(k-d)!}. \tag{6.3}$$

The number of k-block copartitions is given by $\left\{ {n \atop k} \right\}$ and each of these copartitions corresponds in turn to a nonnesting multiset partition. Hence, the total number of regions with d degrees of freedom is $\sum_{k=d}^n \left\{ {n \atop k} \right\} \#\Pi_{\mathbf{r}}$. □

Definition 6.3. The *Ish type* of a region R of $\mathrm{Ish}(\tilde{N})$ is a matrix

$$\begin{pmatrix} a_1 & \cdots & a_c \\ b_1 & \cdots & b_c \end{pmatrix}$$

where $b_1, ..., b_c$ are exactly the numbers in the Ish diagram of R which have dots, and a_i is the number of dots on b_i. Note that c is the number of ceilings of the region R.

Theorem 6.4. *The number of regions in* $\mathrm{Ish}(\tilde{N})$ *with* d *degrees of freedom is*

$$\sum_{c=1}^{n-1} \sum_{\binom{a_1 \ \cdots \ a_c}{b_1 \ \cdots \ b_c}} \frac{d(n-d-1)!(n-c-1)!}{(c-1)!(n-c-d)!}$$

Proof. Fix an Ish type $\begin{pmatrix} a_1 & \cdots & a_c \\ b_1 & \cdots & b_c \end{pmatrix}$. Since all dots must occur to the right of 1, any Ish diagram (π, ε) of this type must satisfy

$$\pi^{-1}(1) < \pi^{-1}(b_1) < \cdots < \pi^{-1}(b_c)$$

Since we are considering regions with d degrees of freedom, we know that $\pi^{-1}(b_c) - \pi^{-1}(1) = n - d$. Thus there are d ways to place the symbols 1 and b_c, and $\binom{n-d-1}{c-1}$ ways to place the symbols $b_2, ..., b_{c-1}$ left to right between them. Finally there are $(n-c-1)!$ ways to place the remaining symbols. Since we have fixed the Ish type (and therefore the vector ε) this completely determines the Ish diagram. $\square$

7. To Do

Problem 7.1. *There exists a bijection between the regions of* $\mathrm{Ish}(N)$ *and* $\mathrm{Ish}(\widetilde{N}_n^m)$ *for arbitrary* $\mathrm{Shi}^{(m)}(n)$*-compatible* N *that preserves degrees of freedom.*

References

1. Takuro Abe, Daisuke Suyama, and Shuhei Tsujie, *The freeness of Ish arrangements*, arXiv preprint arXiv:1410.2084 (2014).
2. Drew Armstrong and Brendon Rhoades, *The Shi arrangement and the Ish arrangement*, Transactions of the American Mathematical Society **364** (2012), no. 3, 1509–1528.
3. Christos A Athanasiadis and Svante Linusson, *A simple bijection for the regions of the Shi arrangement of hyperplanes*, Discrete mathematics **204** (1999), no. 1, 27–39.
4. Emily Leven, Brendon Rhoades, and Andrew Timothy Wilson, *Bijections for the Shi and the Ish arrangements*, European Journal of Combinatorics **39** (2014), no. 1, 1–23.
5. Brendon Rhoades, *Enumeration of connected Catalan objects by type*, European Journal of Combinatorics **32** (2011), no. 2, 330–338.
6. Richard P Stanley, *Hyperplane arrangements, parking functions and tree inversions*, Mathematical Essays in Honor of Gian-Carlo Rota, Springer, 1998, pp. 359–375.
7. Thomas Zaslavsky, *Facing up to arrangements: face-count formulas for partitions of space by hyperplanes*, 1975.

Department of Mathematics, University of California, San Diego, 9500 Gilman Dr., La Jolla, CA 92093