MINIMAL SUBDYNAMICS AND MINIMAL FLOWS WITHOUT CHARACTERISTIC MEASURES

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Abstract. Given a countable group G and a G-flow X, a probability measure μ on X is called characteristic if it is $\mathrm{Aut}(X, G)$-invariant. Frisch and Tamuz asked about the existence of a minimal G-flow, for any group G, which does not admit a characteristic measure. We construct for every countable group G such a minimal flow. Along the way, we are motivated to consider a family of questions we refer to as minimal subdynamics: Given a countable group G and a collection of infinite subgroups $\{\Delta_i : i \in I\}$, when is there a faithful G-flow for which every Δ_i acts minimally?

Given a countable group G and a faithful G-flow X, we write $\mathrm{Aut}(X, G)$ for the group of homeomorphisms of X which commute with the G-action. When G is abelian, $\mathrm{Aut}(X, G)$ contains a natural copy of G resulting from the G-action, but in general this need not be the case. Much is unknown about how the properties of X restrict the complexity of $\mathrm{Aut}(X, G)$; for instance, Cyr and Kra [1] conjecture that when $G = \mathbb{Z}$ and $X \subseteq 2^{\mathbb{Z}}$ is a minimal, 0-entropy subshift, then $\mathrm{Aut}(X, \mathbb{Z})$ must be amenable. In fact, no counterexample is known even when restricting to any two of the three properties "minimal," "0-entropy," or "subshift." In an effort to shed light on this question, Frisch and Tamuz [3] define a probability measure μ on X to be *characteristic* if it is $\mathrm{Aut}(X, G)$-invariant. They show that 0-entropy subshifts always admit characteristic measures. More recently, Cyr and Kra [2] provide several examples of flows which admit characteristic measures for non-trivial reasons, even in cases where $\mathrm{Aut}(X, G)$ is non-amenable. Frisch and Tamuz asked (Question 1.5, [3]) whether there exists, for any countable group G, some minimal G-flow without a characteristic measure. We give a strong affirmative answer.

Theorem 1. *For any countably infinite group G, there is a free minimal G-flow X so that X does not admit a characteristic measure. More precisely, there is a free $(G \times F_2)$-flow X which is minimal as a G-flow and with no F_2-invariant measure.*

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We remark that the X we construct will not in general be a subshift.

Over the course of proving Theorem 1, there are two main difficulties to overcome. The first difficulty is a collection of dynamical problems we refer to as *minimal subdynamics*. The general template of these questions is as follows.

Question 2. Given a countably infinite group Γ and a collection $\{\Delta_i : i \in I\}$ of infinite subgroups of Γ, when is there a faithful (or essentially free, or free) minimal Γ-flow for which the action of each Δ_i is also minimal? Is there a natural space of actions in which such flows are generic?

In [8], the author showed that this was possible in the case $\Gamma = G \times H$ and $\Delta = G$ for any countably infinite groups G and H. We manage to strengthen this result considerably.

Theorem 3. *For any countably infinite group Γ and any collection $\{\Delta_n : n \in \mathbb{N}\}$ of infinite normal subgroups of Γ, there is a free Γ-flow which is minimal as a Δ_n-flow for every $n \in \mathbb{N}$.*

In fact, what we show when proving Theorem 3 is considerably stronger. Recall that given a countably infinite group Γ, a subshift $X \subseteq 2^\Gamma$ is *strongly irreducible* if there is some finite symmetric $D \subseteq \Gamma$ so that whenever $S_0, S_1 \subseteq \Gamma$ satisfy $DS_0 \cap S_1 = \emptyset$ (i.e. S_0 and S_1 are D-*apart*), then for any $x_0, x_1 \in X$, there is $y \in X$ with $y|_{S_i} = x_i|_{S_i}$ for each $i < 2$. Write $\mathcal{S}$ for the set of strongly irreducible subshifts, and write $\overline{\mathcal{S}}$ for its Vietoris closure. Frisch, Tamuz, and Vahidi-Ferdowsi [5] show that in $\overline{\mathcal{S}}$, the minimal subshifts form a dense G_δ subset. In our proof of Theorem 3, we show that the shifts in $\overline{\mathcal{S}}$ which are Δ_n-minimal for each $n \in \mathbb{N}$ also form a dense G_δ subset.

This brings us to the second main difficulty in the proof of Theorem 1. Using this stronger form of Theorem 3, one could easily prove Theorem 1 by finding a strongly irreducible F_2-subshift which does not admit an invariant measure. This would imply the existence of a strongly irreducible $(G \times F_2)$-subshift without an F_2-invariant measure. As not admitting an F_2-invariant measure is a Vietoris-open condition, the genericity of G-minimal subshifts would then be enough to obtain the desired result. Unfortunately whether such a strongly irreducible subshift can exist (for any non-amenable group) is an open question. To overcome this, we introduce a flexible weakening of the notion of a strongly irreducible shift.

The paper is organized as follows. Section 1 is a very brief background section on subsets of groups, subshifts, and strong irreducibility. Section 2 introduces the notion of a UFO, a useful combinatorial gadget for constructing shifts where subgroups act minimally; Theorem 3 answers Question 3.6 from [8]. Section 3 introduces the notion of $\mathcal{B}$-irreduciblity for any group H, where $\mathcal{B} \subseteq \mathcal{P}_f(H)$ is a right-invariant collection of finite subsets of H. When $H = F_2$, we will be interested in the case when $\mathcal{B}$ is the collection of finite subsets of F_2 which are connected in the standard left Cayley graph. Section 4 gives the proof of Theorem 1.

