TEMPERLEY-LIEB IMMANANTS

BRENDON RHOADES AND MARK SKANDERA

Abstract. We use the Temperley-Lieb algebra to define a family of totally nonnegative polynomials of the form $\sum_{\sigma \in S_n} f(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$. The cone generated by these polynomials contains all totally nonnegative polynomials of the form $\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x)$, where $\Delta_{I,I'}(x), \ldots, \Delta_{K,K'}(x)$ are matrix minors. We also give new conditions on the eight sets $I, \ldots, K'$ which characterize differences of products of minors which are totally nonnegative.

1. Introduction

A real matrix is called *totally nonnegative* (TNN) if the determinant of each of its square submatrices is nonnegative. Such matrices appear in many areas of mathematics and the concept of total nonnegativity has been generalized to apply not only to matrices, but also to other mathematical objects. (See e.g. [15] and references there.) In particular, a polynomial $p(x)$ in n^2 variables $x = (x_{1,1}, \ldots, x_{n,n})$ is called *totally nonnegative* if it satisfies

$$p(A) \underset{\text{def}}{=} p(a_{1,1}, \ldots, a_{n,n}) \geq 0 \tag{1.1}$$

for every $n \times n$ TNN matrix $A = (a_{i,j})$. Obvious examples are the $n \times n$ determinant and the $k \times k$ *minors*, i.e. the determinants of $k \times k$ submatrices.

In light of (1.1) it will be convenient to consider $x = (x_{i,j})$ to be a *matrix* of n^2 variables. For each pair (I, I') of subsets of $[n] = \{1, \ldots, n\}$ we will define the (I, I') *submatrix* of x and (I, I') *minor* of x to be

$$\begin{aligned} x_{I,I'} &\underset{\text{def}}{=} (x_{i,j})_{i \in I, j \in I'}, \\ \Delta_{I,I'}(x) &\underset{\text{def}}{=} \det(x_{I,I'}). \end{aligned} \tag{1.2}$$

Thus $\Delta_{I,I'}(x)$ is the determinant of the submatrix of x corresponding to rows I and columns I'. In writing $\Delta_{I,I'}(x)$ we will tacitly assume that we have $|I| = |I'|$.

Some recent interest in TNN polynomials concerns a collection of polynomials arising in the study of canonical bases of quantum groups [2]. While this collection, known as the *dual canonical basis* of type A_{n-1}, currently has no simple description, Lusztig [29] has proved that it consists entirely of TNN polynomials. Berenstein,

Date: March 3, 2005.

Gelfand, and Zelevinsky [3, 16] have developed machinery to enumerate the dual canonical basis elements for small n, and further investigation suggests that these polynomials are expressable as subtraction-free Laurent expressions in matrix minors [12]. Progress on the problem of describing the dual canonical basis is obstructed somewhat by the scarcity of nontrivial families of polynomials which are known to be TNN.

Providing examples of such families of TNN polynomials, several authors have studied polynomials constructed from functions $f : S_n \rightarrow \mathbb{R}$ by

$$\mathrm{Imm}_f(x) = \sum_{\sigma \in S_n} f(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}. \tag{1.3}$$

For a fixed function f, this polynomial is called the the f-*immanant* in [34, Sec. 3]. We will refer to all elements of $\mathrm{span}_{\mathbb{R}}\{x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \,|\, \sigma \in S_n\}$ as *immanants*.

Stembridge proved the total nonnegativity of the immanants $\mathrm{Imm}_{\chi^\lambda}(x)$ constructed from the irreducible characters $\chi^\lambda : S_n \rightarrow \mathbb{R}$ of S_n [36, Cor. 3.3]. (See also [26].) These immanants are usually abbreviated $\mathrm{Imm}_\lambda(x)$,

$$\mathrm{Imm}_\lambda(x) = \sum_{\sigma \in S_n} \chi^\lambda(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}. \tag{1.4}$$

Stembridge also proved the stronger result [36, Cor. 3.4] that the immanants

$$\mathrm{Imm}_\lambda(x) - \deg(\chi^\lambda) \det(x) \tag{1.5}$$

are TNN, and posed several related questions which remain open [37]. (See also [19, 20, 21, 35].)

Discovering a very different family of TNN immanants, Fallat et. al. [10, Thm. 4.6] characterized all TNN immanants of the form

$$\Delta_{J,J}(x)\Delta_{\overline{J},\overline{J}}(x) - \Delta_{I,I}(x)\Delta_{\overline{I},\overline{I}}(x),$$

where $\overline{I} = [n] \smallsetminus I$, $\overline{J} = [n] \smallsetminus J$. This result was later strengthened [31, Thm. 3.2] to include products of nonprincipal minors

$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x). \tag{1.6}$$

The results in both papers [10], [31] generalize to polynomials which are not immanants.

While the coefficients in (1.6) do not correspond to *class* functions on S_n as do those in (1.4), certain quotients of the symmetric group algebra provide a link between the two families of immanants. Indeed the methods in [36] imply that such quotients provide important information about TNN polynomials in general. In this paper, we use such a quotient which is isomorphic to the Temperley-Lieb algebra $T_n(2)$ to define a family of functions

$$\{f_\tau : S_n \rightarrow \mathbb{R} \,|\, \tau \text{ a basis element of } T_n(2)\}$$

and a family of corresponding TNN immanants

$$\mathrm{Imm}_\tau(x) \underset{\mathrm{def}}{=} \sum_{\sigma \in S_n} f_\tau(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$$

whose cone contains all immanants in the family (1.6). We begin in Section 2 with some of the well-known combinatorics of total nonnegativity. In Section 3 we define Temperley-Lieb immanants, which generalize the determinant, and give a combinatorial proof that these immanants are TNN. We also present some of their basic properties, including generalizations of determinantal properties. In Sections 4 and 5 we give improved criteria for deciding the total nonnegativity of immanants of the form (1.6) and of related polynomials. Finally in Section 6 we discuss connections between the Temperley-Lieb immanants and symmetric functions.

2. Total nonnegativity and planar networks

It is possible to prove that some polynomials $p(x)$ are TNN by providing a combinatorial interpretation for $p(A)$ whenever A is a TNN matrix. Typically such a combinatorial interpretation involves a class of directed graphs which we call planar networks. After reviewing the basis facts results concerning planar networks, we will state and prove some results concerning special planar networks which we will call wiring diagrams and generalized wiring diagrams.

We define a *planar network of order* n to be an acyclic planar directed multigraph $G = (V, E)$ in which $2n$ boundary vertices are labeled counterclockwise as $v_1, \dots, v_n$, $v'_n, \dots, v'_1$. We call the vertices $v_1, \dots, v_n$ *sources* and the vertices $v'_1, \dots, v'_n$ *sinks.* Each edge $e \in E$ is weighted by a complex number $\omega(e)$, and we define the weight of a set F of edges to be the product of weights of edges in F,

$$\omega(F) = \prod_{e \in F} \omega(e). \tag{2.1}$$

More generally, we define the weight of a *multiset* of edges to be the analogous product in which weights of edges may appear with multiplicities greater than one. If $m = (m_e)_{e \in F}$ is a vector of multiplicities which defines a multiset of edges in F, we denote the weight of this multiset by $\omega(F, m)$.

Given a planar network G of order n, we define a subgraph H of G to be a *planar subnetwork* of G if it is a planar network whose sources and sinks are precisely those of G. We will economize notation by writing $H \subset G$ to denote that H is a planar subnetwork of G.

We define the *path matrix* $A = (a_{i,j})$ of a planar network G by letting $a_{i,j}$ be the sum of path weights

$$a_{i,j} = \sum_F \omega(F),$$

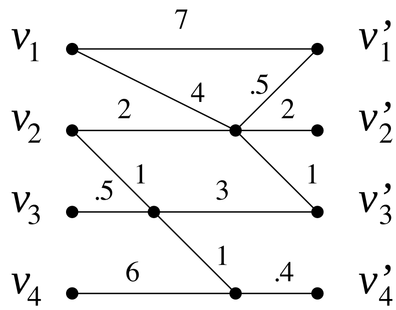


FIGURE 2.1. A planar network

over all paths F from source v_i to sink v'_j. The reader may verify that the path matrix of the planar network in Figure 2.1 is

$$\begin{bmatrix} 9 & 8 & 4 & 0 \\ 1 & 4 & 5 & .4 \\ 0 & 0 & 3 & .2 \\ 0 & 0 & 0 & 2.4 \end{bmatrix}, \tag{2.2}$$

and that this matrix is TNN. In figures we will assume that all edges are directed from left to right.

If $\pi = (\pi_1, \ldots, \pi_k)$ is a family of k paths in a planar network, we define the weight of π to be the weight of the multiset of edges contained in the paths,

$$\omega(\pi) = \omega(\pi_1) \cdots \omega(\pi_k).$$

Thus in Figure 2.1, the unique pair $\pi = (\pi_1, \pi_2)$ of paths from source v_2 to sinks (v'_1, v'_2) has weight $(2 \cdot .5)(2 \cdot 2) = 4$.

The following famous theorem of Lindström and others [1] [6] [7] [23] [27] [28] explains the connection between planar networks and TNN matrices. (See also [15].)

Theorem 2.1. *An $n \times n$ matrix A is totally nonnegative if and only if it is the path matrix of a planar network G of order n in which all edge weights are nonnegative real numbers. Furthermore, for any k-element subsets I, I' of $[n]$,*

$$I = \{i_1, \ldots, i_k\}, \qquad i_1 < \cdots < i_k,$$
$$I' = \{i'_1, \ldots, i'_k\}, \qquad i'_1 < \cdots < i'_k,$$

the (I, I') minor of A has the combinatorial interpretation

$$\Delta_{I,I'}(A) = \sum_{\pi} \omega(\pi),$$

where the sum is over all families $\pi = (\pi_1, \ldots, \pi_k)$ of k nonintersecting paths in G which satisfy

(1) π_j *is a path from* v_{i_j} *to* $v'_{i'_j}$.
(2) π_j *and* π_ℓ *do not intersect for* $j \neq \ell$.

The reader may verify that the graph in Figure 2.1 has three nonintersecting path families from $\{v_1, v_2\}$ to $\{v'_1, v'_3\}$, and that these families have weights 14, 21, and 6. Correspondingly, the $(12, 13)$ minor of the path matrix (2.2) is $41 = 14 + 21 + 6$.

Given two planar networks G, G' of order n, we define their *concatenation* GG' to be the planar network created by superimposing the sinks of G upon the sources of G'. It is easy to see that if A and A' are the path matrices of G and G', then AA' is the path matrix of GG'. Using this fact, one may generalize the combinatorial interpretation in Theorem 2.1 to arbitrary complex $n \times n$ matrices.

Observation 2.2. *Every complex* $n \times n$ *matrix* A *is the path matrix of a planar network* G. *Furthermore, the minor* $\Delta_{I,I'}(A)\Delta_{\overline{I},\overline{I'}}(A)$ *has the same interpretation as in Theorem 2.1.*

Proof. Let A be an $n \times n$ matrix. We may factor A as

$$A = MJM^{-1}$$

where J is a block-diagonal matrix composed of Jordan blocks. Since M and M^{-1} belong to $GL_n(\mathbb{R})$, each factors as a product of matrices of the forms $I + cE_{i,i+1}$, $I + cE_{i+1,i}$, $I + (c-1)E_{i,i}$, where $E_{i,j}$ is the $n \times n$ matrix whose unique nonzero entry is a 1 in position i, j, and c is a complex number. It is easy to see that J factors similarly. These elementary factors are the weighted path matrices of certain planar networks $G_1, \ldots, G_r$ and their product is therefore the path matrix of the concatenation of planar networks $G_1 \cdots G_r$. (See e.g. [14, Sec. 4.2].) $\square$

Immediate consequences of Theorem 2.1 (or Observation 2.2) are combinatorial interpretations for certain TNN immanants. Fix a planar network G and its path matrix A. The application of the monomial $x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ to A has the interpretation

$$a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} = \sum_{\pi} \omega(\pi),$$

where the sum is over path families $\pi = (\pi_1, \ldots, \pi_n)$ in G in which π_i is a path from v_i to $v'_{\sigma(i)}$. We will say that such a path family has *type* σ. Also, by choosing $I = I' = [n]$ in Theorem 2.1, we have that

$$\det(A) = \sum_{H \subset G} \omega(H),$$

where the sum is over all planar subnetworks H of G which are unions of n nonintersecting paths. With a bit more work, one can derive a similar combinatorial interpretation for the TNN immanants (1.6),

$$\Delta_{J,J'}(A)\Delta_{\overline{J},\overline{J'}}(A) - \Delta_{I,I'}(A)\Delta_{\overline{I},\overline{I'}}(A) = \sum_{H\in\mathcal{H}} c_H\omega(H),$$

for appropriate collections $\mathcal{H}$ of planar subnetworks which depend on the index sets I, J, etc., and for appropriate constants c_H. (See [31, Cor. 3.3].) No analogous combinatorial interpretation for the TNN immanants (1.4) and (1.5) is known.

To construct more TNN polynomials, we shall examine the planar networks of order n which are unions of n paths. We will say that a path family π *covers* a planar network $H = (V, E)$ if every edge in E belongs to a path in π. Since two different path families may cover the edges of a planar network with different multiplicities, we introduce the following notation. Given a sequence $m = (m_e)_{e\in E}$ of positive multiplicities, we define the group algebra element

$$\beta(H, m) = \sum_{\pi} \text{type}(\pi), \tag{2.3}$$

where the sum is over all path families π which cover the edges of H with multiplicities m. This will serve as an unweighted path generating function for the pair (H, m). Similarly, we define

$$\beta(H) = \sum_{m\in[n]^{|E|}} \beta(H, m). \tag{2.4}$$

Certain planar networks which appear often in conjunction with the symmetric group are called *wiring diagrams.* To the *adjacent transpositions* $s_1, \ldots, s_{n-1}$ we associate the planar networks $H_1, \ldots, H_{n-1}$ in Figure 2.2.

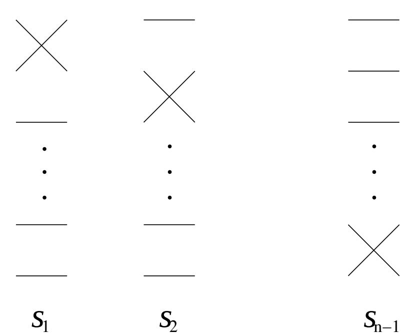
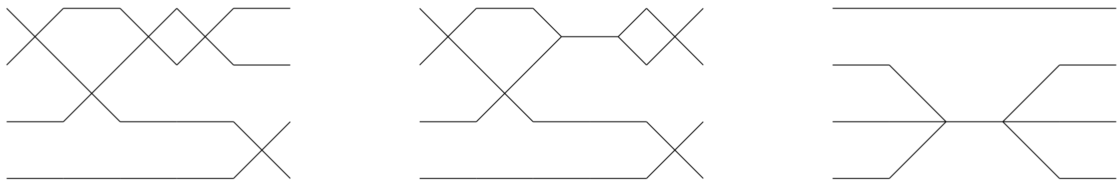


FIGURE 2.2

We then define the wiring diagram of an expression

$$s_{i_1} \cdots s_{i_k} \tag{2.5}$$



(a) (b) (c)

Figure 2.3. A wiring diagram, a generalized wiring diagram, and another planar network.

(not necessarily reduced) to be the concatenation $H_{i_1} \cdots H_{i_k}$. In figures we will omit sources, sinks, and intermediate vertices of wiring diagrams and related planar networks when there is no danger of confusion. Figure 2.3(a) shows the wiring diagram associated to the expression $s_1 s_2 s_1 s_1 s_3$ (in S_4). The reader can verify that the path generating function of this wiring diagram is

$$2(2 + 2s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1)(1 + s_3).$$

We state without proof the following simple properties of wiring diagrams.

Observation 2.3. *Let (2.5) be an expression for $\sigma \in S_n$, and let H be the corresponding wiring diagram. Then H has the following properties.*

1. *There is a unique path family which covers H, in which no two paths cross, and in which no three paths share a vertex.*
2. *Any path family which covers H covers each edge exactly once.*
3. *The path generating function for H is $(1 + s_{i_1}) \cdots (1 + s_{i_k})$.*

Closely related to wiring diagrams are planar networks which are unions of n paths, no three of which share a vertex. We will call such planar networks *generalized wiring diagrams* (of order n). Figure 2.3 (b) shows a generalized wiring diagram of order 4.

The properties of wiring diagrams listed in Observation 2.3 generalize in a straightforward way to generalized wiring diagrams. It is easy to see that statement 1 of the observation remains true for generalized wiring diagrams. Statement 2 generalizes as follows.