1. Background

Let Γ be a countably infinite group. Given $U, S \subseteq \Gamma$ with U finite, then we call S a (one-sided) *U-spaced* set if for every $g \neq h \in S$ we have $h \notin Ug$, and we call S a *U-syndetic* set if $US = \Gamma$. A *maximal U-spaced set* is simply a U-spaced set which is maximal under inclusion. We remark that if S is a maximal U-spaced set, then S is $(U \cup U^{-1})$-syndetic. We say that sets $S, T \subseteq \Gamma$ are (one-sided) *U-apart* if $US \cap T = \emptyset$ and $S \cap UT = \emptyset$. Notice that much of this discussion simplifies when U is symmetric, so we will often assume this. Also notice that the properties of being U-spaced, maximal U-spaced, U-syndetic, and U-apart are all right invariant.

If A is a finite set or *alphabet*, then Γ acts on A^Γ by *right shift*, where given $x \in A^\Gamma$ and $g, h \in \Gamma$, we have $(g{\cdot}x)(h) = x(hg)$. A *subshift* of A^Γ is a non-empty, closed, Γ-invariant subset. Let $\mathrm{Sub}(A^\Gamma)$ denote the space of subshifts of A^Γ endowed with the Vietoris topology. This topology can be described as follows. Given $X \subseteq A^\Gamma$ and a finite $U \subseteq \Gamma$, the set of *U-patterns* of X is the set $P_U(X) = \{x|_U : x \in X\} \subseteq A^U$. Then the typical basic open neighborhood of $X \in \mathrm{Sub}(A^\Gamma)$ is the set $N_U(X) := \{Y \in \mathrm{Sub}(A^\Gamma) : P_U(Y) = P_U(X)\}$, where U ranges over finite subsets of Γ.

A subshift $X \subseteq A^\Gamma$ is *U-irreducible* if for any $x_0, x_1 \in X$ and any $S_0, S_1 \subseteq \Gamma$ which are U-apart, there is $y \in X$ with $y|_{S_i} = x_i|_{S_i}$ for each $i < 2$. If X is U-irreducible and $V \supseteq U$ is finite, then X is also V-irreducible. We call X *strongly irreducible* if there is some finite $U \subseteq \Gamma$ with X U-irreducible. By enlarging U if needed, we can always assume U is symmetric. Let $\mathcal{S}(A^\Gamma) \subseteq \mathrm{Sub}(A^\Gamma)$ denote the set of strongly irreducible subshifts of A^Γ, and let $\overline{\mathcal{S}}(A^\Gamma)$ denote the closure of this set in the Vietoris topology.

More generally, if $2^\mathbb{N}$ denotes Cantor space, then Γ acts on $(2^\mathbb{N})^\Gamma$ by right shift exactly as above. If $k < \omega$, we let $\pi_k \colon 2^\mathbb{N} \to 2^k$ denote the restriction to the first k entries. This induces a factor map $\tilde{\pi}_k \colon (2^\mathbb{N})^\Gamma \to (2^k)^\Gamma$ given by $\tilde{\pi}_k(x)(g) = \pi_k(x(g))$; we also obtain a map $\overline{\pi}_k \colon \mathrm{Sub}((2^\mathbb{N})^\Gamma) \to \mathrm{Sub}((2^k)^\Gamma)$ (where 2^k is viewed as a finite alphabet) given by $\overline{\pi}_k(X) = \tilde{\pi}_k[X]$. The Vietoris topology on $\mathrm{Sub}((2^\mathbb{N})^\Gamma)$ is the coarsest topology making every such $\overline{\pi}_k$ continuous. We call a subflow $X \subseteq (2^\mathbb{N})^\Gamma$ *strongly irreducible* if for every $k < \omega$, the subshift $\overline{\pi}_k(X) \subseteq (2^k)^\Gamma$ is strongly irreducible in the ordinary sense. We let $\mathcal{S}((2^\mathbb{N})^\Gamma) \subseteq \mathrm{Sub}((2^\mathbb{N})^\Gamma)$ denote the set of strongly irreducible subflows of $(2^\mathbb{N})^\Gamma$, and we let $\overline{\mathcal{S}}((2^\mathbb{N})^\Gamma)$ denote its Vietoris closure.

The idea of considering the closure of the strongly irreducible shifts has it roots in [4]. This is made more explicit in [5], where it is shown that in $\overline{\mathcal{S}}(A^\Gamma)$, the minimal subflows form a dense G_δ subset. More or less the same argument shows that the same holds in $\overline{\mathcal{S}}((2^\mathbb{N})^\Gamma)$ (see [6]). Recall that a Γ-flow X is *free* if for every $g \in \Gamma \setminus \{1_\Gamma\}$ and every $x \in X$, we have $gx \neq x$.

The main reason for considering a Cantor space alphabet is the following result, which need not be true for finite alphabets.

Proposition 4. *In $\overline{\mathcal{S}}((2^{\mathbb{N}})^\Gamma)$, the free flows form a dense G_δ subset.*

Proof. Fixing $g \in \Gamma$, the set $\{X \in \mathrm{Sub}((2^{\mathbb{N}})^\Gamma) : \forall x \in X\,(gx \neq x)\}$ is open; indeed, if $X_n \to X$ is a convergent sequence in $\mathrm{Sub}((2^{\mathbb{N}})^\Gamma)$ and $x_n \in X_n$ is a point fixed by g, then passing to a subsequence, we may suppose $x_n \to x \in X$, and we have $gx = x$. Intersecting over all $g \in \Gamma \setminus \{1_\Gamma\}$, we see that freeness is a G_δ condition.

Thus it remains to show that freeness is dense in $\overline{\mathcal{S}}((2^{\mathbb{N}})^\Gamma)$. To that end, we fix $g \in \Gamma \setminus \{1_\Gamma\}$ and show that the set of shifts in $\mathcal{S}((2^{\mathbb{N}})^\Gamma)$ where g acts freely is dense. Fix $X \in \mathcal{S}((2^{\mathbb{N}})^\Gamma)$, $k < \omega$, and a finite $U \subseteq \Gamma$; so a typical open set in $\mathcal{S}((2^{\mathbb{N}})^\Gamma)$ has the form $\{X' \in \mathcal{S}((2^{\mathbb{N}})^\Gamma) : P_U(\overline{\pi}_k(X')) = P_U(\overline{\pi}_k(X))\}$. We want to produce $Y \in \mathrm{Sub}((2^{\mathbb{N}})^\Gamma)$ which is strongly irreducible, g-free, and with $P_U(\overline{\pi}_k(Y)) = P_U(\overline{\pi}_k(X))$. In fact, we will produce such a Y with $\overline{\pi}_k(Y) = \overline{\pi}_k(X)$.

Let $D \subseteq \Gamma$ be a finite symmetric set containing g and 1_Γ. Setting $m = |D|$, consider the subshift $\mathrm{Color}(D, m) \subseteq m^\Gamma$ defined by

$$\mathrm{Color}(D, m) := \{x \in m^\Gamma : \forall\, i < m\, [x^{-1}(\{i\}) \text{ is } D\text{-spaced}]\}.$$

A greedy coloring argument shows that $\mathrm{Color}(D, m)$ is non-empty and D-irreducible. Moreover, g acts freely on $\mathrm{Color}(D, m)$. Inject m into $2^{\{k,\ldots,\ell-1\}}$ for some $\ell > k$ and identify $\mathrm{Color}(D, m)$ as a subflow of $(2^{\{k,\ldots,\ell-1\}})^\Gamma$. Then $Y := \overline{\pi}_k(X) \times \mathrm{Color}(D, m) \subseteq (2^\ell)^\Gamma \subseteq (2^{\mathbb{N}})^\Gamma$, where the last inclusion can be formed by adding strings of zeros to the end. Then Y is strongly irreducible, g-free, and $\overline{\pi}_k(Y) = \overline{\pi}_k(X)$. □

2. UFOs and minimal subdynamics

Much of the construction will require us to reason about the product group $G \times F_2$. So for the time being, fix countably infinite groups $\Delta \subseteq \Gamma$. For our purposes, Γ will be $G \times F_2$, and Δ will be G, where we identify G with a subgroup of $G \times F_2$ in the obvious way. However, for this subsection, we will reason more generally.

Definition 5. Let $\Delta \subseteq \Gamma$ be countably infinite groups. A finite subset $U \subseteq \Gamma$ is called a (Γ, Δ)-*UFO* if for any maximal U-spaced set $S \subseteq \Gamma$, we have that S meets every right coset of Δ in Γ.

We say that the inclusion of groups $\Delta \subseteq \Gamma$ *admits UFOs* if for every finite $U \subseteq \Gamma$, there is a finite $V \subseteq \Gamma$ with $V \supseteq U$ which is a (Γ, Δ)-UFO.

As a word of caution, we note that the property of being a (Γ, Δ)-UFO is not upwards closed.

The terminology comes from considering the case of a product group, i.e. $\Gamma = \mathbb{Z} \times \mathbb{Z}$ and $\Delta = \mathbb{Z} \times \{0\}$. Figure 1 depicts a typical UFO subset of $\mathbb{Z} \times \mathbb{Z}$.

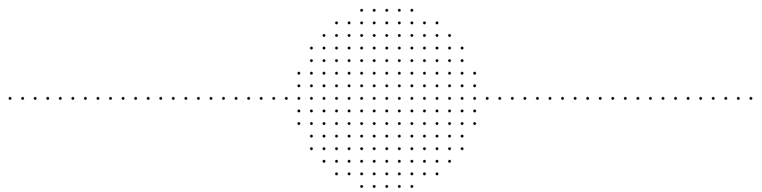

FIGURE 1. Sighting in Roswell; a $(\mathbb{Z}\times\mathbb{Z}, \mathbb{Z}\times\{0\})$-UFO subset of $\mathbb{Z}\times\mathbb{Z}$.

Proposition 6. *Let Δ be a subgroup of Γ. If $|\bigcap_{u\in U} u\Delta u^{-1}|$ is infinite for every finite set $U\subseteq\Gamma$ then $\Delta\subseteq\Gamma$ admits UFOs. In particular, if Δ contains an infinite subgroup that is normal in Γ then $\Delta\subseteq\Gamma$ admits UFOs.*

Proof. We prove the contrapositive. So assume that $\Delta\subseteq\Gamma$ does not admit UFOs. Let $U\subseteq\Gamma$ be a finite symmetric set such that no finite $V\subseteq\Gamma$ containing U is a (Γ,Δ)-UFO. Let $D\subseteq\Delta$ be finite, symmetric, and contain the identity. It will suffice to show that $C=\bigcap_{u\in U} uDu^{-1}$ satisfies $|C|\leq|U|$.