Lemma 2.4. *Let H be a generalized wiring diagram. If a path family π and a path family π' cover the edges of H with multiplicity sequences m and m', respectively, then $m = m'$.*

Proof. Let $H = (V, E)$. Since H is a generalized wiring diagram, there exists a path family $\pi'' = (\pi''_1, \ldots, \pi''_n)$ which covers H with multiplicity sequence m'' and each component of this sequence is 1 or 2. Since the definition of planar networks requires that H have n distinct sources, we have

$$m_e = m'_e = m''_e$$

for each edge e incident upon a source.

Partially order the vertices of H by the edges E (i.e. by defining $u < v$ if it is possible to follow directed edges in E from u to v). Assume that $m \neq m''$, and choose a vertex u_1 such that m and m'' agree on all edges terminating at u_1 but disagree on at least one edge originating at u_1, and such that no other vertex $u_0 < u_1$ has this property. Since we have

$$\sum m''_e = \sum m_e,$$

where the sums are over all edges originating at u_1, there must exist vertices u_2, u_3 which satisfy

$$m''_{(u_1,u_2)} > m_{(u_1,u_2)} > 1,$$
$$m_{(u_1,u_3)} > m''_{(u_1,u_3)} > 1,$$

contradicting the fact that at most two paths in π'' pass through u_1. Thus we must have $m = m''$ and $m' = m''$. □

Since we will always consider the weight of a generalized diagram H in conjunction with the fixed multiplicity vector m described in Lemma 2.4, we will omit this vector from our notation,

$$\omega(H) \underset{\text{def}}{=} \omega(H, m).$$

To generalize statement 3 from Observation 2.3 we will show that the path generating functions of generalized wiring diagrams factor just as those of wiring diagrams. On the other hand, the path generating functions of arbitrary unions of n paths never factor this way. For instance, Figure 2.3 (c) shows a planar network whose path generating function is $1 + s_2 + s_3 + s_2s_3 + s_3s_2 + s_2s_3s_2$. We will denote by $z_{[i,j]}$ the element of $\mathbb{Z}[S_n]$ which is a sum of permutations in the subgroup of S_n generated by $s_i, \ldots, s_{j-1}$. Thus the path generating function of the planar network in Figure 2.3 (c) is $z_{[2,4]}$.

Lemma 2.5. *Let $H = (V, E)$ be a planar network which is a union of n paths. Then H is a generalized wiring diagram if and only if $\beta(H)$ factors as*

$$\beta(H) = (1 + s_{i_1}) \cdots (1 + s_{i_r}),$$

for some generators $s_{i_1}, \ldots, s_{i_r}$ of S_n. In particular, if H is not a generalized wiring diagram then $\beta(H)$ is equal to a nonnegative linear combination of terms of the form

$$z_{[i_1,j_1]} \cdots z_{[i_r,j_r]}, \tag{2.6}$$

and in each such term, at least one pair of indices satisfies $i_k \leq j_k - 2$.

Proof. ($\Rightarrow$) Suppose that $H = (V, E)$ is a generalized wiring diagram, and let m be the unique vector of multiplicities defined as in Lemma 2.4. If $m_e = 2$ for any edge e, then we can contract that edge to obtain a new generalized wiring diagram H' which satisfies

$$\beta(H') = \beta(H).$$

It will suffice therefore to consider the case that $m_e = 1$ for all edges e. Now let $\pi = (\pi_1, \dots, \pi_n)$ be the unique noncrossing path family which covers H, and consider the vertices which belong to two paths of π. Partially order these vertices as in the proof of Lemma 2.4, let $u_1, \dots, u_r$ be a linear extension of this partial order, and define the sequence $(i_1, \dots, i_r)$ by $i_k = c$ if paths π_c, π_{c+1} pass through vertex u_k. Now observe that the wiring diagram $G = H_{i_1} \cdots H_{i_r}$, (where $H_{i_1}, \dots, H_{i_r}$ are defined as in Figure 2.2) satisfies $\beta(G) = \beta(H)$. By Observation 2.3 we have the desired factorization of $\beta(H)$.

($\Leftarrow$) Suppose that $H = (V, E)$ is not a generalized wiring diagram. Fix one vector m which appears in the sum

$$\beta(H) = \sum_m \beta(H, m)$$

and suppose that for some edge e we have $m_e \geq 2$. Replacing the corresponding component of m with m_e new components equal to one, and replacing e with m_e new edges, we obtain a mulitplicity vector m' and a graph H' which satisfsy

$$\beta(H', m') = m_e! \beta(H, m).$$

It is easy to see that H' is not a generalized wiring diagram.

Repeating this process, we eventually obtain a planar network H'' such that $\beta(H, m)$ is equal to a nonnegative multiple of $\beta(H'', m'')$, and H'' is a union of n paths, no two of which share an edge. Let $\pi = (\pi_1, \dots, \pi_n)$ be the unique noncrossing path family which covers H'', and consider the vertices which belong to at least two paths of π. Partially order these vertices as before, let $u_1, \dots, u_r$ be a linear extension of this partial order, and define the pairs $(i_1, j_1), \dots, (i_r, j_r)$ by

$$(i_k, j_k) = (c, d),$$

where the paths which pass through vertex u_k are $\pi_c, \dots, \pi_d$. Then we have

$$\beta(H'', m'') = z_{[i_1,j_1]} \cdots z_{[i_r,j_r]},$$

$\beta(H, m)$ is equal to a nonnegative multiple of this, and $\beta(H)$ is equal to a sum of such terms. Since H'' is not a generalized wiring diagram, one of the vertices u_k belongs to at least three paths in π, and we have $j_k - i_k \geq 3$. □

In studying generalized wiring diagrams, we will make use of the wiring diagrams corresponding to certain reduced expressions for 321-avoiding permutations. (A permutation $\sigma = \sigma_1 \cdots \sigma_n$ is said to be 321-*avoiding* if there are no indices $i < j < k$

for which we have $\sigma_i > \sigma_j > \sigma_k$.) In particular, a well-known property of these permutations is that each has a unique reduced expression of the form

$$\sigma = (s_{a_1} s_{a_1+1} \cdots s_{b_1})(s_{a_2} s_{a_2+1} \cdots s_{b_2}) \cdots (s_{a_r} s_{a_r+1} \cdots s_{b_r}), \tag{2.7}$$

where $[a_1, b_1], \ldots, [a_r, b_r]$ are intervals which satisfy

$$\begin{aligned} a_1 > a_2 > \cdots > a_r, \\ b_1 > b_2 > \cdots > b_r. \end{aligned}$$

(See e.g. [5], [18, Sec. 2.1].)

Lemma 2.6. *Let G be the wiring diagram corresponding to a reduced expression of the form (2.7) for a 321-avoiding permutation σ. The only planar subnetwork of G which is a generalized wiring diagram of order n is G itself.*

Proof. Each of the transpositions in (2.7) corresponds to a vertex of G having indegree and outdegree two. Label these vertices from left to right as $u_1, \ldots, u_\ell$ and let $\pi = (\pi_1, \ldots, \pi_n)$ be the unique noncrossing path family which covers G. Thus the vertices shared by a pair of paths (π_i, π_{i+1}) correspond to occurrences of the transposition s_i in (2.7). Let H be a planar subnetwork of G which is a generalized wiring diagram and let $\pi' = (\pi'_1, \ldots, \pi'_n)$ be the unique noncrossing path family which covers H. Since H is a generalized wiring diagram, no three paths in π' share a vertex.

Suppose that we have $H \neq G$. Then we must have $\pi' \neq \pi$ and there exist some indices j for which $pi'_j \neq \pi_j$. Since π is a noncrossing path family which covers G, each such path π'_j shares an edge with π_{j-1} or with π_{j+1}. Let j be the least index for which π'_j shares an edge with π_{j-1}. Let u_k, u_q be vertices with $k < q$ such that paths π_j and π'_j diverge at u_k and reconverge at u_q. Then the vertices u_k and u_q correspond to occurrences of s_{j-1} in the expression (2.7). Immediately before passing through u_q, path π'_j necessarily passes through the vertex u_{m-1} which by (2.7) corresponds to the transposition s_{j-2}. Since π' is a noncrossing path family, and since we have chosen j so that paths π'_{j-1} and π'_{j-2} do not share edges with lesser indexed paths in π, these two paths must also pass through u_{m-1}. This contradicts the fact that H is a generalized wiring diagram, and we conclude that no path in π' shares an edge with a lesser indexed path in π. By symmetry, no path in π' shares an edge with a greater indexed path in π. It follows that $H = G$. □

3. Main results

Our main results concern the total nonnegativity of immanants related to the Temperley-Lieb algebra. After defining these immanants, which generalize the determinant, we will interpret them combinatorially and state their basic properties.

Given a complex number ξ, we define the *Temperley-Lieb algebra* $T_n(\xi)$ to be the $\mathbb{C}$-algebra generated by elements $t_1, \ldots, t_{n-1}$ subject to the relations

$$\begin{aligned} t_i^2 &= \xi t_i, && \text{for } i = 1, \ldots, n-1, \\ t_i t_j t_i &= t_i, && \text{if } |i-j| = 1, \\ t_i t_j &= t_j t_i, && \text{if } |i-j| \geq 2. \end{aligned}$$

This algebra is often defined as a quotient of the *Hecke algebra* $H_n(q)$, the $\mathbb{C}$-algebra with complex parameter q generated by $s_1, \ldots, s_{n-1}$ subject to the relations

$$\begin{aligned} s_i^2 &= (q-1)s_i + q, && \text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i &= s_j s_i s_j, && \text{if } |i-j| = 1, \\ s_i s_j &= s_j s_i, && \text{if } |i-j| \geq 2. \end{aligned}$$

Specifically, we have

$$\begin{aligned} &H_n(q)/(z_{[1,3]}) \cong T_n(q^{1/2} + q^{-1/2}) \\ &q^{-1/2}(s_i + 1) \mapsto t_i. \end{aligned}$$

(See e.g. [11], [18, Sec. 2.1, Sec. 2.11], [39, Sec. 7].) Specializing at $q = 1$, we have the isomorphism of $\mathbb{C}[S_n]/(z_{[1,3]})$ with $T_n(2)$. Equivalently, the ideal $(z_{[1,3]})$ is the kernel of the homomorphism

$$\begin{aligned} \theta : \mathbb{C}[S_n] &\to T_n(2), \\ s_i &\mapsto t_i - 1. \end{aligned}$$

We will call the elements of the multiplicative monoid generated by $t_1, \ldots, t_{n-1}$ the standard basis of $T_n(2)$, or simply *the basis elements* of $T_n(2)$. The dimension of $T_n(2)$ (and of $T_n(\xi)$) as a complex vector space is well known to be the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. A natural bijection between basis elements of $T_n(2)$ and 321-avoiding permutations in S_n is given by the correspondence of generators $s_i \leftrightarrow t_i$.

We will use $T_n(2)$ to classify planar networks as follows. Given a planar network H which is a union of n paths, define the element $\phi(H)$ of $T_n(2)$ by

$$\phi(H) = \theta(\beta(H)).$$

If H is a generalized wiring diagram, then by Lemma 2.5 we have that

$$\phi(H) = \theta(1 + s_{i_1}) \cdots \theta(1 + s_{i_k}) = t_{i_1} \cdots t_{i_k},$$

for some indices $i_1, \ldots, i_k \in [n]$ and therefore that

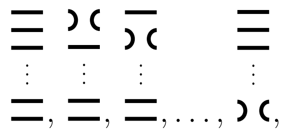
$$\phi(H) = 2^j \tau,$$

for some nonnegative integer j and some basis element τ of $T_n(2)$. We will denote the exponent and basis element by $\epsilon(H)$ and $\psi(H)$ respectively,

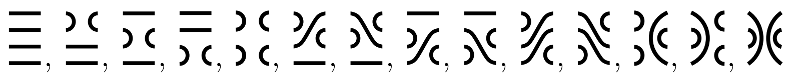
$$\phi(H) = 2^{\epsilon(H)} \psi(H).$$

If on the other hand H is not a generalized wiring diagram, then by Lemma 2.5 we have that $\beta(H)$ is equal to a sum of $\mathbb{C}[S_n]$ elements which belong to the kernel of θ. It follows in this case that $\phi(H) = 0$.

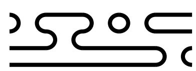
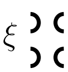
Diagrams of the basis elements of $T_n(\xi)$, made popular by Kauffman [25, Sec. 4], can aid in the calculation of $\phi(H)$. The identity and generators $1, t_1, \ldots, t_{n-1}$ are represented by



and multiplication of these elements corresponds to concatenation of diagrams, with cycles contributing a factor of ξ. For instance, the fourteen basis elements of $T_4(\xi)$ are



and the equality $t_1t_2t_1t_1t_3 = \xi t_1t_3$ in $T_4(\xi)$ is represented by

(3.1)  $= \xi$ .

If H is a generalized wiring diagram, then $\phi(H)$ can be computed graphically as follows.

(1) Contract any doubly covered subpath to a single vertex.
(2) For each vertex v of indegree two and outdegree two, create vertex v' with indegree two and vertex v'' with outdegree two.
(3) Interpret the resulting graph as an element of $T_n(2)$. (Compare Figure 2.3(a) and Equation 3.1.)

Analogous to the determinant, which counts families of nonintersecting paths in a planar network G, we will define for each basis element τ of $T_n(2)$ a *Temperley-Lieb immanant* which essentially counts subnetworks H of G which satisfy $\psi(H) = \tau$. The coefficients of this immanant are given by the function $f_\tau : S_n \to \mathbb{R}$, which maps σ to the coefficient of τ in $\theta(\sigma)$. To economize notation, we will write Imm_τ instead of Imm_{f_τ},

$$\mathrm{Imm}_\tau(x) = \sum_{\sigma \in S_n} f_\tau(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}.$$

When convenient we will tacitly extend f_τ linearly to $\mathbb{C}[S_n]$. Note that in the special case $\tau = 1$, the function f_τ maps a permutation σ to $(-1)^{\ell(\sigma)}$. Thus the determinant is a Temperley-Lieb immanant,

$$\det(x) = \mathrm{Imm}_1(x).$$

Theorem 3.1. *For any basis element τ of $T_n(2)$, $\mathrm{Imm}_\tau(x)$ is totally nonnegative. In particular, let G be a planar network of order n and let A be its path matrix. Then we have*

$$\mathrm{Imm}_\tau(A) = \sum_{H \subset G} 2^{\epsilon(H)} \omega(H), \tag{3.2}$$

where the sum is over all planar subnetworks H of G which are generalized wiring diagrams and which satisfy $\psi(H) = \tau$.

Proof. We have

$$\begin{aligned} \mathrm{Imm}_\tau(A) &= \sum_{\sigma \in S_n} f_\tau(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \\ &= \sum_{(H,m)} \omega(H,m) f_\tau(\beta(H,m)). \end{aligned}$$

where the inner sum is over all planar subnetworks H of G which are unions of n paths. If H is not a generalized wiring diagram, then $\beta(H,m)$ belongs to the kernel of θ, and we have $f_\tau(\beta(H,m)) = 0$. If on the other hand H is a generalized wiring diagram, then $\beta(H,m) = \beta(H)$ and $f_\tau(\beta(H))$ is equal to the coefficient of τ in

$$\begin{aligned} \theta(\beta(H)) &= \theta((1 + s_{i_1}) \cdots (1 + s_{i_k})) \\ &= 2^{\epsilon(H)} \psi(H). \end{aligned}$$

This coefficient is $2^{\epsilon(H)}$ if $\psi(H) = \tau$ and is zero otherwise. □

Note that the combinatorial interpretation of $\mathrm{Imm}_\tau(A)$ given by Theorem 3.1 generalizes that of $\det(A)$ given by Lindström's Lemma (Theorem 2.1): $\mathrm{Imm}_1(A)$ is equal to the sum of weights of nonintersecting path families in G. To generalize Lindström's interpretation of matrix minors, we will consider *generalized submatrices* of a matrix. Let M, M' be multisets of $[n]$, $[n']$, given by the sequences

$$\begin{aligned} &\mu(1) \leq \cdots \leq \mu(k), \\ &\mu'(1) \leq \cdots \leq \mu'(k), \end{aligned} \tag{3.3}$$

respectively. In analogy to (1.2), we define the (M, M') *generalized submatrix* of the $n \times n'$ matrix $x = (x_{i,j})$ to be

$$x_{M,M'} = \begin{bmatrix} x_{\mu(1),\mu'(1)} & \cdots & x_{\mu(1),\mu'(k)} \\ \vdots & & \vdots \\ x_{\mu(k),\mu'(1)} & \cdots & x_{\mu(k),\mu'(k)} \end{bmatrix}.$$

It is easy to see that for any TNN matrix A, each generalized submatrix $A_{M,M'}$ is again TNN. We also have the following.

Observation 3.2. *Let $p(y) = p(y_{1,1}, \ldots, y_{n,n})$ be a TNN polynomial, let M, M' be multisets of $[n]$, and define the polynomial $q(x)$ by $q(x) = p(x_{M,M'})$. Then $q(x)$ is totally nonnegative.*

Proof. For each TNN matrix A, $A_{M,M'}$ is also TNN and we have

$$q(A) = p(A_{M,M'}) \geq 0.$$

□

Given a planar network G for A, we may construct a planar network $\hat{G}$ for $A_{M,M'}$ as follows. For each source v_i of G, if i appears with multiplicity m_i in M, introduce m_i new sources and m_i directed edges from these to v_i. Introduce new sinks similarly. This construction gives a bijection between unions of k paths in G, no three of which share a vertex, and generalized wiring diagrams which are planar subnetworks of $\hat{G}$. We will denote the correspondence by $H \leftrightarrow \hat{H}$. By this correspondence, it is clear that $\mathrm{Imm}_\tau(x_{M,M'})$ is zero whenever M or M' contains an element with multiplicity greater than 2.

Theorem 3.3. *Let G be a planar network of order n with path matrix A, let τ be a basis element of $T_k(2)$, and let M, M' be cardinality-k multisets of $[n]$ given by (3.3). Then we have*

$$\mathrm{Imm}_\tau(A_{M,M'}) = \sum_H 2^{\epsilon(\hat{H})}\omega(H), \tag{3.4}$$

where the sum is over planar networks H which are unions of k paths $\pi = (\pi_1, \ldots, \pi_k)$ in G and which satisfy

(1) π_j is a path from $v_{\mu(j)}$ to $v'_{\mu'(j)}$.
(2) $\psi(\hat{H}) = \tau$.

Proof. Let $\hat{G}$ be the planar network of order k constructed from G as above so that $A_{M,M'}$ is the path matrix of $\hat{G}$. By Theorem 3.1 we have

$$\mathrm{Imm}_\tau(A_{M,M'}) = \sum_F 2^{\epsilon(F)}\omega(F),$$

where the sum is over planar subnetworks F of $\hat{G}$ which are generalized wiring diagrams and which satisfy $\psi(F) = \tau$. Since the correspondence $H \leftrightarrow \hat{H}$ satisfies $\omega(H) = \omega(\hat{H})$, we have the desired result. □

We may simplify Theorem 3.1 somewhat by replacing planar subnetworks with path families of type 1. (Each path π_i begins at source i and ends at sink i.) We will extend the definitions of ϕ, ϵ, and ψ to path such families in the natural way,

$$\phi(\pi) = \phi(H),$$
$$\epsilon(\pi) = \epsilon(H),$$
$$\psi(\pi) = \psi(H),$$

where H is the set of edges covered by π. Furthermore, we define a path family $\pi = (\pi_1, \dots, \pi_n)$ to be *nonintersecting (mod 2)* if the paths π_i and π_j do not intersect whenever $i \equiv j \pmod 2$.

Theorem 3.4. *For any basis element τ of $T_n(2)$, $\mathrm{Imm}_\tau(x)$ is totally nonnegative. In particular, let G be a planar network of order n and let A be its path matrix. Then we have*

$$\mathrm{Imm}_\tau(A) = \sum_\pi \omega(\pi), \tag{3.5}$$

where the sum is over all path families $\pi = (\pi_1, \dots, \pi_n)$ of type 1 in G which are nonintersecting (mod 2) and which satisfy $\psi(\pi) = \tau$.

Proof. By [31, Prop. 2.1] we may interpret the coefficient $2^{\epsilon(H)}$ in Equation (3.2) as the number of path families of type 1 which are nonintersecting (mod 2) and which cover H. □

The following consequences of Theorem 3.1 show that Temperley-Lieb immanants are not only TNN, but that they also play an important role in characterizing immanants which are TNN.

Corollary 3.5. *Let σ be a 321-avoiding permutation in S_n, let G be the wiring diagram corresponding to any reduced expression*

$$\sigma = s_{i_1} \cdots s_{i_\ell}, \tag{3.6}$$

define the basis element ν of $T_n(2)$ by using the same indices as in (3.6),

$$\nu = t_{i_1} \cdots t_{i_\ell}. \tag{3.7}$$

and let $A(\nu)$ be the path matrix of G. Then we have

$$\mathrm{Imm}_\tau(A(\nu)) = \begin{cases} 1 & \text{if } \nu = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since σ is 321-avoiding, any reduced expression for σ can be obtained from any other by a sequence of swaps of the form

$$W_1 s_i s_j W_2 \mapsto W_1 s_j s_i W_2,$$

where W_1 and W_2 are words in $s_1, \dots, s_{n-1}$ and $|i - j| \geq 2$. (See [38].) While such a swap maps a wiring diagram to a distinct wiring diagram, it is easy to see that the corresponding path matrices are equal. Thus $A(\nu)$ is the path matrix of an unweighted wiring diagram H of the form given by Equation (2.7).

By Lemma 2.6, the only planar subnetwork of H which is a generalized wiring diagram is H itself. Thus Equation (3.2) gives

$$\mathrm{Imm}_\tau(A(\nu)) = \begin{cases} 2^{\epsilon(H)} & \text{if } \nu = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Since (3.6) is a reduced expression, (3.7) is as well and we have $\epsilon(H) = 0$, as desired. □

Corollary 3.6. *A linear combination of immanants*

$$p(x) = \sum_\tau d_\tau \, \mathrm{Imm}_\tau(x),$$

is totally nonnegative if and only if each coefficient d_τ is nonnegative.

Proof. If the coefficients in this expression are all nonnegative, then obviously $p(x)$ is TNN. Suppose therefore that we have $d_\nu < 0$ for some basis element ν of $T_n(2)$. Defining the TNN matrix $A(\nu)$ as in Corollary 3.5, we have

$$p(A(\nu)) = d_\nu < 0,$$

and $p(x)$ is not TNN. □

The authors do not know an explicit formula for the coefficients $\{f_\tau(\rho) \,|\, \rho \in S_n\}$ occurring in $\mathrm{Imm}_\tau(x)$. However, it is easy to describe in terms of the Bruhat order which of these coefficients are zero. (See [9] for another connection between TNN immanants and the Bruhat order.)

Proposition 3.7. *Let τ be a basis element of $T_n(2)$, let $t_{i_1} \cdots t_{i_\ell}$ be a reduced expression for τ and let $\sigma = s_{i_1} \cdots s_{i_\ell}$ be the corresponding 321-avoiding permutation in S_n. Then we have $f_\tau(\sigma) = 1$ and*

$$\mathrm{Imm}_\tau(x) = \sum_{\rho \geq \sigma} f_\tau(\rho) x_{1,\rho(1)} \cdots x_{n,\rho(n)},$$

where the comparison of permutations is in the Bruhat order.

Proof. Let ρ be a permutation which is *less* than or equal to σ in the Bruhat order. Then each reduced expression $s_{j_1} \cdots s_{j_k}$ for ρ satisfies $k \leq \ell$. Since $f_\tau(\rho)$ is equal to the coefficient of τ in

$$\theta(\rho) = (t_{j_1} - 1) \cdots (t_{j_k} - 1),$$

and no expression for τ has length less than ℓ, we must have

$$f_\tau(\rho) = \begin{cases} 1 & \text{if } \rho = \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

□

Corollary 3.8. *The set* $\{\, \mathrm{Imm}_\tau(x) \,|\, \tau$ *a basis element of* $T_n(2)\}$ *is linearly independent.*

Straightforward but somewhat tedious computations give the following formulas for Temperley-Lieb immanants when $n \leq 4$. In each formula, the permutation σ should be understood to be the 321-avoiding permutation which corresponds to τ as in Proposition 3.7. For $\tau \in T_n(2)$ and $n < 4$ we have

$$\mathrm{Imm}_\tau(x) = \sum_{\rho \geq \sigma} (-1)^{\ell(\rho)-\ell(\sigma)} x_{1,\rho(1)} \cdots x_{n,\rho(n)}.$$

For $\tau \in T_4(2)$ we have

$$\begin{aligned} \mathrm{Imm}_\tau(x) &= \sum_{\rho \geq \sigma} (-1)^{\ell(\rho)-\ell(\sigma)} x_{1,\rho(1)} \cdots x_{n,\rho(n)} \qquad \text{if } \tau \notin \{t_2, t_1 t_3\}, \\ \mathrm{Imm}_{t_2}(x) &= \sum_{\rho \geq \sigma} (-1)^{\ell(\rho)-\ell(\sigma)} x_{1,\rho(1)} \cdots x_{n,\rho(n)} - \sum_{\rho \geq 3412} (-1)^{\ell(\rho)-\ell(\sigma)} x_{1,\rho(1)} \cdots x_{n,\rho(n)}, \\ \mathrm{Imm}_{t_1 t_3}(x) &= \sum_{\rho \geq \sigma} (-1)^{\ell(\rho)-\ell(\sigma)} x_{1,\rho(1)} \cdots x_{n,\rho(n)} - \sum_{\rho \geq 4231} (-1)^{\ell(\rho)-\ell(\sigma)} x_{1,\rho(1)} \cdots x_{n,\rho(n)}. \end{aligned}$$

These formulas suggest the following problem.

Problem 3.1. Given a basis element τ of $T_n(2)$, define σ as in Proposition 3.7 and find a family of sets $\{U(\rho, \sigma) \,|\, \rho \geq \sigma\}$ which satisfy

$$f_\tau(\rho) = |U(\rho, \sigma)| (-1)^{\ell(\rho)-\ell(\sigma)}.$$

Since Temperley-Lieb immanants include the determinant as a special case, it is not surprising that several properties of the determinant generalize nicely to Temperley-Lieb immanants.

Similar to the identity $\det(x^T) = \det(x)$ is the following property concerning transposed matrices.

Proposition 3.9. *Let* τ, τ' *be basis elements of* $T_n(2)$ *which satisfy*

$$\begin{aligned} \tau &= t_{i_1} \cdots t_{i_\ell}, \\ \tau' &= t_{i_\ell} \cdots t_{i_1}. \end{aligned}$$

Then we have $\mathrm{Imm}_\tau(x^T) = \mathrm{Imm}_{\tau'}(x)$.

Proof. Let $s_{j_1} \cdots s_{j_k}$ be a reduced expression for a permutation σ in S_n. Then $f_\tau(\sigma^{-1})$ is the coefficient of $t_{i_1} \cdots t_{i_\ell}$ in

$$\theta(\sigma^{-1}) = (t_{j_k} - 1) \cdots (t_{j_1} - 1),$$

which is equal to the coefficient of $t_{i_\ell} \cdots t_{i_1}$ in

$$\theta(\sigma) = (t_{j_1} - 1) \cdots (t_{j_k} - 1),$$

which is $f_{\tau'}(\sigma)$. Thus we have

$$\begin{aligned} \mathrm{Imm}_\tau(x^T) &= \sum_{\sigma \in S_n} f_\tau(\sigma^{-1}) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \\ &= \sum_{\sigma \in S_n} f_{\tau'}(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \\ &= \mathrm{Imm}_{\tau'}(x). \end{aligned}$$

□

A property of determinants known as the *Cauchy-Binet identity* relates the $k \times k$ minors of a product of two matrices to the $k \times k$ minors of the individual matrices. Letting A, B, C be $n \times n$ matrices which satisfy $A = BC$ and letting I, I' be k-element subsets of $[n]$, we have

$$\det(A_{I,I'}) = \sum_K \det(B_{I,K}) \det(C_{K,I'}), \tag{3.8}$$

where the sum is over all k-element subsets K of $[n]$. Many generalizations of the Cauchy-Binet identity exist in the literature. (See e.g. [14, p. 379], [24], [30, Lem. 2.3], [36, Thm. 1.1].) Our generalization is as follows.

Proposition 3.10. *Let τ be a basis element of $T_k(2)$, let A, B, C be $n \times n$ matrices which satisfy $A = BC$, and let M, M' be cardinality-k multisets of $[n]$. Then we have*

$$\mathrm{Imm}_\tau(A_{M,M'}) = \sum_N \sum_{(\tau_1,\tau_2)} 2^{d(\tau_1,\tau_2) - e(N)} \mathrm{Imm}_{\tau_1}(B_{M,N}) \mathrm{Imm}_{\tau_2}(C_{N,M'}),$$

where the outer sum is over cardinality-k multisets N of $[n]$, the inner sum is over pairs (τ_1, τ_2) of basis elements of $T_k(2)$ which satisfy $\tau_1\tau_2 = 2^{d(\tau_1,\tau_2)}\tau$, and $e(N)$ is the number of elements appearing twice in N.

Proof. Let G_1, G_2 be planar networks with path matrices $B_{M,[n]}$, $C_{[n],M'}$, and define $G = G_1 G_2$, so that the path matrix of G is $A_{M,M'}$. Then we have

$$\mathrm{Imm}_\tau(A_{M,M'}) = \sum_H 2^{\epsilon(H)} \omega(H),$$

where the sum is over all planar subnetworks H of G which are generalized wiring diagrams and which satisfy $\psi(H) = \tau$. Each planar subnetwork H appearing in this sum determines a multiset $N(H)$ as follows. Let $\pi(H) = (\pi_1, \dots, \pi_k)$ be the unique noncrossing path family which covers H, let $U = \{u_1, \dots, u_n\}$ be the vertices created by identifying the sinks of G_1 with the sources of G_2, and define $N(H) = 1^{m_1} \cdots n^{m_n}$ by

$$m_i = \#\{j \,|\, \pi_j \text{ contains } u_i\}.$$

We therefore have

$$\mathrm{Imm}_\tau(A_{M,M'}) = \sum_N \sum_H 2^{\epsilon(H)} \omega(H),$$

where the planar networks H in the inner sum satisfy $N(H) = N$. It is clear that such planar networks correspond bijectively with pairs (H_1, H_2) of planar networks satisfying the following conditions.

(1) The sources of H_1 are those of G.
(2) The sinks of H_2 are those of G.
(3) The sinks of H_1 and the sources of H_2 are $\{u_i \,|\, i \in N\}$.
(4) $\psi(H_1 H_2) = \tau$.
(5) There exist a unique noncrossing path family covering H_1, and a unique noncrossing path family covering H_2. For $i = 1, \dots, n$, the vertex u_i belongs to exactly m_i paths in each of these families.

Thus we have

$$\mathrm{Imm}_\tau(A_{M,M'}) = \sum_N \sum_{(H_1,H_2)} 2^{\epsilon(H_1 H_2)} \omega(H_1)\omega(H_2), \tag{3.9}$$

where the inner sum is over all pairs (H_1, H_2) which satisfy the above conditions.

Now fix a multiset N. For each pair (H_1, H_2) appearing in the inner sum of (3.9), there exists a pair (τ_1, τ_2) of basis elements of $T_k(2)$ and a nonnegative integer $d(\tau_1, \tau_2)$ satisfying

$$\tau_1 = \psi(\hat{H}_1), \qquad \tau_2 = \psi(\hat{H}_2),$$
$$\tau_1 \tau_2 = 2^{d(\tau_1,\tau_2)},$$

where $\hat{H}_1$, $\hat{H}_2$ are constructed from N, H_1, H_2 as preceeding Theorem 3.3. As a consequence, we have

$$\epsilon(\hat{H}_1 \hat{H}_2) = \epsilon(\hat{H}_1) + \epsilon(\hat{H}_2) + d(\tau_1, \tau_2).$$

Because of the additional vertices and edges introduced by the construction $H_j \mapsto \hat{H}_j$, we also have

$$\epsilon(\hat{H}_1 \hat{H}_2) = \epsilon(H_1 H_2) + e(N),$$

where $e(H)$ is the number of indices which appear twice in N. Applying this information and Theorem 3.3 to Equation (3.9), we obtain

$$\begin{aligned}\mathrm{Imm}_\tau(A_{M,M'}) &= \sum_N \sum_{(\tau_1,\tau_2)} 2^{d(\tau_1,\tau_2)-e(N)} \sum_{(H_1,H_2)} 2^{\epsilon(\hat{H}_1\hat{H}_2)}\omega(H_1)\omega(H_2)\\ &= \sum_N \sum_{(\tau_1,\tau_2)} 2^{d(\tau_1,\tau_2)-e(N)} \sum_{H_1} 2^{\epsilon(\hat{H}_1)}\omega(H_1) \sum_{H_2} 2^{\epsilon(\hat{H}_2)}\omega(H_2)\\ &= \sum_N \sum_{(\tau_1,\tau_2)} 2^{d(\tau_1,\tau_2)-e(N)}\, \mathrm{Imm}_{\tau_1}(B_{M,N})\, \mathrm{Imm}_{\tau_2}(C_{N,M'}).\end{aligned}$$

□

Using this generalized Cauchy-Binet identity and the following observation, we can generalize other well-known facts concerning the determinant.