Set $V=U\cup D^2$. Since V is not a (Γ,Δ)-UFO, there is a maximal V-spaced set $S\subseteq\Gamma$ and $g\in\Gamma$ with $S\cap\Delta g=\varnothing$. Since S is V-spaced and $u^{-1}C^2u\subseteq D^2\subseteq V$, the set $C_u=(uS)\cap(Cg)$ is C^2-spaced for every $u\in U$. Of course, any C^2-spaced subset of Cg is empty or a singleton, so $|C_u|\leq 1$ for each $u\in U$. On the other hand, since S is maximal we have $VS=\Gamma$, and since $S\cap\Delta g=\varnothing$ we must have $Cg\subseteq US$. Therefore $|C|=|Cg|=\sum_{u\in U}|C_u|\leq|U|$. □

In the spaces $\overline{\mathcal{S}}(k^\Gamma)$ and $\overline{\mathcal{S}}((2^{\mathbb{N}})^\Gamma)$, the minimal flows form a dense G_δ. However, when $\Delta\subseteq\Gamma$ is a subgroup, we can ask about the properties of members of $\overline{\mathcal{S}}(k^\Gamma)$ and $\overline{\mathcal{S}}((2^{\mathbb{N}})^\Gamma)$ viewed as Δ-flows.

Definition 7. Given a subshift $X\subseteq k^\Gamma$ and a finite $E\subseteq\Gamma$, we say that X is (Δ,E)-*minimal* if for every $x\in X$ and every $p\in P_E(X)$, there is $g\in\Delta$ with $(gx)|_E=p$. Given a subflow $X\subseteq(2^{\mathbb{N}})^\Gamma$ and $n\in\mathbb{N}$, we say that X is (Δ,E,n)-*minimal* if $\overline{\pi}_n(X)\subseteq(2^n)^\Gamma$ is (Δ,E)-minimal. When $\Delta=\Gamma$, we simply say that X is E-*minimal* or (E,n)-*minimal*.

The set of (Δ,E)-minimal flows is open in $\mathrm{Sub}(k^\Gamma)$, and $X\subseteq k^\Gamma$ is minimal as a Δ-flow iff it is (Δ,E)-minimal for every finite $E\subseteq\Gamma$. Similarly, the set of (Δ,E,n)-minimal flows is open in $\mathrm{Sub}((2^{\mathbb{N}})^\Gamma)$, and $X\subseteq(2^{\mathbb{N}})^\Gamma$ is minimal as a Δ-flow iff it is (Δ,E,n) minimal for every finite $E\subseteq\Gamma$ and every $n\in\mathbb{N}$.

In the proof of Proposition 8, it will be helpful to extend conventions about the shift action to subsets of Γ. If $U\subseteq\Gamma$, $g\in G$, and $p\in k^U$, we write $g\cdot p\in k^{Ug^{-1}}$ for the function where given $h\in Ug^{-1}$, we have $(g\cdot p)(h)=p(hg)$.

Proposition 8. *Suppose $\Delta \subseteq \Gamma$ are countably infinite groups and that $\Delta \subseteq \Gamma$ admits UFOs. Then $\{X \in \overline{\mathcal{S}}(k^\Gamma) : X$ is minimal as a Δ-flow$\}$ is a dense G_δ subset. Similarly, $\{X \in \overline{\mathcal{S}}(2^\mathbb{N})^\Gamma : X$ is minimal as a Δ-flow$\}$ is a dense G_δ subset.*

Proof. We give the arguments for k^Γ, as those for $(2^\mathbb{N})^\Gamma$ are very similar.

It suffices to show for a given finite $E \subseteq \Gamma$ that the collection of (Δ, E)-minimal flows is dense in $\overline{\mathcal{S}}(k^\Gamma)$. By enlarging E if needed, we can assume that E is symmetric.

Consider a non-empty open $O \subseteq \overline{\mathcal{S}}(k^\Gamma)$. By shrinking O and/or enlarging E if needed, we can assume that for some $X \in \mathcal{S}(k^\Gamma)$, we have $O = N_E(X) \cap \overline{\mathcal{S}}(k^\Gamma)$. We will build a (Δ, E)-minimal shift Y with $Y \in N_E(X) \cap \mathcal{S}(k^\Gamma)$. Fix a finite symmetric $D \subseteq \Gamma$ so that X is D-irreducible. Then fix a finite $U \subseteq \Gamma$ which is large enough to contain an EDE-spaced set $Q \subseteq U \cap \Delta$ of cardinality $|P_E(X)|$, and enlarging U if needed, choose such a Q with $EQ \subseteq U$. Fix a bijection $Q \to P_E(X)$ by writing $P_E(X) = \{p_g : g \in Q\}$. Because X is D-irreducible, we can find $\alpha \in P_U(X)$ so that $(gq)|_E = p_g$ for every $g \in Q$. By Proposition 6, fix a finite $V \subseteq \Gamma$ with $V \supseteq UDU$ which is a (Γ, Δ)-UFO. We now form the shift

$$Y = \{y \in X : \exists \text{ a max. } V\text{-spaced set } T \text{ so that } \forall g \in T\, (g{\cdot}y)|_U = \alpha\}.$$

Because $V = UDU$ and X is D-irreducible, we have that $Y \neq \emptyset$. In particular, for any maximal V-spaced set $T \subseteq \Gamma$, we can find $y \in Y$ so that $(gy)|_U = \alpha$ for every $g \in T$. We also note that $Y \in N_E(X)$ by our construction of α.

To see that Y is (Δ, E)-minimal, fix $y \in Y$ and $p \in P_E(Y)$. Suppose this is witnessed by the maximal V-spaced set $T \subseteq \Gamma$. Because V is a (Γ, Δ)-UFO, find $h \in \Delta \cap T$. So $(hy)|_U = \alpha$. Now suppose $g \in Q$ is such that $p = p_g$. We have $(ghy)|_E = (g \cdot ((hy)|_U)|_E = p_g$.

To see that $Y \in \mathcal{S}(k^\Gamma)$, we will show that Y is $DUVUD$-irreducible. Suppose $y_0, y_1 \in Y$ and $S_0, S_1 \subseteq \Gamma$ are $DUVUD$-apart. For each $i < 2$, fix $T_i \subseteq \Gamma$ a maximal V-spaced set which witnesses that y_i is in Y. Set $B_i = \{g \in T_i : DUg \cap S_i \neq \emptyset\}$. Notice that $B_i \subseteq UDS_i$. It follows that $B_0 \cup B_1$ is V-spaced, so extend to a maximal V-spaced set B. It also follows that $S_i \cup UB_i \subseteq U^2DS_i$. Since $V \supseteq UDU$ and by the definition of B_i, the collection of sets $\{S_i \cup UB_i : i < 2\} \cup \{Ug : g \in B \setminus (B_0 \cup B_1)\}$ is pairwise D-apart. By the D-irreducibility of X, we can find $y \in X$ with $y|_{S_i \cup UB_i} = y_i|_{S_i \cup UB_i}$ for each $i < 2$ and with $(gy)|_U = \alpha$ for each $g \in B \setminus (B_0 \cup B_1)$. Since $B_i \subseteq T_i$, we actually have $(gy)|_U = \alpha$ for each $g \in B$. So $y \in Y$ and $y|_{S_i} = y_i|_{S_i}$ as desired. ☐

Proof of Theorem 3. By Proposition 8, the generic member of $\overline{\mathcal{S}}((2^\mathbb{N})^\Gamma)$ is minimal as a Δ_n-flow for each $n \in \mathbb{N}$, and by Proposition 4, the generic member of $\overline{\mathcal{S}}((2^\mathbb{N})^\Gamma$ is free. ☐

In contrast to Theorem 1, the next example shows that Question 2 is non-trivial to answer in full generality.

Theorem 9. *Let $G = \sum_{\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})$ and let X be a G flow with infinite underlying space. Then there exists an infinite subgroup H such that X is not minimal as an H flow.*

Proof. We may assume that X is a minimal G-flow, as otherwise we may take $H = G$. We construct a sequence $X \supsetneq K_0 \supseteq K_1 \supseteq \cdots$ of proper, non-empty, closed subsets of X and a sequence of group elements $\{g_n : n \in \mathbb{N}\}$ so that by setting $K = \bigcap_{\mathbb{N}} K_n$ and $H = \langle g_n : n \in \mathbb{N}\rangle$, then K will be a minimal H-flow. Start by fixing a closed, proper subset $K_0 \subsetneq X$ with non-empty interior. Suppose K_n has been created and is $\langle g_0, ..., g_{n-1}\rangle$-invariant. As X is a minimal G-flow, the set $S_n := \{g \in G : \mathrm{Int}(gK_n \cap K_n) \neq \emptyset\}$ is infinite. Pick any $g_n \in S_n \setminus \{1_G\}$, and set $K_{n+1} = g_n K_n \cap K_n$. As $g_n^2 = 1_G$, we see that K_{n+1} is g_n-invariant, and as G is abelian, we see that K_{n+1} is also g_i-invariant for each $i < n$. It follows that K will be H-invariant as desired. □

Before moving on, we give a conditional proof of Theorem 1, which works as long as some non-amenable group admits a strongly irreducible shift without an invariant measure. It is the inspiration for our overall construction.

Proposition 10. *Let G and H be countably infinite groups, and suppose that for some $k < \omega$ and some strongly irreducible flow $Y \subseteq k^H$ that Y does not admit an H-invariant measure. Then there is a minimal G-flow which does not admit a characteristic measure.*

Proof. Viewing $Z = k^G \times Y$ as a subshift of $k^{G\times H}$, then Z is strongly irreducible and does not admit an H-invariant probability measure. The property of not possessing an H-invariant measure is an open condition in $\mathrm{Sub}(k^{G\times H})$; indeed, if $X_n \to X$ is a convergent sequence in $\mathrm{Sub}(k^{G\times H})$ and μ_n is an H-invariant probability measure supported on X_n, then by passing to a subsequence, we may suppose that the μ_n weak*-converge to some H-invariant probability measure μ supported on X. By Proposition 8, we can therefore find $X \subseteq k^{G\times H}$ which is minimal as a G-flow and which does not admit an H-invariant measure. As H acts by G-flow automorphisms on X, we see that X does not admit a characteristic measure. □

Unfortunately, the question of if there exists any countable group H and a strongly irreducible H-subshift Y with no H-invariant measure is an open problem. Therefore our construction proceeds by considering the free group F_2 and defining a suitable weakening of strongly irreducible subshift which is strong enough for G-minimality to be generic in $(G \times F_2)$-subshifts, but weak enough for $(G \times F_2)$-subshifts without F_2-invariant measures to exist.

3. Variants of strong irreducibility

In this section, we investigate a weakening of strong irreducibility that one can define given any right-invariant collection $\mathcal{B}$ of finite subsets of a given countable group. For our overall construction, we will consider F_2 and $G \times F_2$, but we give the definitions for any countably infinite group Γ. Write $\mathcal{P}_f(\Gamma)$ for the collection of finite subsets of Γ.

Definition 11. Fix a right-invariant subset $\mathcal{B} \subseteq \mathcal{P}_f(\Gamma)$. Given $k \in \mathbb{N}$, we say that a subshift $X \subseteq k^\Gamma$ is *$\mathcal{B}$-irreducible* if there is a finite $D \subseteq \Gamma$ so that for any $m < \omega$, any $B_0, ..., B_{m-1} \in \mathcal{B}$, and any $x_0, ..., x_{m-1} \in X$, if the sets $\{B_0, ..., B_{m-1}\}$ are pairwise D-apart, then there is $y \in X$ with $y|_{B_i} = x_i|_{B_i}$ for each $i < m$. We call D the *witness* to $\mathcal{B}$-irreducibility. If we have D in mind, we can say that X is *$\mathcal{B}$-D-irreducible.*

We say that a subflow $X \subseteq (2^\mathbb{N})^\Gamma$ is *$\mathcal{B}$-irreducible* if for each $k \in \mathbb{N}$, the subshift $\overline{\pi}_k(X) \subseteq (2^k)^\Gamma$ is $\mathcal{B}$-irreducible.