Observation 3.11. *Let P be the permutation matrix corresponding to the adjacent transposition s_i of S_n. Then we have*

$$\mathrm{Imm}_\tau(P_{[n],M}) = \begin{cases} 1 & \text{if } \tau = t_i \text{ and } M = [n],\\ -1 & \text{if } \tau = 1 \text{ and } M = [n],\\ 0 & \text{otherwise.}\end{cases}$$

Proof. The permutation matrices corresponding to $s_1, \ldots, s_{n-1}$ are the path matrices of the planar networks shown in Figure 3.1. Unlabeled edges have weight 1. □

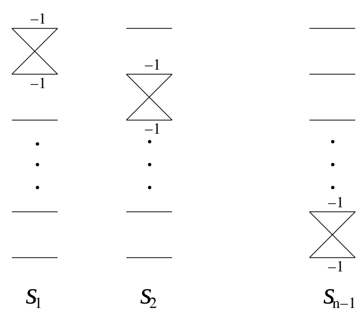


FIGURE 3.1

Recall that swapping two rows of a matrix changes the sign of its determinant. Such a swap changes the sign of some, but not all, of the Temperley-Lieb immanants. Similarly, the equality of two rows, which causes the determinant of a matrix to vanish, causes only some of the Temperley-Lieb immanants of a matrix to vanish. In the following two results, let the left vetices of a generic basis element of $T_n(2)$ be labeled $v_1, \ldots, v_n$, and the right vertices be labeled $v'_1, \ldots, v'_n$, from top to bottom in both cases.

Proposition 3.12. *Let τ be a basis element of $T_n(2)$, let A be an $n \times n$ matrix, and let P be the permutation matrix corresponding to the adjacent transposition s_i. Then we have*

$$\mathrm{Imm}_\tau(PA) = \begin{cases} -\mathrm{Imm}_\tau(A) & \text{if } \tau \text{ does not contain the edge } (v_i, v_{i+1}), \\ \mathrm{Imm}_\tau(A) + \sum_\nu \mathrm{Imm}_\nu(A) & \text{otherwise,} \end{cases}$$

where the sum is over basis elements ν of $T_n(2)$ which satisfy $t_i\nu = \tau$, and

$$\mathrm{Imm}_\tau(AP) = \begin{cases} -\mathrm{Imm}_\tau(A) & \text{if } \tau \text{ does not contain the edge } (v'_i, v'_{i+1}), \\ \mathrm{Imm}_\tau(A) + \sum_\nu \mathrm{Imm}_\nu(A) & \text{otherwise,} \end{cases}$$

where the sum is over basis elements ν of $T_n(2)$ which satisfy $\nu t_i = \tau$.

Proof. By Proposition 3.10 and Observation 3.11, we have

$$\begin{aligned} \mathrm{Imm}_\tau(PA) &= \mathrm{Imm}_1(P)\,\mathrm{Imm}_\tau(A) + \sum_\nu 2^{d(t_i,\nu)}\,\mathrm{Imm}_{t_i}(P)\,\mathrm{Imm}_\nu(A), \\ &= -\mathrm{Imm}_\tau(A) + \sum_\nu 2^{d(t_i,\nu)}\,\mathrm{Imm}_\nu(A), \end{aligned}$$

where the sum is over all basis elements ν of $T_n(2)$ for which $t_i\nu$ is a multiple of τ. If τ does not contain the edge (v_i, v_{i+1}) then there is no such element ν and we have

$$\mathrm{Imm}_\tau(PA) = -\mathrm{Imm}_\tau(A).$$

Suppose therefore that τ does contain the edge (v_i, v_{i+1}). Then we have

$$\begin{aligned} \mathrm{Imm}_\tau(PA) &= -\mathrm{Imm}_\tau(A) + 2\,\mathrm{Imm}_\tau(A) + \sum_\nu \mathrm{Imm}_\nu(A), \\ &= \mathrm{Imm}_\tau(A) + \sum_\nu \mathrm{Imm}_\nu(A), \end{aligned}$$

where the sum is over basis elements $\nu \neq \tau$ of $T_n(2)$ which satisfy $t_i\nu = \tau$.

A similar argument applies to $\mathrm{Imm}_\tau(AP)$. ☐

Corollary 3.13. *Let τ be a basis element of $T_n(2)$, let A be an $n \times n$ matrix, and let $i < j$ be indices in $[n]$. Then we have $\mathrm{Imm}_\tau(A) = 0$ if rows i and j of A are equal and τ contains none of the edges $(v_i, v_{i+1}), \ldots, (v_{j-1}, v_j)$, or if columns i and j of A are equal and τ contains none of the edges $(v'_i, v'_{i+1}), \ldots, (v'_{j-1}, v'_j)$.*

While the equality of two rows or two columns of a matrix doesn't cause all Temperley-Lieb immanants of that matrix to vanish, the equality of *three* rows or three columns does.

Proposition 3.14. *Let A be an $n \times n$ matrix and let τ be a basis element of $T_n(2)$. If any three rows or any three columns of A are equal, then we have $\mathrm{Imm}_\tau(A) = 0$.*

Proof. Let $i_1 < i_2 < i_3$ be the indices of three rows of A that are equal, and use induction on $i_3 - i_1$.

By Corollary 3.13, the statement is true when $i_3 - i_1 = 2$. Now suppose that we have $i_3 - i_1 = k$ and assume that the statement is true when this difference is less than k. Define P to be the permutation matrix corresponding to the adjacent transposition s_{i_1} if $i_2 = i_3 - 1$, or corresponding to s_{i_3-1} otherwise, so that PA satisfies the induction hypothesis. Applying Proposition 3.12, we see that $\mathrm{Imm}_\tau(A)$ is equal to a sum of immanants of PA, and therefore is equal to zero.

A similar argument applies to columns that are equal. □

To generalize the identity

$$\det \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \det(B)\det(D)$$

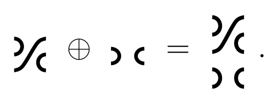
concerning block-upper-triangular matrices, we introduce the following operation on basis elements of Temperley-Lieb algebras. Given basis elements

$$\tau_1 = t_{i_1} \cdots t_{i_k} \in T_n(2),$$
$$\tau_2 = t_{j_1} \cdots t_{j_\ell} \in T_r(2),$$

we define the basis element $\tau_1 \oplus \tau_2$ of $T_{n+r}(2)$ by

$$\tau_1 \oplus \tau_2 = t_{i_1} \cdots t_{i_k} t_{n+j_1} \cdots t_{n+j_\ell}.$$

Using diagrams we construct $\tau_1 \oplus \tau_2$ by drawing τ_1 above τ_2. For instance, we have



Thus a basis element of $T_{n+r}(2)$ decomposes as $\tau = \tau_1 \oplus \tau_2$ with $\tau_1 \in T_n(2)$, $\tau_2 \in T_r(2)$ if and only if no edge of its diagram connects any of the top $2n$ vertices to any of the bottom $2r$ vertices. Equivalently, we have such a decomposition if and only if the generator t_n does not appear in a reduced expression for τ.

Proposition 3.15. *Let τ be a basis element of $T_{n+r}(2)$, and let A be an $(n+r) \times (n+r)$ block-upper-triangular matrix of the form*

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}, \qquad (3.10)$$

with B an $n \times n$ matrix and D an $r \times r$ matrix. Then we have

$$\mathrm{Imm}_\tau(A) = \begin{cases} \mathrm{Imm}_{\tau_1}(B)\,\mathrm{Imm}_{\tau_2}(D) & \text{if } \tau = \tau_1 \oplus \tau_2 \text{ for some } \tau_1 \in T_n(2),\ \tau_2 \in T_r(2), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let G be a planar network of order $n+r$ whose path matrix is A. By Theorem 3.4 we have

$$\mathrm{Imm}_\tau(A) = \sum_\pi \omega(\pi), \tag{3.11}$$

where the sum is over path families of type 1 in G which are nonintersecting (mod 2) and which satisfy $\psi(\pi) = \tau$. Let $\pi = (\pi_1, \dots, \pi_{n+r})$ be such a path family. Since there is no path in G from source $n+1$ to sink n, each path $\pi_1, \dots, \pi_n$ intersects none of the paths $\pi_{n+1}, \dots, \pi_{n+r}$. It follows that $\psi(\pi) = \tau$ if only if there exist basis elements $\tau_1 \in T_n(2)$, $\tau_2 \in T_r(2)$ satisfying

$$\begin{aligned} \psi(\pi_1, \dots, \pi_n) &= \tau_1, \\ \psi(\pi_{n+1}, \dots, \pi_{n+r}) &= \tau_2. \end{aligned} \tag{3.12}$$

Thus the sum in (3.11) is over all path families which satisfy (3.12), and is equal to zero if there is no such family.

Assume therefore that τ decomposes as $\tau_1 \oplus \tau_2$ for $\tau_1 \in T_n(2)$, $\tau_2 \in T_r(2)$. Now observe that for each path family $\pi' = (\pi'_1, \dots, \pi'_n)$ from sources $1, \dots, n$ to sinks $1, \dots, n$ in G and each path family $\pi'' = (\pi''_{n+1}, \dots, \pi''_{n+r})$ from sources $n+1, \dots, n+r$ to sinks $n+1, \dots, n+r$ in G, the combined family satisfies

$$\psi(\pi'_1, \dots, \pi'_n, \pi''_{n+1}, \dots, \pi''_{n+r}) = \tau$$

if and only if we have $\psi(\pi') = \tau_1$, $\psi(\pi'') = \tau_2$. Thus we have

$$\mathrm{Imm}_\tau(A) = \sum_{(\pi', \pi'')} \omega(\pi')\omega(\pi'') = \sum_{\pi'} \omega(\pi') \sum_{\pi''} \omega(\pi''),$$

where the sums are over path families as immediately above. This expression is clearly equal to $\mathrm{Imm}_{\tau_1}(B)\,\mathrm{Imm}_{\tau_2}(D)$. □

4. Products of two complementary minors

Deciding if an immanant of the form

$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \tag{4.1}$$

is TNN reduces to deciding if the index sets satisfy a certain system of inequalities. After reviewing this result (Proposition 4.1), we will state and prove the equivalence of two new combinatorial tests to decide the total nonnegativity of immanants of the form (4.1). Consequently we will see that the space of all immanants of the form

$$\sum_{(I,I')} c_{I,I'} \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$$

has dimension equal to a Catalan number, and that an element of this space is TNN if and only if it is equal to a nonnegative linear combination of Temperley-Lieb immanants.

In stating these results it will be convenient to associate sets I'', $\overline{I''}$, J'', $\overline{J''}$ to the products $\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x), \Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x)$ as follows. Given subsets I, I', J, J' of $[n]$, we define the subsets I'', $\overline{I''}$, J'', $\overline{J''}$ of $[2n]$ by

$$\begin{aligned} I'' &= I \cup \{2n+1-i \,|\, i \in \overline{I'}\}, \qquad & \overline{I''} &= [2n] \smallsetminus I'', \\ J'' &= J \cup \{2n+1-i \,|\, i \in \overline{J'}\}, \qquad & \overline{J''} &= [2n] \smallsetminus J''. \end{aligned} \tag{4.2}$$

The following result was proved in [31, Thm. 3.2]. (See also [10, Thm. 4.6].)

Proposition 4.1. *Let I, I', J, J' be subsets of $[n]$ and define the subsets $I'', \overline{I''}, J'', \overline{J''}$ of $[2n]$ as in (4.2). The immanant $\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$ is totally nonnegative if and only if for each subinterval B of $[2n]$ the sets $I'', \overline{I''}, J'', \overline{J''}$ satisfy*

$$\max\{|B \cap J''|, |B \cap \overline{J''}|\} \leq \max\{|B \cap I''|, |B \cap \overline{I''}|\}. \tag{4.3}$$

Of course Proposition 4.1 applies more generally to polynomials of the form (4.1) in which I, $\overline{I}$, etc. are complements within some row set and some column set, each of cardinality n. By deleting appropriate rows and columns and renumbering those which remain, we obtain an immanant.

To state a combinatorial alternative to the system of inequalities (4.3), we will associate lattice paths and set partitions to the sets I'', J'', and will show that the the total nonnegativity of the immanant (4.1) is equivalent to a refinement relation between the two set partitions.

Let m be an integer. Given a subset S of $[m]$, define the sequence $P(S, \overline{S}) = (p_1, \ldots, p_m)$ by

$$p_i = \begin{cases} \overrightarrow{(1,1)} & \text{if } i \in S, \\ \overrightarrow{(1,-1)} & \text{if } i \in \overline{S}. \end{cases}$$

Thus $P(S, \overline{S})$ may be interpreted as a lattice path in the plane, beginning at the origin and terminating at the point $(m, 2|S| - m)$. Now given such a lattice path we define an equivalence relation on $[m]$ by $i \sim j$ if steps p_i and p_j of the path are equally high above the x-axis. (That is, if their projections onto the y-axis are equal.) Let $\Pi(S, \overline{S})$ be the set partition of $[m]$ whose blocks are the equivalence classes of this relation.

To the products of minors $\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$ and $\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x)$ we will associate the lattice paths $P(I'', \overline{I''})$, $P(J'', \overline{J''})$ and the set partitions $\Pi(I'', \overline{I''})$, $\Pi(J'', \overline{J''})$.

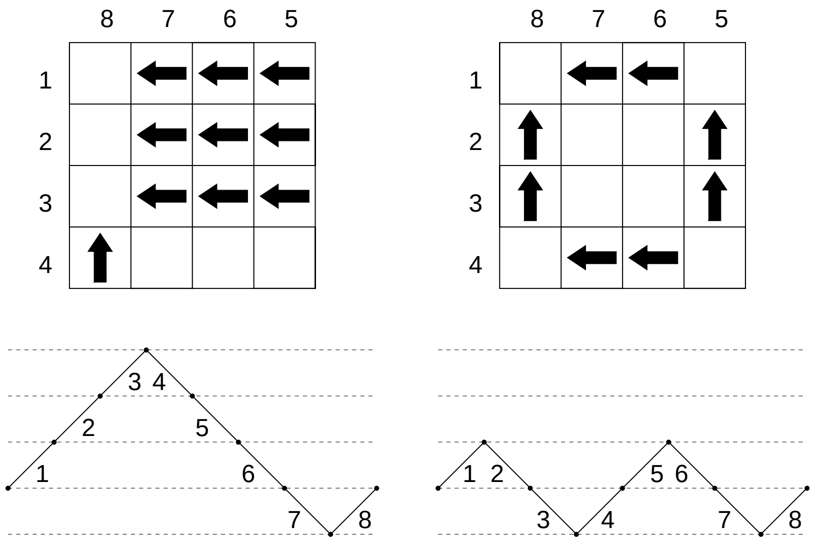


FIGURE 4.1. Lattice paths corresponding to $\Delta_{123,234}(x)\Delta_{4,1}(x)$ and $\Delta_{14,23}(x)\Delta_{23,14}(x)$

Consider for example the products of minors

$$\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) = \Delta_{123,234}(x)\Delta_{4,1}(x),$$
$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) = \Delta_{14,23}(x)\Delta_{23,14}(x),$$

shown in Figure 4.1. Defining the sets

$$I'' = 1238, \qquad \overline{I''} = 4567,$$
$$J'' = 1458, \qquad \overline{J''} = 2367,$$

as in (4.2), we obtain the lattice paths $P(I'', \overline{I''})$, $P(J'', \overline{J''})$ shown in the figure. To aid in drawing the path we have labeled matrix rows and columns by $1, \ldots, 8$ and we have marked matrix entries participating in the minors by arrows so that labels pointed to by arrows correspond to steps up in the lattice paths. Inspection of the lattice paths gives the set partitions

$$\Pi(I'', \overline{I''}) = 16|25|34|78,$$
$$\Pi(J'', \overline{J''}) = 1256|3478.$$

Notice that each block of $\Pi(I'', \overline{I''})$ is contained in a block of $\Pi(J'', \overline{J''})$. We therefore say that $\Pi(I'', \overline{I''})$ *refines* $\Pi(J'', \overline{J''})$.

Deciding the total nonnegativity of (4.1) is equivalent to deciding if there is a refinement relation between $\Pi(I'', \overline{I''})$ and $\Pi(J'', \overline{J''})$.

Theorem 4.2. *Let I ,I', J, J' be subsets of $[n]$ and define I'', $\overline{I''}$, J'', $\overline{J''}$ by (4.2). The immanant $\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$ is totally nonnegative if and only if $\Pi(I'', \overline{I''})$ refines $\Pi(J'', \overline{J''})$.*

Proof. We will prove by induction on m that for any subsets S, T of $[m]$, the relation $\Pi(S, \overline{S})$ refines the relation $\Pi(T, \overline{T})$ if and only if for each subinterval B of $[m]$ the sets S, $\overline{S}$, T, $\overline{T}$ satsify

$$\max\{|B \cap T|, |B \cap \overline{T}|\} \leq \max\{|B \cap S|, |B \cap \overline{S}|\}. \tag{4.4}$$

Note that the statement is trivially true for $m = 1$ and assume that it is true when S, T are subsets of $[2], \ldots, [m-1]$. We may assume that $1 \in S$, $1 \in T$ by swapping S with $\overline{S}$ (T with $\overline{T}$) if necessary.