We write $\mathcal{S}_\mathcal{B}(k^\Gamma)$ or $\mathcal{S}_\mathcal{B}((2^\mathbb{N})^\Gamma)$ for the set of $\mathcal{B}$-irreducible subflows of k^Γ or $(2^\mathbb{N})^\Gamma$, respectively, and we write $\overline{\mathcal{S}}_\mathcal{B}(k^\Gamma)$ or $\overline{\mathcal{S}}_\mathcal{B}((2^\mathbb{N})^\Gamma)$ for the Vietoris closures.

Remark.

(1) If $\mathcal{B}$ is closed under unions, it is enough to consider $m = 2$. However, this will often not be the case.
(2) By compactness, if $X \subseteq k^\Gamma$ is $\mathcal{B}$-D-irreducible, $\{B_n : n < \omega\} \subseteq \mathcal{B}$ is pairwise D-apart, and $\{x_n : n < \omega\} \subseteq X$, then there is $y \in X$ with $y|_{B_i} = x_i|_{B_i}$.
(3) If $\mathcal{B} \subseteq \mathcal{B}'$, then $\mathcal{S}_{\mathcal{B}'}(k^\Gamma) \subseteq \mathcal{S}_\mathcal{B}(k^\Gamma)$ and $\mathcal{S}_{\mathcal{B}'}((2^\mathbb{N})^\Gamma) \subseteq \mathcal{S}_\mathcal{B}((2^\mathbb{N})^\Gamma)$

When $\mathcal{B}$ is the collection of all finite subsets of H, then we recover the notion of a strongly irreducible shift. Again, we consider Cantor space alphabets to obtain freeness.

Proposition 12. *For any right-invariant collection $\mathcal{B} \subseteq \mathcal{P}_f(\Gamma)$, the generic member of $\overline{\mathcal{S}}_\mathcal{B}((2^\mathbb{N})^\Gamma)$ is free.*

Proof. Analyzing the proof of Proposition 4, we see that the only properties that we need of the collections $\mathcal{S}_\mathcal{B}(k^\Gamma)$ and $\mathcal{S}_\mathcal{B}((2^\mathbb{N})^\Gamma)$ for the proof to generalize are that they are closed under products and contain the flows $\mathrm{Color}(D, m)$. If $k, \ell \in \mathbb{N}$ an $X \subseteq k^\Gamma$ and $Y \subseteq \ell^\Gamma$ are $\mathcal{B}$-D-irreducible and $\mathcal{B}$-E-irreducible for some finite $D, E \subseteq \Gamma$, then $X \times Y \subseteq (k \times \ell)^\Gamma$ will be $\mathcal{B}$-$(D \cup E)$-irreducible. And as $\mathrm{Color}(D, m)$ is strongly irreducible, it is $\mathcal{B}$-irreducible. □

Now we consider the group F_2. We consider the left Cayley graph of F_2 with respect to the standard generating set $A := \{a, b, a^{-1}, b^{-1}\}$. We let

$d\colon F_2 \times F_2 \to \omega$ denote the graph metric. Write $D_n = \{s \in F_2 : d(s, 1_{F_2}) \leq n\}$.

Definition 13. Given n with $1 \leq n < \omega$, we set

$$\mathcal{B}_n = \{D \in \mathcal{P}_f(F_2) : \text{ connected components of } D \text{ are pairwise } D_n\text{-apart}\}.$$

Write $\mathcal{B}_\omega$ for the collection of finite, connected subsets of F_2.

Proposition 14. *Suppose $X \subseteq k^{F_2}$ is $\mathcal{B}_\omega$-irreducible. Then there is some $n < \omega$ for which X is $\mathcal{B}_n$-irreducible.*

Proof. Suppose X is $\mathcal{B}_\omega$-D_n-irreducible. We claim X is $\mathcal{B}_n$-D_n-irreducible. Suppose $m < \omega$, $B_0, ..., B_{m-1} \in \mathcal{B}_n$ are pairwise D_n-apart, and $x_0, ..., x_{m-1} \in X$. For each $i < m$, we suppose B_i has n_i-many connected componenets, and we write $\{C_{i,j} : j < n_i\}$ for these components. Then the collection of connected sets $\bigcup_{i<m}\{C_{i,j} : j < n_i\}$ is pairwise D_n-apart. As X is $\mathcal{B}_\omega$-D_n-irreducible, we can find $y \in X$ so that for each $i < m$ and $j < n_i$, we have $y|_{C_{i,j}} = x_i|_{C_{i,j}}$. Hence $y|_{B_i} = x_i|_{B_i}$, showing that X is $\mathcal{B}_n$-D_n-irreducible. $\square$

We now construct a $\mathcal{B}_\omega$-irreducible subshift with no F_2-invariant measure. We consider the alphabet A^2, and write $\pi_0, \pi_1\colon A^2 \to A$ for the projections. We set

$$\begin{aligned} X_{pdox} = \{x \in (A^2)^{F_2} : &\forall g, h \in F_2\, \forall i, j < 2 \\ &(i, g) \neq (j, h) \Rightarrow \pi_i(x(g)) \cdot g \neq \pi_j(x(h)) \cdot h\}. \end{aligned}$$

More informally, the flow X_{pdox} is the space of "2-to-1 paradoxical decompositions" of F_2 using A. We remark that here, our decomposition need not be a partition of F_2; we just ask for disjoint $S_0, S_1 \subseteq F_2$ such that for every $g \in G$ and $i < 2$, we have $Ag \cap S_i \neq \emptyset$. This is in some sense the prototypical example of an F_2-shift with no F_2-invariant measure.