($\Rightarrow$) Suppose that we have (4.4) for each subinterval B of $[m]$ and that $\Pi(S, \overline{S})$ does not refine $\Pi(T, \overline{T})$. Then there exists a pair of numbers belonging to a single block of $\Pi(S, \overline{S})$ and to distinct blocks of $\Pi(T, \overline{T})$. Applying the induction hypothesis to the subintervals $[m-1]$ and $[2, m]$ of $[m]$ we see that this pair of numbers must be $(1, m)$.

Note that if any element c in $[2, m-1]$ satisfies $1 \sim c \sim m$ in $\Pi(S, \overline{S})$, then the induction hypothesis applied to the intervals $[m-1]$ and $[2, m]$ gives $1 \sim c \sim m$ in $\Pi(T, \overline{T})$, a contradiction. Thus $\{1, m\}$ is a block of $\Pi(S, \overline{S})$, and steps $2, \ldots, m-1$ of $P(S, \overline{S})$ define a lattice path from the point $(1, 1)$ to the point $(m-1, 1)$. It follows that steps 1, 2 of $P(S, \overline{S})$ are equal to $\overrightarrow{(1, 1)}$, and that steps $m-1$, m of $P(S, \overline{S})$ are equal to $\overrightarrow{(1, -1)}$. Clearly, $P(S, \overline{S})$ ends on the x-axis, and we have

$$\max\{|[m] \cap S|, |[m] \cap \overline{S}|\} = \tfrac{m}{2}.$$

Now suppose that $P(T, \overline{T})$ also ends on the x-axis. Since $1 \nsim m$ in $\Pi(T, \overline{T})$, step m of $P(T, \overline{T})$ must be equal to $\overrightarrow{(1, 1)}$ and steps $2, \ldots, m-1$. of $P(T, \overline{T})$ must define a lattice path from the point $(1, 1)$ to the point $(m, -1)$. It follows that we have $2 \nsim m-1$ in $\Pi(T, \overline{T})$, contradicting the fact that the restriction of $\Pi(S, \overline{S})$ to $[m-1]$ refines the restriction of $\Pi(T, \overline{T})$ to this interval. Thus $P(T, \overline{T})$ does *not* end on the x-axis, and we have

$$\max\{|[m] \cap T|, |[m] \cap \overline{T}|\} > \tfrac{m}{2},$$

contradicting (4.4).

($\Leftarrow$) Suppose that $\Pi(S, \overline{S})$ refines $\Pi(T, \overline{T})$ and that for some subinterval B of $[m]$ we have

$$\max\{|B \cap T|, |B \cap \overline{T}|\} > \max\{|B \cap S|, |B \cap \overline{S}|\}.$$

The induction hypothesis implies that B is equal to $[m]$ and therefore that

$$\max\{|T|, |\overline{T}|\} > \max\{|S|, |\overline{S}|\}.$$

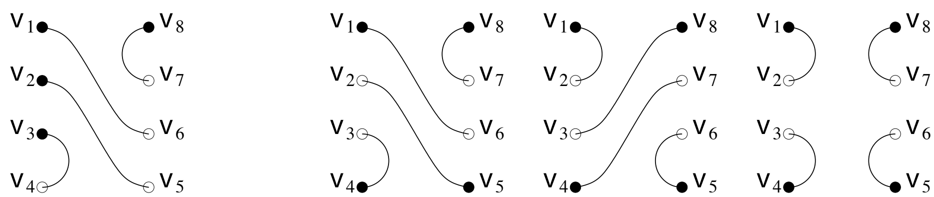


FIGURE 4.2. A one-element subset of $T_4(2)$ corresponding to $\Delta_{123,234}(x)\Delta_{4,1}(x)$ and a three-element subset of $T_4(2)$ corresponding to $\Delta_{14,23}(x)\Delta_{23,14}(x)$.

The result [31, Lem. 3.1] then implies that m is even, that m belongs to $T \smallsetminus S$, and that we have

$$|[2, m-1] \cap T| = |[2, m-1] \cap S| = \tfrac{m-2}{2}.$$

Thus $P(T, \overline{T})$ ends at the point $(m, 2)$ while $P(S, \overline{S})$ ends on the x-axis, and we have $1 \nsim m$ in $\Pi(T, \overline{T})$ while $1 \sim m$ in $P(S, \overline{S})$, a contradiction. $\square$

To state a second combinatorial alternative to the system of inequalities (4.3), and to reveal several interesting properties of the Temperley-Lieb immanants, we will associate a subset of the basis elements of $T_n(2)$ to each product $\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$. For the remainder of this section, we will label the vertices of a generic basis element of $T_n(2)$ by $v_1, \ldots, v_{2n}$, beginning in the upper left and continuing counterclockwise to the upper right.

Definition 4.1. Let S be an n-element subset of $[2n]$ and let τ be a basis element of $T_n(2)$. Call τ *compatible with* the pair $(S, \overline{S})$ if each edge of τ is incident upon exactly one of the vertices $\{v_i \,|\, i \in S\}$. Define $\Phi(S, \overline{S})$ to be the set of basis elements of $T_n(2)$ which are compatible with $(S, \overline{S})$.

To enumerate the elements of $\Phi(S, \overline{S})$, draw the vertices of a generic basis element of $T_n(2)$, assign a color to each of the sets $\{v_i \,|\, i \in S\}$, $\{v_i \,|\, i \in \overline{S}\}$, and draw edges so that no edge is monochromatic.

To the products of minors $\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$ and $\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x)$ we will associate the sets $\Phi(I'', \overline{I''}), \Phi(J'', \overline{J''})$ of basis elements of $T_m(2)$. For example consider again the products of minors

$$\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) = \Delta_{123,234}(x)\Delta_{4,1}(x),$$
$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) = \Delta_{14,23}(x)\Delta_{23,14}(x)$$

and the corresponding sets $I'' = 1238$, $J'' = 1458$. Figure 4.2 shows the subsets

$$\Phi(I'', \overline{I''}) = \{t_3t_2t_1\},$$
$$\Phi(J'', \overline{J''}) = \{t_3t_2t_1, t_1t_2t_3, t_1t_3\}.$$

The vertices $\{v_i \,|\, i \in I''\}$ $(\{v_i \,|\, i \in J''\})$ of each basis element are colored black so that compatibility is equivalent to the requirement that each edge be incident upon exactly one black vertex. Notice that the black vertices correspond to the steps equal to $\overrightarrow{(1,1)}$ of the lattice paths in Figure 4.1.

The following result shows that the relationship between products of complementary minors and immanants of the form $\mathrm{Imm}_\tau(x)$ is characterized by our above definition of compatibility.

Proposition 4.3. *Let I, I' be subsets of $[n]$ and define I'', $\overline{I''}$ as in (4.2). Then we have*

$$\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) = \sum_{\tau \in \Phi(I'',\overline{I''})} \mathrm{Imm}_\tau(x). \tag{4.5}$$

Proof. Let σ be a permutation in S_n, let A be the permutation matrix corresponding to σ, and let G be a planar network whose path matrix is A. Then the coefficient of $x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ on the left hand side of (4.5) is $\Delta_{I,I'}(A)\Delta_{\overline{I},\overline{I'}}(A)$. We can deduce from [31, Prop. 2.1, Thm. 3.2] that this coefficient is equal to

$$\sum_H \sum_\pi \omega(\pi),$$

where the outer sum is over all planar subnetworks H of G for which $\psi(H)$ belongs to $\Phi(I'',\overline{I''})$, and the inner sum is over path families $\pi = (\pi_1, \ldots, \pi_n)$ of type 1 in G which are nonintersecting (mod 2) and which cover H. By Theorem 3.4, this is precisely

$$\sum_{\tau \in \Phi(I'',\overline{I''})} \mathrm{Imm}_\tau(A),$$

which is equal to the coefficient of $x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ on the right hand side of (4.5). □

We will now use Proposition 4.3 to justify several total nonnegativity tests for immanants which are equal to arbitrary linear combinations of products of complementary minors. By statement 3 of the following theorem, we may test such an immanant $\mathrm{Imm}_f(x)$ for total nonnegativity by applying it to a set of $\frac{1}{n+1}\binom{2n}{n}$ matrices, and by checking the inequality

$$\mathrm{Imm}_f(A) \geq 0$$

for each matrix A on this list. (See [22, Thm. 1] for an analogous result concerning immanants which are nonnegative on positive semidefinite Hermitian matrices.) By statement 4 of the theorem, we may test such an immanant for total nonnegativity by applying f to a set of $\frac{1}{n+1}\binom{2n}{n}$ elements of $\mathbb{Z}[S_n]$, and by checking the inequality

$$f(z) \geq 0$$

for each element z on this list. (See [19, Sec. 6], [20, Sec. 2], [21, Thm. 1.5], [36, Cor. 3.3] for analogous methods of proving nonnegativity properties of irreducible character immanants.)

Theorem 4.4. *Let $f : S_n \to \mathbb{R}$ be a function which satisfies*

$$\mathrm{Imm}_f(x) = \sum_{(I,I')} c_{I,I'} \Delta_{I,I'}(x) \Delta_{\overline{I},\overline{I'}}(x). \tag{4.6}$$

The following conditions on f are equivalent:

1. $\mathrm{Imm}_f(x)$ *is totally nonnegative.*
2. *There exist nonnegative constants $\{d_\tau \mid \tau \in T_n(2)\}$ such that we have*
$$\mathrm{Imm}_f(x) = \sum_\tau d_\tau \,\mathrm{Imm}_\tau(x).$$
3. *For each basis element $\tau \in T_n(2)$ and the corresponding matrix $A(\tau)$ defined in Corollary 3.5, we have we have*
$$\mathrm{Imm}_f(A(\tau)) \geq 0.$$
4. *For each 321-avoiding permutation σ and each (equivalently, any) reduced expression $\sigma = s_{i_1} \cdots s_{i_\ell}$, we have*
$$f((s_{i_1} + 1) \cdots (s_{i_\ell} + 1)) \geq 0.$$
5. *For each basis element τ of $T_n(2)$ we have*
$$\sum c_{I,I'} \geq 0, \tag{4.7}$$
where the sum is over all pairs $\{(I, I') \mid \tau \in \Phi(I'', \overline{I''})\}$.

Proof. $(2 \Rightarrow 1 \Rightarrow 3)$ Obvious.

$(2 \Leftrightarrow 3 \Leftrightarrow 5)$ Corollary 3.5 gives

$$\mathrm{Imm}_f(A(\tau)) = d_\tau,$$

and we also have

$$\sum_{(I,I')} c_{I,I'} = d_\tau.$$

$(4 \Leftrightarrow 2)$ Proposition 4.3 implies that there exist real numbers $\{d_\tau \mid \tau \in T_n(2)\}$ which satisfy

$$f = \sum_\tau d_\tau f_\tau.$$

Let τ be any basis element of $T_n(2)$ and let $t_{i_1} \cdots t_{i_\ell}$ be a reduced expression for τ. Then we have

$$f((s_{i_1} + 1) \cdots (s_{i_\ell} + 1)) = \sum_\tau d_\tau f_\tau((s_{i_1} + 1) \cdots (s_{i_\ell} + 1)) = d_\tau.$$

□

A special case of statement (5) of Theorem 4.4 is the following.

Corollary 4.5. *Let* I, I', J, J' *be subsets of* $[n]$ *and define* $I'', \overline{I''}, J'', \overline{J''}$ *by (4.2). The immanant* $\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$ *is totally nonnegative if and only if* $\Phi(I'', \overline{I''})$ *is contained in* $\Phi(J'', \overline{J''})$*. In this case we have*

$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) = \sum_{\tau} \mathrm{Imm}_{\tau}(x),$$

where the sum is over all elements τ *of* $\Phi(J'', \overline{J''}) \smallsetminus \Phi(I'', \overline{I''})$*.*

For example, Figure 4.2 shows that we have

$$\Delta_{14,23}(x)\Delta_{23,14}(x) - \Delta_{123,234}(x)\Delta_{4,1}(x) = \mathrm{Imm}_{t_1t_2t_3}(x) + \mathrm{Imm}_{t_1t_3}(x).$$

Corollary 4.5 suggests defining a poset $\mathcal{P}_n$ on products $\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$ of complementary minors of $n \times n$ matrices by

$$\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \leq \Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x)$$

if $\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$ is TNN. Figure 4.3 shows the poset $\mathcal{P}_3$. Each product of minors $\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$ is accompanied by the lattice path $P(I'', \overline{I''})$ and the set $\Phi(I'', \overline{I''})$. Since the elements of $\mathcal{P}_n$ are ordered by refinement of related set partitions, $\mathcal{P}_n$ is a subposet of the poset of all set partitions of $[2n]$. (See [32, p. 97].) The maximal element of $\mathcal{P}_n$, which corresponds to the single-block partition of $[2n]$, has the following simple description. (See also [10, Cor. 4.14].)

Proposition 4.6. *For any* n*, the unique maximal element* $\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$ *of the poset* $\mathcal{P}_n$ *is given by*

$$I = I' = \{i \in [n] \mid i \text{ odd }\},$$
$$\overline{I} = \overline{I''} = \{i \in [n] \mid i \text{ even }\}.$$

and this product of minors is equal to

$$\sum_{\tau} \mathrm{Imm}_{\tau}(x), \tag{4.8}$$

where the sum is over all basis elements τ *of* $T_n(2)$*.*

Proof. By Proposition 4.3, we have

$$\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) = \sum_{\tau \in \Phi(I'', \overline{I''})} \mathrm{Imm}_{\tau}(x),$$

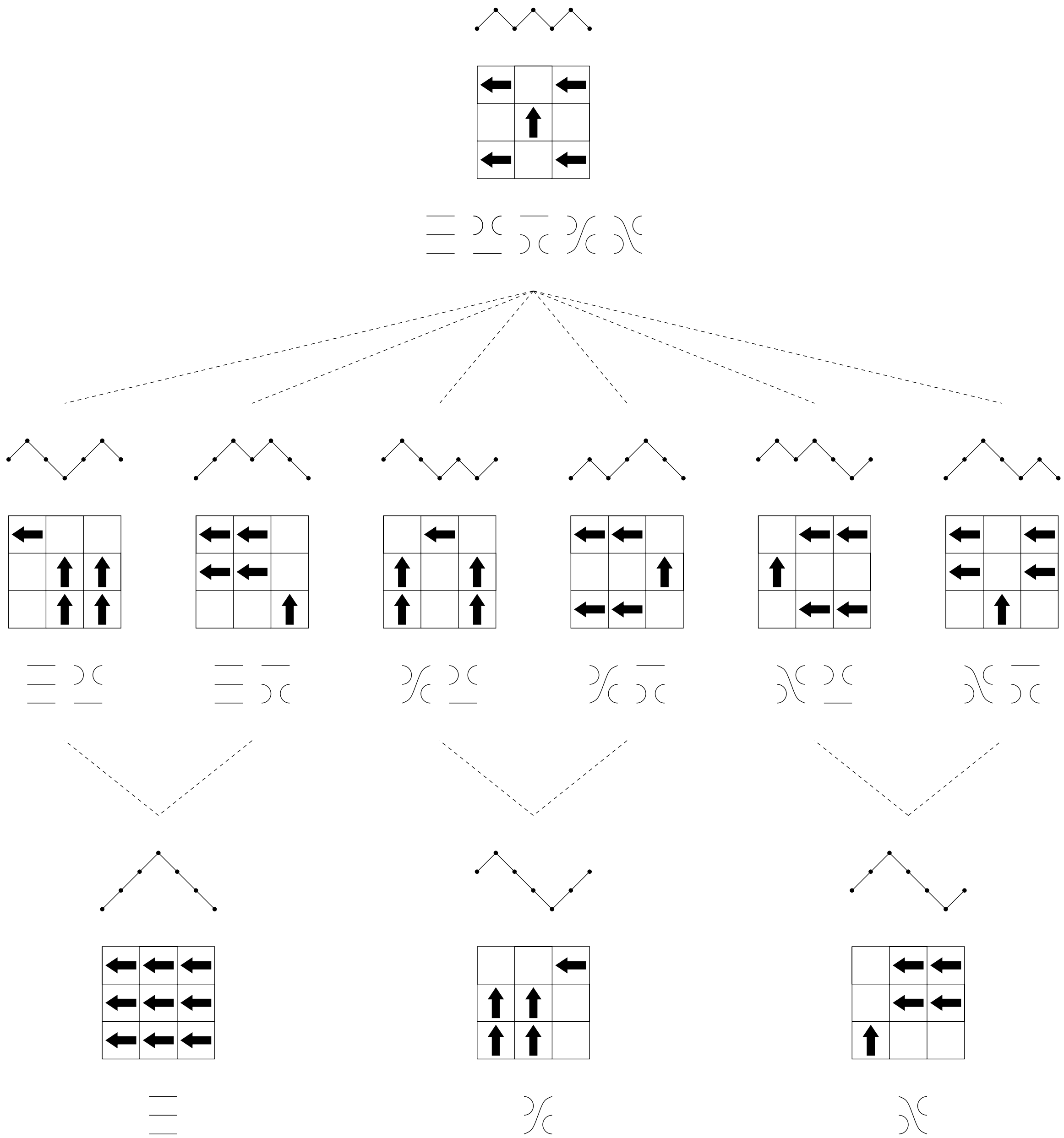


FIGURE 4.3. The poset of products of complementary minors of 3×3 matrices.

where

$$I'' = \{1, 3, \ldots, 2n-1\},$$
$$\overline{I''} = \{2, 4, \ldots, 2n\}.$$

A basis element τ of $T_n(2)$ belongs to $\Phi(I'', \overline{I''})$ if for each edge (v_i, v_j) of τ, the number $i - j$ is odd. Equivalently, we have

$$\Phi(I'', \overline{I''}) = T_n(2),$$

as desired. $\square$

Returning to Proposition 4.3, we see that the linear span of products of complementary minors of $n \times n$ matrices has dimension no greater than the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. We will show using Dyck paths that this dimension is in fact equal to C_n. A lattice path beginning at $(0,0)$ in the plane and consisting of equal numbers of steps of type $\overrightarrow{(1,1)}$ and $\overrightarrow{(1,-1)}$ is called a *Dyck path* if it never passes below the x-axis. The number of Dyck paths having $2n$ steps is well known to be C_n.

Proposition 4.7. *The collection of immanants*

$$\{\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \,|\, P(I,I') \text{ is a Dyck path } \} \tag{4.9}$$

is linearly independent. In particular, we have

$$\dim \operatorname{span}_{\mathbb{R}}\{\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \,|\, I, I' \subset [n]\} = C_n.$$

Proof. Assume that $1 \in I$. The well-known bijection between Dyck paths having $2n$ steps and basis elements of $T_n(2)$ may be described by

$$P = P(I'', \overline{I''}) \mapsto \tau(P),$$

where $\tau(P)$ is the unique element of $T_n(2)$ in which each edge (v_k, v_ℓ) with $k < \ell$ satisfies $k \in I''$, $\ell \in \overline{I''}$. Thus $\tau(P)$ belongs to $\Phi(I'', \overline{I''})$, and for each edge (v_k, v_ℓ) of $\tau(P)$, steps k and ℓ of P are equal to $\overrightarrow{(1,1)}$ and $\overrightarrow{(1,-1)}$, respectively.

Now define a partial order on the set of Dyck paths having $2n$ steps by $P_i < P_j$ if P_j fits under P_i. (This is a subposet of Young's lattice. See [33, p. 263].) Let $P_1, \ldots, P_{C_n}$ be a linear extension of this partial order, and let (I_i, I'_i) be the pair of subsets of $[n]$ which satisfies

$$P_i = P({I''}_i, \overline{I''}_i).$$

We claim that the $C_n \times C_n$ matrix $A = (a_{i,j})$ defined by the equations

$$\Delta_{I_i,I'_i}(x)\Delta_{\overline{I_i},\overline{I'_i}}(x) = \sum_{j=1}^{C_n} a_{i,j} \operatorname{Imm}_{\tau(P_j)}(x), \quad i = 1, \ldots, C_n$$

is lower triangular with ones on the diagonal. From Proposition 4.3 we have

$$a_{i,j} = \begin{cases} 1 & \text{if } \tau(P_j) \in \Phi({I''}_i, \overline{I''}_i), \\ 0 & \text{otherwise,} \end{cases}$$

and from the definition of $\tau(P_i)$ it is clear that $a_{i,i} = 1$ for all i.

Now fix indices $i < j$. From the definition of our partial order, P_i does not fit under P_j. Let ℓ be the smallest index such that step ℓ of P_i lies above step ℓ of P_j. These steps must be equal to $\overrightarrow{(1,1)}$ and $\overrightarrow{(1,-1)}$ respectively. Let k be the index such that (v_k, v_ℓ) is an edge of $\tau(P_j)$. Then half of the steps $k, \dots, \ell$ of P_j are equal to $\overrightarrow{(1,1)}$. By the minimality of ℓ, step k of P_i lies at or below step k of P_j, and *more* than half of the steps $k, \dots, \ell$ of P_i are equal to $\overrightarrow{(1,1)}$. There must therefore exist an edge (v_a, v_b) of $\tau(P_j)$ with $k \leq a, b \leq \ell$ and $a, b \in {I''}_i$. It follows that $\tau(P_j)$ does not belong to $\Phi({I''}_i, \overline{I''}_i)$.

Since A is invertible, the collection (4.9) of immanants is linearly independent. $\square$

5. General products of two minors

To generalize the results of Section 4 we will let x be an $n \times n'$ matrix of variables and we will consider polynomials of the form

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x), \tag{5.1}$$

in which row and column sets are not necessarily complementary. In this more general setting we will compare multisets and will write $S \Cup T$ for the multiset union of (multi)sets S and T. For example,

$$123 \Cup 234 = 122334.$$

In addition to stating results concerning the total nonnegativity of polynomials of the form (5.1), we will consider the space of all polynomials of the form

$$\sum_{(I,I',K,K')} c_{I,I',K,K'} \Delta_{I,I'}(x)\Delta_{K,K'}(x) \tag{5.2}$$

where the sum is over all quadruples (I, I', K, K') which satisfy

$$I \Cup K = M, \qquad I' \Cup K' = M'$$

for a fixed pair of multisets M, M'. As in the previous section, we will show that the dimension of this space is equal to a Catalan number and that an element of this space is TNN if and only if it is equal to a nonnegative linear combination of Temperley-Lieb immanants of the matrix $x_{M,M'}$. (See definitions following Theorem 3.1.)

The sets $I'', \overline{I''}, J'', \overline{J''}$ which we associated to products of complementary minors have the following analogs with respect to the more general products $\Delta_{I,I'}(x)\Delta_{K,K'}(x)$, $\Delta_{J,J'}(x)\Delta_{L,L'}(x)$. Letting I, J, K, L be subsets of $[n]$ and I', J', K', L' be subsets of

$[n']$, we define the subsets I'', J'', K'', L'' of $[n+n']$ by

$$\begin{aligned} I'' &= I \cup \{n+n'+1-i \,|\, i \in K'\}, \\ K'' &= K \cup \{n+n'+1-i \,|\, i \in I'\}, \\ J'' &= J \cup \{n+n'+1-i \,|\, i \in L'\}, \\ L'' &= L \cup \{n+n'+1-i \,|\, i \in J'\}. \end{aligned} \tag{5.3}$$

Note that this does not conflict with our earlier notation (4.2) in the event that n, K, L are equal to n', $\overline{I}$, $\overline{J}$, respectively.

The following result [31, Thm. 4.2] generalizes Proposition 4.1.

Proposition 5.1. *Given subsets I, J, K, L of $[n]$ and subsets I', J', K', L' of $[n']$, define the subsets I'', J'', K'', L'' of $[n+n']$ by (5.3). Then the polynomial*

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x)$$

is totally nonnegative if and only if we have

$$I \uplus K = J \uplus L, \qquad I' \uplus K' = J' \uplus L', \tag{5.4}$$

and for each subinterval B of $[n+n']$ the sets I'', J'', K'', L'' satisfy

$$\max\{|B \cap J''|, |B \cap L''|\} \leq \max\{|B \cap I''|, |B \cap K''|\}. \tag{5.5}$$

Note that the multiset equalities (5.4) allow us to restrict our attention to differences (5.1) in which the index sets $I, I', \dots, L, L'$ satisfy

$$\begin{aligned} I \cup K = J \cup L = [n], &\qquad I' \cup K' = J' \cup L' = [n'], \\ I \cap K = J \cap L, &\qquad I' \cap K' = J' \cap L'. \end{aligned} \tag{5.6}$$

In particular, if the unions of rows and columns in (5.1) are not $[n]$ and $[n']$ as above, we may delete certain rows and columns and renumber those remaining to obtain (5.6).

Note also that we may modify the system (5.5) of inequalities by replacing the sets I'', J'', K'', L'' with the differences

$$\begin{aligned} I''' = I'' \smallsetminus K'', &\qquad K''' = K'' \smallsetminus I'', \\ J''' = J'' \smallsetminus L'', &\qquad L''' = L'' \smallsetminus J'', \end{aligned} \tag{5.7}$$

since this replacement has the effect of reducing both maxima by the same constant. We will state an analog of Theorem 4.2 in terms of these differences. In particular, we will modify the lattice paths and set partitions we defined in Section 4 as follows.

Given disjoint sets S, T with $|S|+|T| = m$, define the sequence $P(S,T) = (p_i)_{i \in S \cup T}$ by

$$p_i = \begin{cases} \overrightarrow{(1,1)} & \text{if } i \in S, \\ \overrightarrow{(1,-1)} & \text{if } i \in T, \end{cases}$$

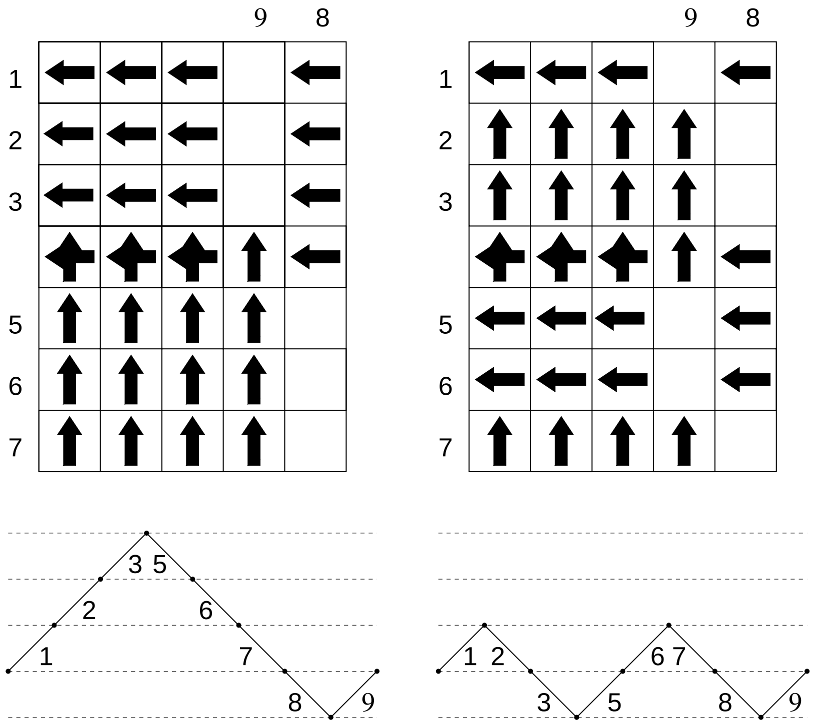


FIGURE 5.1. Lattice paths corresponding to $\Delta_{1234,1235}(x)\Delta_{4567,1234}(x)$ and $\Delta_{1456,1235}(x)\Delta_{2347,1234}(x)$.

where the indices are in increasing order, but not necessarily consecutive. Thus $P(S,T)$ may be interpreted as a lattice path in the plane, beginning at the origin and terminating at the point $(m, |S| - |T|)$. Now given such a lattice path we define an equivalence relation on $S \cup T$ as in Section 4 and we let $\Pi(S,T)$ be the set partition of $S \cup T$ whose blocks are the equivalence classes of this relation.

To the products of minors $\Delta_{I,I'}(x)\Delta_{K,K'}(x)$ and $\Delta_{J,J'}(x)\Delta_{L,L'}(x)$ we will associate the lattice paths $P(I''', K''')$, $P(J''', L''')$ and the set partitions $\Pi(I''', K''')$, $\Pi(J''', L''')$. Consider for example the products of minors

$$\Delta_{I,I'}(x)\Delta_{K,K'}(x) = \Delta_{1234,1235}(x)\Delta_{4567,1234}(x),$$
$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) = \Delta_{1456,1235}(x)\Delta_{2347,1234}(x)$$

shown in Figure 5.1. The unions of row indices and column indices are [7] and [5] respectively, so we define the subsets I''', J''', K''', L''' of [12] as in (5.7) by

$$I''' = 1239, \qquad K''' = 5678,$$
$$J''' = 1569, \qquad J''' = 2378,$$

and we obtain the lattice paths $P(I''', K''')$, $P(J''', L''')$ shown in the figure. To aid in computing the sets I''', J''', K''', L''' and in drawing the paths we have marked matrix entries participating in the minors by arrows and have labeled only those matrix rows and columns containing one type of arrow. Thus I''', J''' consist of labels pointed to by arrows and correspond to steps up in the lattice paths. Similarly, K''', L''' consist of the remaining labels and correspond to steps down. Inspection of the lattice paths gives the set partitions

$$\Pi(I''', K''') = 17|26|35|89,$$
$$\Pi(J''', L''') = 1267|3589.$$

Notice that $\Pi(I''', K''')$ refines $\Pi(J''', L''')$. In analogy to Theorem 4.2, we have the following combinatorial alternative to the system of inequalities (5.5).

Theorem 5.2. *Let subsets I, J, K, L of $[n]$ and subsets I', J', K', L' of $[n']$ satisfy (5.6), and define I''', J''', K''', L''' as in (5.7). The polynomial*

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x)$$

is totally nonnegative if and only if $\Pi(I''', K''')$ refines $\Pi(J''', L''')$.

Proof. By Proposition 5.1 and the comment preceding (5.7), the polynomial

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x)$$

is TNN if and only if for each subinterval B of $[n+n']$, the sets I''', J''', K''', L''' satisfy

$$\max\{|B \cap J'''|, |B \cap L'''|\} \leq \max\{|B \cap I'''|, |B \cap K'''|\}.$$

Letting η be the unique order preserving map

$$\eta : I''' \cup K''' \to [|I'''| + |K'''|],$$

we see that this condition is satisfied if and only if for each subinterval B of $[|I''|+|K''|]$, the sets $\eta(I'''), \eta(J''')$ satisfy

$$\max\{|B \cap \eta(J''')|, |B \cap \overline{\eta(J''')}|\} \leq \max\{|B \cap \eta(I''')|, |B \cap \overline{\eta(I''')}|\},$$

where the set complements are within $[|I''| + |K''|]$. By Proposition 4.1, this is equivalent to the condition that $\Pi(\eta(I'''), \overline{\eta(I''')})$ refines $\Pi(\eta(J'''), \overline{\eta(J''')})$, which is clearly equivalent to the condition that $\Pi(I''', K''')$ refines $\Pi(J''', L''')$. □

Although the polynomials (5.1) are not in general immanants, a result analgous to Proposition 4.3 reveals a connection between these polynomials and Temperley-Lieb immanants. To state these and other results, we will associate to each product $\Delta_{I,I'}(x)\Delta_{K,K'}(x)$ a subset of the basis elements of an appropriate Temperley-Lieb algebra as follows. As in the previous section, we will label the vertices of a generic basis element of $T_r(2)$ by $v_1, \ldots, v_{2r}$, beginning in the upper left and continuing counterclockwise to the upper right.

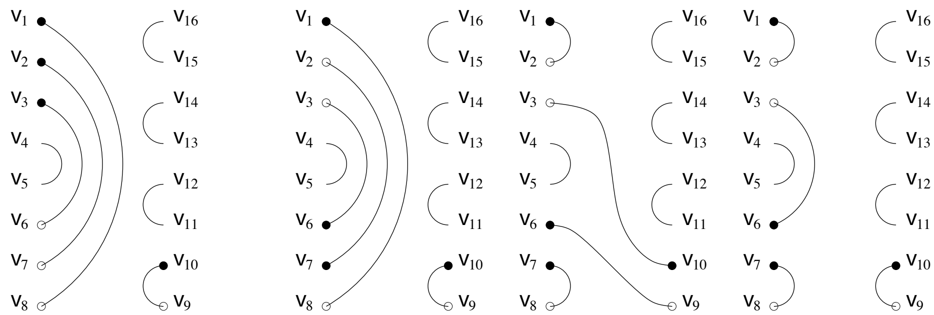


FIGURE 5.2. A one-element subset of $T_8(2)$ corresponding to $\Delta_{1234,1235}(x)\Delta_{4567,1234}(x)$ and a three-element subset of $T_8(2)$ corresponding to $\Delta_{1456,1235}(x)\Delta_{2347,1234}(x)$.