Lemma 15. *X_{pdox} has no F_2-invariant measure.*

Proof. For $u \in A^2$ set $Y_u = \{x \in X_{pdox} : x(1_G) = u\}$. Notice that if $y \in Y_u$, $i < 2$, and $x = \pi_i(u)y$, then $x(\pi_i(u)^{-1}) = y(1_G) = u$. Consequently, if $u, v \in A^2$, $x \in \pi_i(u)Y_u \cap \pi_j(v)Y_v$ then, since $x \in X_{pdox}$ and

$$\pi_i(x(\pi_i(u)^{-1}))\pi_i(u)^{-1} = 1_G = \pi_j(x(\pi_j(v)^{-1}))\pi_j(v)^{-1},$$

we must have that $(i, \pi_i(u)) = (j, \pi_j(v))$, and hence also

$$\pi_{1-i}(u) = \pi_{1-i}(x(\pi_i(u)^{-1})) = \pi_{1-j}(x(\pi_j(v)^{-1})) = \pi_{1-j}(v).$$

Therefore $\pi_i(u)Y_u \cap \pi_j(v)Y_v = \varnothing$ whenever $(i, u) \neq (j, v)$.

If μ were an invariant Borel probability measure on X_{pdox} then we would have

$$2\mu(X_{pdox}) = 2\sum_{u \in A^2} \mu(Y_u) = \sum_{i<2}\sum_{u \in A^2} \mu(\pi_i(u)Y_u) \leq \mu(X)$$

which is a contradiction. □

When proving that X_{pdox} is $\mathcal{B}_\omega$-irreducible, note that $D_1 = A \cup \{1_{F_2}\}$.

Proposition 16. *X_{pdox} is $\mathcal{B}_\omega$-D_4-irreducible.*

Proof. The proof will use a 2-to-1 instance of Hall's matching criterion [7] which we briefly describe. Fix a bipartite graph $\mathbb{G} = (V, E)$ with partition $V = V_0 \sqcup V_1$. Given $S \subseteq V_0$, write $N_\mathbb{G}(S) = \{v \in V_1 : \exists u \in S(u, v) \in E\}$. Then the matching condition we need states that if for every finite $S \subseteq V_0$, we have $|N_\mathbb{G}(S)| \geq 2S$, then there is $E' \subseteq E$ so that in the graph $\mathbb{G}' := (V, E')$, $d_{\mathbb{G}'}(u) = 2$ for every $u \in V_0$.

Let $B_0, ..., B_{k-1} \in \mathcal{B}_\omega$ be pairwise D_4-apart. Let $x_0, ..., x_{k-1} \in X_{pdox}$. To construct $y \in X_{pdox}$ with $y|_{B_i} = x_i|_{B_i}$ for each $i < k$, we need to verify a 2-to-1 Hall's matching criterion on every finite subset of $F_2 \setminus \bigcup_{i<k} B_i$. Call $s \in F_2$ *matched* if for some $i < k$, some $g \in B_i$, and some $j < 2$, we have $s = \pi_j(x_i(g)) \cdot g$. So we need for every finite $E \in \mathcal{P}_f(F_2 \setminus \bigcup_{i<k} B_i)$ that AE contains at least $2|E|$-many unmatched elements. Towards a contradiction, let $E \in \mathcal{P}_f(F_2 \setminus \bigcup_{i<k} B_i)$ be a minimal failure of the Hall condition.

In the left Cayley graph of F_2, given a reduced word w in alphabet $A = \{a, b, a^{-1}, b^{-1}\}$, write N_w for the set of reduced words which *end* with w. Now find $t \in E$ (let us assume the leftmost character of t is a) so that all of $E \cap N_{at}$, $E \cap N_{bt}$ and $E \cap N_{b^{-1}t}$ are empty. If any two of at, bt and $b^{-1}t$ is an unmatched point in AE, then $E \setminus \{t\}$ is a smaller failure of Hall's criterion. So there must be some $i < k$, some $g \in B_i$, and some $j < 2$, we have $\pi_j(x_i(g)) \cdot g \in \{at, bt, b^{-1}t\}$. Let us suppose $\pi_j(x_i(g)) \cdot g = at$. Note that since $g \notin E$, we must have $g \in \{bat, a^2t, b^{-1}at\}$. But then since B_i is connected, we have $D_1 B_i \cap \{bt, b^{-1}t\} = \emptyset$, and since the other B_q are at least distance 5 from B_i, we have $D_1 B_q \cap \{bt, b^{-1}t\} = \emptyset$ for every $q \in k \setminus \{i\}$. In particular, bt and $b^{-1}t$ are unmatched points in AE, a contradiction. □

We remark that X_{pdox} is not D_n-irreducible for any $n \in \mathbb{N}$. See Figure 2.

4. The construction

Our goal for the rest of the paper is to use X_{pdox} to build a subflow of $(2^\mathbb{N})^{G \times F_2}$ which is free, G-minimal, and with no F_2-invariant measure. In what follows, given an F_2-coset $\{g\} \times F_2$, we endow this coset with the left Cayley graph for F_2 using the generating set A exactly as above. We extend the definition of $\mathcal{B}_n$ to refer to finite subsets of any given F_2-coset.