Definition 5.1. Let integers n, n', r and sets S, T satisfy $S \cup T = [n+n']$ and $|S|+|T| = 2r$, and let $w_1 \cdots w_{2r}$ be the nondecreasing rearrangement of $S \uplus T$. Call a basis element τ of $T_r(2)$ *compatible with* the pair (S,T) if for each edge (v_i, v_j) of τ the numbers w_i, w_j satisfy one of the following conditions.

(1) w_i and w_j are equal, and belong to $S \cap T$.
(2) One of the numbers w_i, w_j belongs to $S \smallsetminus T$ and the other belongs to $T \smallsetminus S$.

Define $\Phi(S,T)$ to be the set of all basis elements of $T_r(2)$ which are compatible with (S,T).

To enumerate the elements of $\Phi(S,T)$, draw the vertices of a generic basis element of $T_r(2)$ and the mandatory edges (v_i, v_{i+1}) for each equality $w_i = w_{i+1}$. Assign a color to each of sets $\{v_i \,|\, w_i \in S \smallsetminus T\}$, $\{v_i \,|\, w_i \in T \smallsetminus S\}$ and draw the remaining edges so that no edge is monochromatic.

To the products of minors $\Delta_{I,I'}(x)\Delta_{K,K'}(x)$ and $\Delta_{J,J'}(x)\Delta_{L,L'}(x)$ we will associate the sets $\Phi(I'',K'')$, $\Phi(J'',L'')$ of basis elements of $T_r(2)$. For example consider again the product of minors

$$\Delta_{I,I'}(x)\Delta_{K,K'}(x) = \Delta_{1234,1235}(x)\Delta_{4567,1234}(x)$$

shown in Figure 5.1. Defining the subsets I'', K'' of $[12]$ as in (5.3), we have

$$I'' = \{1,2,3,4,9,10,11,12\},$$
$$K'' = \{4,5,6,7,8,10,11,12\},$$

and $|I''|+|K''| = 16$. The nondecreasing rearrangement $(w_1,\ldots,w_{16})$ of $I'' \uplus K''$ is then given by the table

$$\begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 & w_{10} & w_{11} & w_{12} & w_{13} & w_{14} & w_{15} & w_{16} \\ 1 & 2 & 3 & 4 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 10 & 11 & 11 & 12 & 12 \end{bmatrix}.$$

Labeling the vertices of a generic basis element of $T_8(2)$ by $v_1, \ldots, v_{16}$ as in Figure 5.2, we note the equalities

$$\begin{aligned} w_4 &= w_5 = 4, \\ w_{11} &= w_{12} = 10, \\ w_{13} &= w_{14} = 11, \\ w_{15} &= w_{16} = 12, \end{aligned}$$

and we draw the mandatory edges (v_4, v_5), (v_{11}, v_{12}), (v_{13}, v_{14}), (v_{15}, v_{16}). Then coloring the vertices

$$\{v_i \,|\, w_i \in I'' \smallsetminus K''\} = \{v_1, v_2, v_3, v_9\}$$

black, and the vertices

$$\{v_i \,|\, w_i \in K'' \smallsetminus I''\} = \{v_5, v_6, v_7, v_8\}$$

white, we find that the basis element $t_4t_5t_6t_7t_3t_4t_5t_2t_3t_1$ shown on the left in the figure is the only basis element which is compatible with the pair (I'', K''). Notice that for each black vertex v_i of this basis element, there is a step up in the lattice path $P(I''', K''')$ shown in Figure 5.1, and that this step is labeled by w_i. The reader may repeat the above process for the product of minors

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) = \Delta_{1456,1235}(x)\Delta_{2347,1234}(x)$$

to verify that the subset $\Phi(J'', L'')$ of $T_8(2)$ consists of the three basis elements shown on the right of Figure 5.2, and to see that black vertices in these basis elements correspond to steps up in the lattice path $P(J''', L''')$ shown in Figure 5.1.

While the polynomials $\Delta_{I,I'}(x)\Delta_{K,K'}(x)$ are not immanants of the matrix x, they are related to Temperley-Lieb immanants of generalized submatrices of x. (See Section 3.)

Proposition 5.3. *Let I, I', K, K' be sets which satisfy $I \cup K = [n]$, $I' \cup K' = [n']$, and define I'', K'' as in* (5.3). *Then we have*

$$\Delta_{I,I'}(x)\Delta_{K,K'}(x) = \sum_{\tau} \mathrm{Imm}_{\tau}(x_{I \Cup K, I' \Cup K'}), \tag{5.8}$$

where the sum is over elements τ of $\Phi(I'', K'')$.

Proof. Let $w_1, \cdots w_{2r}$ be the nondecreasing rearrangement of $I'' \Cup K''$, Defining

$$\begin{aligned} J &= \{i \,|\, w_i \in I, w_{i-1} \neq w_i\}, & J' &= \{2r + 1 - i \,|\, n + n' + 1 - w_i \in I', w_{i-1} \neq w_i\}, \\ \overline{J} &= \{i \,|\, w_i \in K, w_i \neq w_{i+1}\}, & \overline{J'} &= \{2r + 1 - i \,|\, n + n' + 1 - w_i \in K', w_i \neq w_{i+1}\}, \end{aligned}$$

we may express the polynomial $\Delta_{I,I'}(x)\Delta_{K,K'}(x)$ as the product of complementary minors of the $r \times r$ matrix $x_{I \Cup K, I' \Cup K'}$,

$$\Delta_{I,I'}(x)\Delta_{K,K'}(x) = \Delta_{J,J'}(x_{I \Cup K, I' \Cup K'})\Delta_{\overline{J},\overline{J'}}(x_{I \Cup K, I' \Cup K'}).$$

By Proposition 4.3, this product is therefore equal to the polynomial

$$\sum_{\tau \in \Phi(J'', \overline{J''})} \mathrm{Imm}_\tau(x_{I \Cup K, I' \Cup K'}),$$

where $J'', \overline{J''}$ are defined by

$$\begin{aligned} J'' &= J \cup \{2r + 1 - i \,|\, i \in \overline{J'}\}, \\ &= \{i \,|\, w_i \in I, w_{i-1} \neq w_i\} \cup \{i \,|\, n + n' + 1 - w_i \in K', w_i \neq w_{i+1}\}, \end{aligned} \tag{5.9}$$

and $\overline{J''} = [2r] \smallsetminus J''$.

To see that $\Phi(I'', K'')$ is contained in $\Phi(J'', \overline{J''})$, let τ be an element of $\Phi(I'', K'')$, and consider an edge (v_i, v_j) of τ. Suppose that $w_i = w_j$ for some $i < j$. Then $j = i + 1$ and $i \neq r$. By (5.9), exactly one of these indices belongs to J''. On the other hand, suppose that $w_i \neq w_j$. Then we do not have $w_{i-1} = w_i$ or $w_i = w_{i+1}$. Since τ belongs to $\Phi(I'', K'')$, then one of the numbers w_i, w_j belongs to $I'' \smallsetminus K''$ and the other belongs to $K'' \smallsetminus I''$. Since I'', K'' are defined by

$$\begin{aligned} I'' &= I \cup \{n + n' + 1 - i \,|\, i \in K'\}, \\ K'' &= K \cup \{n + n' + 1 - i \,|\, i \in I'\}, \end{aligned}$$

we again have that exactly one of the indices i, j belongs to J''. Since these are the only possibilities for an edge (v_i, v_j) of an element in $\Phi(I'', K'')$, we conclude that τ belongs to $\Phi(J'', \overline{J''})$.

To see that the sum (5.8) need not include the elements of $\Phi(J'', \overline{J''}) \smallsetminus \Phi(I'', K'')$, let τ be such an element. In order not to be compatible with (I'', K''), τ must have an edge (v_i, v_j) which satisfies one of the following conditions.

(1) Both w_i, w_j belong to $I'' \cap K''$, but $w_i \neq w_j$.
(2) Exactly one of the numbers w_i, w_j belongs $I'' \cap K''$.

In either case, there must be a pair $\ell, \ell + 1$ of indices such that $w_\ell = w_{\ell+1}$ and τ does not contain the edge $(v_\ell, v_{\ell+1})$. Corollary 3.13 then implies that $\mathrm{Imm}_\tau(x_{I \Cup K, I' \Cup K'}) = 0$. □

We now turn to the problem of characterizing TNN polynomials of the form

$$\sum c_{I,I',K,K'} \Delta_{I,I'}(x) \Delta_{K,K'}(x).$$

It is not in general true that such a polynomial is TNN if and only if it is a nonnegative linear combination of TNN immanants. For instance the polynomial

$$p(x) = \Delta_{1,1}(x)\Delta_{1,1}(x) - 2\Delta_{1,1}(x)\Delta_{2,2}(x) + \Delta_{2,2}(x)\Delta_{2,2}(x)$$

is TNN because it is equal to $(\Delta_{1,1}(x) - \Delta_{2,2}(x))^2$. Nevertheless, when it is expanded in terms of immanants of generalized submatrices of x it has negative coefficients,

$$p(x) = \operatorname{Imm}_{t_1}(x_{11,11}) + \operatorname{Imm}_{t_1}(x_{22,22}) - 2\operatorname{Imm}_1(x_{12,12}) - 2\operatorname{Imm}_{t_1}(x_{12,12}).$$

In order to generalize Theorem 4.4, we will need to restrict our attention to polynomials which are homogeneous in the sense of (5.2). (This homogeneity property is implicit in the statements of Proposition 5.1, Theorem 5.2, and Proposition 5.3.) Let us fix integers n, n', r, subsets $S \subset [n]$, $S' \subset [n']$ satisfying

$$|S| + n = |S'| + n' = r, \tag{5.10}$$

and consider a polynomial

$$p(x) = \sum c_{I,I',K,K'} \Delta_{I,I'}(x) \Delta_{K,K'}(x) \tag{5.11}$$

where the sum is over quadruples (I, I', K, K') which satisfy

$$\begin{aligned} I \cup K &= [n], \qquad & I' \cup K' &= [n'], \\ I \cap K &= S, \qquad & I' \cap K' &= S'. \end{aligned} \tag{5.12}$$

By Proposition 5.3 we have

$$p(x) = \operatorname{Imm}_f(x_{[n]\uplus S,[n']\uplus S'}) = \sum d_\tau \operatorname{Imm}_\tau(x_{[n]\uplus S,[n']\uplus S'}), \tag{5.13}$$

for some function $f : S_r \to \mathbb{R}$ and some real numbers $\{d_\tau \,|\, \tau \in T_r(2)\}$. We will show in Theorem 5.4 that $p(x)$ is TNN if and only if the coefficients d_τ are nonnegative.

Note that each coefficient d_τ appearing in (5.13) is zero unless τ belongs to $\Phi(I'', K'')$ for some quadruple (I, I', K, K') satisfying (5.12). Let $R(I'', K'')$ be this subset of basis elements of $T_r(2)$. We may characterize $R(S, S')$ by letting $w = w_1 \cdots w_{2r}$ be the nondecreasing rearrangement of

$$[n + n'] \uplus S \uplus \{n + n' + 1 - i \,|\, i \in S'\}, \tag{5.14}$$

and by usin Definition 5.1. Specifically, the elements of $R(S, S')$ are those which contain all edges (v_i, v_{i+1}) for which $w_i = w_{i+1}$,

$$R(S, S') = \{\tau \in T_m(2) \,|\, w_i = w_{i+1} \Rightarrow \tau \text{ contains } (v_i, v_{i+1})\}. \tag{5.15}$$

The set $R(S, S')$ provides an analog of the matrices $A(\tau)$ defined in Corollary 3.5 which are used to test the total nonnegativity of the immanants in Theorem 4.4. Since the mandatory edges (v_i, v_{i+1}) force rows i and $i+1$ of $A(\tau)$ to be equal if $i < r$, and columns $2r - i$ and $2r - i + 1$ to be equal if $i > r$, there exists an $n \times n'$ matrix $B(\tau)$ which satisfies

$$A(\tau) = B(\tau)_{[n]\uplus S,[n']\uplus S'}. \tag{5.16}$$

Since $B(\tau)$ is a submatrix of the TNN matrix $A(\tau)$, it too is TNN. We will show in Theorem 5.4 that $p(x)$ is TNN if and only if $p(B(\tau))$ is nonnegative for each τ in $R(S, S')$.

Theorem 5.4. *Fix subsets $S \subset [n]$, $S' \subset [n']$ satisfying $|S|+n = |S'|+n' = r$, define the subset $R(S, S')$ as in (5.15), and let $p(x)$ be a polynomial of the form*

$$p(x) = \sum c_{I,I',K,K'} \Delta_{I,I'}(x) \Delta_{K,K'}(x), \tag{5.17}$$

where the sum is over quadruples (I, I', K, K') which satisfy (5.12). The following conditions on $p(x)$ are equivalent.

(1) $p(x)$ is totally nonnegative.
(2) There exist nonnegative constants $\{d_\tau \,|\, \tau \in R(S, S')\}$ such that we have

$$p(x) = \sum_\tau d_\tau \operatorname{Imm}_\tau(x_{[n] \uplus S, [n'] \uplus S'}).$$

(3) For each element τ of $R(S, S')$, the matrix $B(\tau)$ defined in (5.16) satisfies

$$p(B(\tau)) \geq 0. \tag{5.18}$$

(4) For each 321-avoiding permutation σ in S_r and each (equivalently, any) reduced expression $\sigma = s_{j_1} \cdots s_{j_\ell}$, the function $f : \mathbb{R}[S_n] \to \mathbb{R}$ defined by (5.13) satisfies

$$f((s_{j_1} + 1) \cdots (s_{j_\ell} + 1)) \geq 0. \tag{5.19}$$

(5) For each element τ of $R(S, S')$ we have

$$\sum c_{I,I',K,K'} \geq 0, \tag{5.20}$$

where the sum is over all quadruples $\{(I, I', K, K') \,|\, \tau \in \Phi(I'', K'')\}$.

Proof. $(2 \Rightarrow 1)$ By Theorem 3.1, $\operatorname{Imm}_\tau(y)$ is a TNN polynomial in the m^2 variables $y = (y_{i,j})$. Therefore by Observation 3.2, $\operatorname{Imm}_\tau(x_{[n] \uplus S, [n'] \uplus S'})$ is a TNN polynomial in the $n \times n'$ variables $x = (x_{i,j})$, as is $p(x)$.

$(1 \Rightarrow 3)$ Since $B(\tau)$ is TNN, we have $p(B(\tau)) \geq 0$.

$(2 \Leftrightarrow 3 \Leftrightarrow 5)$ If τ belongs to $R(S, S')$, we have

$$\sum c_{I,I',K,K'} = d_\tau,$$

and

$$p(B) = \sum_\tau \operatorname{Imm}_\tau(B_{[n] \uplus S, [n'] \uplus S'}) = \sum_\tau \operatorname{Imm}_\tau(A(\tau)) = d_\tau.$$

$(2 \Leftrightarrow 4)$ By Proposition 5.3, we have

$$f(z) = \sum_{\tau \in R(S,S')} d_\tau f_\tau(z)$$

for all z in $\mathbb{R}[S_r]$. Thus we have

$$\begin{aligned} f((s_{i_1}+1)\cdots(s_{i_\ell}+1)) &= \sum_{\tau\in R(S,S')} d_\tau f_\tau((s_{i_1}+1)\cdots(s_{i_\ell}+1)) \\ &= \begin{cases} d_\tau & \text{if } \tau = t_{i_1}\cdots t_{i_\ell} \text{ belongs to } R(S,S'), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus the condition that $f((s_{i_1}+1)\cdots(s_{i_\ell}+1))$ be nonnegative is equivalent to the condition that d_τ be nonnegative if $\tau = t_{i_1}\cdots t_{i_\ell}$ belongs to $R(S,S')$. $\square$

One sees from the proof of Theorem 5.4 that to test the total nonnegativity of a polynomial of the form (5.17) using statements $(2),(3),(5)$ of the theorem, it is sufficient to consider only those basis elements τ of $T_r(2)$ which satisfy

$$w_i = w_{i+1} \Rightarrow \tau \text{ contains the edge } (v_i, v_{i+1}).$$

A similar shortcut applies to statement (4) of the theorem. Defining the *left descent set* and *right descent set* of a permutation σ to be the set of adjacent transpositions s which satisfy $\ell(s\sigma) = \ell(\sigma) - 1$ and $\ell(\sigma s) = \ell(\sigma) - 1$, respectively, we may consider only those permutations σ having left and right descent sets equal to

$$\{s_i \,|\, w_i = w_{i+1}, i < r\}, \qquad \{s_{2r-i} \,|\, w_i = w_{i+1}, i > r\},$$

respectively.

The following special case of statement (5) is analogous to Corollary 4.5.

Corollary 5.5. *The polynomial $\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x)$ is totally nonnegative if and only if $I \Cup K = J \Cup L$, $I' \Cup K' = J' \Cup L'$, and $\Phi(I'',K'')$ is contained in $\Phi(J'',L'')$. In this case we have*

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x) = \sum_\tau \mathrm{Imm}_\tau(x_{I\Cup K, I'\Cup K'}),$$

where the sum is over all elements τ of $\Phi(J'',L'') \smallsetminus \Phi(I'',K'')$.

For example, Figure 5.2 shows that we have

$$\Delta_{1456,1235}(x)\Delta_{2347,1234}(x) - \Delta_{1234,1235}(x)\Delta_{4567,1234}(x) = \mathrm{Imm}_{\tau_1}(x) + \mathrm{Imm}_{\tau_2}(x),$$

where

$$\begin{aligned} \tau_1 &= t_1t_4t_7t_3t_6t_5, \\ \tau_2 &= t_1t_4t_7t_3t_5. \end{aligned}$$

Generalizing the poset $\mathcal{P}_n$ of products of complementary minors of $n\times n$ matrices, we have posets of products of overlapping minors. Since two such products are comparable only if they involve the same multisets of matrix rows and columns, we will fix subsets $S \subset [n]$ and $S' \subset [n']$ and define $\mathcal{P}([n] \uplus S, [n'] \uplus S')$ to be the poset of all products of the form $\Delta_{I,I'}(x)\Delta_{K,K'}(x)$, where $I \cap K = S$, $I' \cap K' = S'$.

Proposition 5.6. *Fix subsets $S \subset [n]$, $S' \subset [n']$ satisfying $n + |S| = n' + |S'| = r$, and let $w_1 \cdots w_r$, $w'_1 \cdots w'_r$ be the nondecreasing rearrangements of $S \uplus [n]$, $S' \uplus [n']$, respectively. Then the poset $\mathcal{P}([n] \uplus S, [n'] \uplus S')$ is isomorphic to $\mathcal{P}_{2r-|S|-|S'|}$. Its unique maximal element $\Delta_{I,I'}(x)\Delta_{K,K'}(x)$ is given by*

$$\begin{aligned} I &= \{w_i \,|\, i \text{ odd } \}, \qquad & I' &= \{w'_i \,|\, i \text{ odd } \}, \\ K &= \{w_i \,|\, i \text{ even } \}, \qquad & K' &= \{w'_i \,|\, i \text{ even } \}, \end{aligned} \tag{5.21}$$

and satisfies

$$\Delta_{I,I'}(x)\Delta_{K,K'}(x) = \sum_{\tau} \mathrm{Imm}_\tau(x_{[n]\uplus S,[n']\uplus S'}),$$

where the sum is over all basis elements τ of $T_r(2)$ containing the edges

$$\{(v_i, v_{i+1}) \,|\, w_i = w_{i+1} \in S\} \cup \{(v_{2r+1-i}, v_{2r-i}) \,|\, w_i = w_{i+1} \in S'\}. \tag{5.22}$$

Proof. The isomorphism of posets follows immediately from Theorems 4.2 and 5.2. Corresponding elements in the two posets are those which share a given lattice path.

By Proposition 5.3 and Equation 5.3, we have

$$\Delta_{I,I'}(x)\Delta_{K,K'}(x) = \sum_{\tau \in \Phi(I'',K'')} \mathrm{Imm}_\tau(x_{[n]\uplus S,[n']\uplus S'}),$$

where

$$\begin{aligned} I'' &= \{w_i \,|\, i \text{ odd } \} \cup \{n + n' + 1 - w'_i \,|\, i \text{ even } \}, \\ K'' &= \{w_i \,|\, i \text{ even } \} \cup \{n + n' + 1 - w'_i \,|\, i \text{ odd } \}. \end{aligned}$$

By Definition 5.1, $\Phi(I'', K'')$ is a subset of the basis elements of $T_r(2)$. Precisely, it consists of those basis elements which contain the $|S| + |S'|$ edges in the set (5.22) and $r - |S| - |S'|$ edges from the set

$$\begin{aligned} &\{(v_i, v_j) \,|\, w_i \in I \smallsetminus K, w_j \in K \smallsetminus I\} \cup \{(v_i, v_j) \,|\, w'_{2r+1-i} \in I' \smallsetminus K', w'_{2r+1-j} \in K' \smallsetminus I'\} \\ \cup &\{(v_i, v'_j) \,|\, w_i \in I \smallsetminus K, w'_{2r+1-j} \in I' \smallsetminus K'\} \cup \{(v_i, v'_j) \,|\, w_i \in K \smallsetminus I, w'_{2r+1-j} \in K' \smallsetminus I'\}. \end{aligned}$$

This union is equal to the set of edges (v_i, v_j) which satisfy

(1) $i - j$ is odd.
(2) $w_i \notin S$ if $i \leq r$; $w_j \notin S$ if $j \leq r$.
(3) $w_{2r+1-i} \notin S'$ if $i > r$; $w_{2r+1-j} \notin S'$ if $j > r$.

Now let τ be a basis element of $T_r(2)$ which contains the edges (5.22) and suppose that some edge (v_i, v_j) fails to satisfy one of the above conditions. A failure to satisfy the first condition contradicts the fact that τ is a basis element of $T_r(2)$. A failure to satisfy the second or third condition contradicts the fact that τ contains the edges

(5.22). We conclude that every basis element of $T_r(2)$ which contains the edges (5.22) belongs to $\Phi(I'', K'')$. By Corollary 5.5, the element $\Delta_{I,I'}(x)\Delta_{K,K'}(x)$ is maximal in $\mathcal{P}([n] \uplus S, [n'] \uplus S')$. □

Proposition 5.7. *Fix subsets $S \subset [n]$, $S' \subset [n']$ which satisfy $|S| + n = |S'| + n' = r$, and define the vector space*

$$V(S, S') = \{\Delta_{I,I'}(x)\Delta_{K,K'}(x) \mid I \uplus K = [n] \uplus S, I' \uplus K' = [n'] \uplus S'\}.$$

Then a basis of $V(S, S')$ is given by

$$\{\Delta_{I,I'}(x)\Delta_{K,K'}(x) \in V(S, S') \mid P(I''', \overline{I'''}) \text{ is a Dyck path } \}. \tag{5.23}$$

In particular, the dimension of $V(S, S')$ is C_r.

Proof. Let $w = w_1 \cdots w_{2r}$ be the nondecreasing rearrangement of

$$[n + n'] \uplus S \uplus \{n + n' + 1 - i \mid i \in S'\}.$$

Given a Dyck path $P = (p_i)_{i \in I''' \cup K'''}$, we may define a map

$$P = P(I''', K''') \mapsto \tau(P)$$

by letting $\tau(P)$ be the unique element in $T_r(2)$ in which each edge (v_i, v_j) with $i < j$ satisfies one of the conditions

(1) $w_i = w_j$.
(2) $w_i \in I'''$, $w_j \in K'''$.

We may then proceed as in the proof of Proposition 4.7. □

6. More nonnegativity properties

The nonnegativity properties of Temperley-Lieb immanants seem to extend beyond that of total nonnegativity.

For example, it is possible to show that for each basis element τ of $T_n(2)$, $n \leq 5$ that $\mathrm{Imm}_\tau(x)$ may be expressed as a subtraction-free Laurent polynomial in matrix minors. Stronger than total nonnegativity, this property seems to be shared by elements of the dual canonical basis of type A_{n-1} and by other polynomials which arise in the study of cluster algebras. (See [14], [15].) Since this *subtraction-free Laurent* property is not well understood, it would be interesting to exhibit classes of TNN polynomials that have this property, or even classes of polynomials which may be expressed as *subtraction-free rational* expressions in matrix minors [12].

Question 6.1. Are Temperley-Lieb immanants equal to subtraction-free Laurent polynomials (or subtraction-free rational expressions) in matrix minors?

Another nonnegativity property of the Temperley-Lieb immanants may be defined in terms of symmetric functions. (See [33, Ch. 7] for definitions.) We define a polynomial $p(x)$ to be *monomial nonnegative* (MNN) if for every Jacobi-Trudi matrix A, the symmetric function $p(A)$ is equal to a nonnegative linear combination of monomial symmetric functions. An example of a family of MNN polynomials is provided by the (TNN) irreducible character immanants mentioned in Section 1. (See [19], [20].) Related polynomials conjectured to be MNN and TNN are the monomial immanants. (See [35], [37]).

While neither the MNN or the TNN property is known to imply the other, a link between the two properties was established by the Gessel-Viennot interpretation of Jacobi-Trudi matrices [17]. In particular, a polynomial which is proved to be TNN by a path family argument, such as that of Theorem 3.1, must also be MNN. We therefore have the following corollary of Theorem 3.1.

Corollary 6.1. *Temperley-Lieb immanants are monomial nonnegative.*

Contained in the class of MNN polynomials is the class of *Schur nonnegative* (SNN) polynomials. These are polynomials $p(x)$ with the property that for each Jacobi-Trudi matrix A, the symmetric function $p(A)$ is equal to a nonnegative linear combination of Schur functions. Somewhat less well understood than MNN and TNN polynomials, SNN polynomials seem nevertheless to arise where MNN and TNN polynomials do. The irreducible character immanants are SNN [21], and the monomial immanants are conjectured to be SNN as well [37]. Furthermore, the Bruhat order has three very similar characterizations in terms of TNN, MNN, and SNN immanants. (See [8], [9].)

Since questions of Schur nonnegativity have applications in algebraic geometry, it would be desirable to have a better understanding of SNN polynomals. In particular, several recent conjectures [4, 13] concern polynomials of the form

$$\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x),$$

and it is interesting that Temperley-Lieb immanants seem to be SNN for small n.

Question 6.2. Are Temperley-Lieb immanants Schur nonnegative?

It is peculiar that examples of polynomials which have the above five nonnegativity properties are provided by essentially the same immanants. Even the above mentioned characterizations of the Bruhat order may be trivially restated in terms of the subtraction-free Laurent property and subtraction-free rational function property. Apparently there is some connection between these properties which remains to be discovered.

Problem 6.3. State the implications which exist among the above nonnegativity properties of polynomials. Are these implications different if we restrict our attention to immanants?

7. Acknowledgements

The authors are grateful to Sergey Fomin, Rosa Orellana, Richard Stanley, John Stembridge, Brian Drake, Sean Gerrish, Michal Ostrowski, Nina Palmo, Zachary Pavlov, Randy Pistor, and Andrew Wagner for many helpful conversations.

References

[1] M. Aissen, I. J. Schoenberg, and A. Whitney. On generating functions of totally positive sequences. *J. Anal. Math.*, **2** (1952) pp. 93–103.

[2] A. Berenstein and A. Zelevinsky. String bases for quantum groups of type A_r. *Adv. Sov. Math.*, **16** (1993) pp. 51–89.

[3] A. Berenstein and A. Zelevinsky. Canonical bases for the quantum groups of type A_r and piecewise linear combinatorics. *Duke Math. J.*, **82** (1996) pp. 473–502.

[4] F. Bergeron, R. Biagioli, and M. Rosas. On a conjecture concerning Littlewood-Richardson coefficients, 2003. In progress.

[5] S. Billey, W. Jockusch, and R. Stanley. Some combinatorial properties of Schubert polynomials. *J. Alg. Comb.*, **2** (1993) pp. 345–374.

[6] F. Brenti. Combinatorics and total positivity. *J. Combin. Theory Ser. A*, **71** (1995) pp. 175–218.

[7] C. W. Cryer. Some properties of totally positive matrices. *Linear Algebra Appl.*, **15** (1976) pp. 1–25.

[8] B. Drake, S. Gerrish, and M. Skandera. Nonnegativity properties and the Bruhat order, 2004. In progress.

[9] B. Drake, S. Gerrish, and M. Skandera. Two new criteria for comparison in the Bruhat order. *Electron. J. Combin.*, **11** (2004). Note 6, 4 pp. (electronic).

[10] S. M. Fallat, M. I. Gekhtman, and C. R. Johnson. Multiplicative principal-minor inequalities for totally nonnegative matrices. *Adv. Appl. Math.*, **30** (2003) pp. 442–470.

[11] K. Fan and R. M. Green. Monomials and Temperley-Lieb algebras. *J. Algebra*, **190** (1997) pp. 498–517.

[12] S. Fomin, 2002. Personal communication.

[13] S. Fomin, W. Fulton, C. Li, and Y. Poon. Eigenvalues, singular values, and Littlewood-Richardson coefficients, 2003. Preprint math.AG/03013078 on ArXiv.

[14] S. Fomin and A. Zelevinsky. Double Bruhat cells and total positivity. *J. Amer. Math. Soc.*, **12** (1999) pp. 335–380.

[15] S. Fomin and A. Zelevinsky. Total positivity: Tests and parametrizations. *Math. Intelligencer*, **22** (2000) pp. 23–33.

[16] I. Gelfand and A. Zelevinsky. Canonical basis in irreducible representations of gl_3 and its applications. In *Group Theoretical Methods in Physics, Vol. II (Jurmala, 1985)*. VNU Sci. Press, Utrecht, 1986 pp. 127–146.

[17] I. Gessel and G. Viennot. Determinants and plane partitions, 1989. Preprint.

[18] F. M. Goodman, P. de la Harpe, and V. Jones. *Coxeter Graphs and Towers of Algebras*. Springer–Verlag, New York, 1989.

[19] I. P. Goulden and D. M. Jackson. Immanants of combinatorial matrices. *J. Algebra*, **148** (1992) pp. 305–324.

[20] C. Greene. Proof of a conjecture on immanants of the Jacobi-Trudi matrix. *Linear Algebra Appl.*, **171** (1992) pp. 65–79.

[21] M. Haiman. Hecke algebra characters and immanant conjectures. *J. Amer. Math. Soc.*, **6** (1993) pp. 569–595.
[22] G. James. Immanants. *Linear and Multilinear Algebra*, **32** (1992) pp. 197–210.
[23] S. Karlin and G. McGregor. Coincidence probabilities. *Pacific J. Math.*, **9** (1959) pp. 1141–1164.
[24] S. Karlin and Y. Rinott. A generalized Cauchy-Binet formula and applications to total positivity and majorization. *J. Multivariate Anal.*, **27** (1988) pp. 284–299.
[25] L. Kauffman. State models and the Jones polynomial. *Topology*, **26** (1987) pp. 395–407.
[26] B. Kostant. Immanant inequalities and 0-weight spaces. *J. Amer. Math. Soc.*, **8** (1995) pp. 181–186.
[27] B. Lindström. On the vector representations of induced matroids. *Bull. London Math. Soc.*, **5** (1973) pp. 85–90.
[28] C. Loewner. On totally positive matrices. *Math. Z.*, **63** (1955) pp. 338–340.
[29] G. Lusztig. Total positivity in reductive groups. In *Lie Theory and Geometry: in Honor of Bertram Kostant*, vol. 123 of *Progress in Mathematics*. Birkhäuser, Boston, 1994 pp. 531–568.
[30] M. Marcus and H. Minc. Generalized matrix functions. *Trans. Amer. Math. Soc.*, **116** (1965) pp. 316–329.
[31] M. Skandera. Inequalities in products of minors of totally nonnegative matrices. *J. Alg. Combin.*, **20** (2004).
[32] R. Stanley. *Enumerative Combinatorics*, vol. 1. Cambridge University Press, Cambridge, 1997.
[33] R. Stanley. *Enumerative Combinatorics*, vol. 2. Cambridge University Press, Cambridge, 1999.
[34] R. Stanley. Positivity problems and conjectures. In *Mathematics: Frontiers and Perspectives* (V. Arnold, M. Atiyah, P. Lax, and B. Mazur, eds.). American Mathematical Society, Providence, RI, 2000 pp. 295–319.
[35] R. Stanley and J. R. Stembridge. On immanants of Jacobi-Trudi matrices and permutations with restricted positions. *J. Combin. Theory Ser. A*, **62** (1993) pp. 261–279.
[36] J. Stembridge. Immanants of totally positive matrices are nonnegative. *Bull. London Math. Soc.*, **23** (1991) pp. 422–428.
[37] J. Stembridge. Some conjectures for immanants. *Can. J. Math.*, **44** (1992) pp. 1079–1099.
[38] J. Tits. Le problème des mots dans les groupes de Coxeter. In *Symposia Mathematica (INDAM, Rome, 1967/68), Vol. 1*, vol. 1. Academic Press, London, 1969 pp. 175–185.
[39] B. W. Westbury. The representation theory of the Temperley-Lieb algebras. *Math. Z.*, **219** (1995) pp. 539–565.

Authors' address: Dept. of Mathematics, 6188 Bradley Hall, Dartmouth College, Hanover, NH 03755-3551

E-mail address: `brhoades@umich.edu, Mark.Skandera@dartmouth.edu`