Definition 17. Given n with $1 \leq n \leq \omega$, we set

$$\mathcal{B}_n^* = \{D \in \mathcal{P}_f(G \times F_2) : \text{ for each } F_2\text{-coset } C,\ D \cap C \in \mathcal{B}_n\}.$$

Given $y \in k^{G \times F_2}$ and $g \in G$, we define $y_g \in k^{F_2}$ where given $s \in F_2$, we set $y_g(s) = y(g, s)$. If $X \subseteq k^{F_2}$ is $\mathcal{B}_n$-irreducible, then the subshift $X^G \subseteq k^{G \times F_2}$

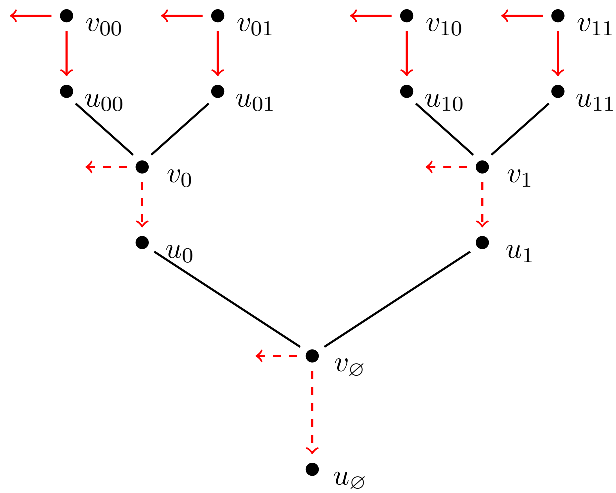

FIGURE 2. A pair of outgoing edges, drawn in solid red, is chosen at each of v_{00}, v_{01}, v_{10}, and v_{11}. Edges which must consequently be oriented in a particular direction are indicated with dashed red arrows. Most importantly, $v_\varnothing$ is forced to direct an edge to $u_\varnothing$. By considering the generalization of this picture for any length of binary string, we see that X_{pdox} cannot be D_n-irreducible for any $n \in \mathbb{N}$.

is in $\mathcal{S}_{\mathcal{B}_n^*}$, where we view X^G as the set $\{y \in k^{G\times F_2} : \forall g \in G\,(y_g \in X)\}$. In particular, $(X_{pdox})^G$ is $\mathcal{B}_4^*$-irreducible. By encoding $(X_{pdox})^G$ as a subshift of $(2^m)^{G\times F_2}$ for some $m \in \mathbb{N}$ and considering $\tilde{\pi}_m^{-1}((X_{pdox})^G) \subseteq (2^{\mathbb{N}})^{G\times F_2}$, we see that there is a $\mathcal{B}_4^*$-irreducible subflow of $(2^{\mathbb{N}})^{G\times F_2}$ for which the F_2-action doesn't fix a measure. It follows that such subflows constitute a non-empty open subset of $\Phi := \overline{\bigcup_n \mathcal{S}_{\mathcal{B}_n^*}((2^{\mathbb{N}})^{G\times F_2})}$. Combining the next result with Proposition 12, we will complete the proof of Theorem 1.

Proposition 18. *With Φ as above, the G-minimal flows are dense G_δ in Φ.*

Proof. We show the result for $\Phi_k := \overline{\bigcup_n \mathcal{S}_{\mathcal{B}_n^*}(k^{G\times F_2})}$ to simplify notation; the proof in full generality is almost identical.

We only need to show density. To that end, fix a finite symmetric $E \subseteq G \times F_2$ which is connected in each F_2-coset. It is enough to show that the (G, E)-minimal subshifts are dense in Φ_k. Fix some non-empty open $O \subseteq \Phi_k$. By enlarging E and/or shrinking O, we may assume that for some $n < \omega$ and $X \in \mathcal{S}_{\mathcal{B}_n^*}(k^{G\times F_2})$ that $O = \{X' \in \Phi_k : P_E(X') = P_E(X)\}$. We will build a (G, E)-minimal subshift $Y \subseteq k^{G\times F_2}$ so that $P_E(Y) = P_E(X)$ and so that for some $N < \omega$, we have $Y \in \mathcal{S}_{\mathcal{B}_N^*}(k^{G\times F_2})$.

Recall that $D_n \subseteq F_2$ denotes the ball of radius n. Fix a finite, symmetric $D \subseteq G \times F_2$ so that $\{1_G\} \times D_{2n} \subseteq D$ and X is $\mathcal{B}_n^*$-D-irreducible. Find a finite symmetric $U_0 \subseteq G$ with $1_G \subseteq U_0$ and $r < \omega$ so that upon setting $U = U_0 \times D_r \subseteq G \times F_2$, then U is large enough to contain an EDE-spaced set $Q \subseteq G$ with $EQ \subseteq U$. As X is $\mathcal{B}_n^*$-D-irreducible, there is a pattern $\alpha \in P_U(X)$ so that $\{(g\alpha)|_E : g \in Q\} = P_E(X)$.

Let $V \supseteq UD^2U$ be a $(G \times F_2, G)$-UFO. We remark that for most of the remainder of the proof, it would be enough to have $V \supseteq UDU$; we only use the stronger assumption $V \supseteq UD^2U$ in the proof of the final claim. Consider the following subshift:

$$Y = \{y \in X : \exists \text{ a max. } V\text{-spaced set } T \text{ so that } \forall g \in T\, (gy)|_U = \alpha\}.$$

The proof that Y is non-empty and (G, E)-minimal is exactly the same as the analogous proof from Proposition 8. Note that by construction, we have $P_E(Y) = P_E(X)$.

We now show that $Y \in \mathcal{S}_{\mathcal{B}_N^*}(k^{G \times F_2})$ for $N = 4r + 3n$. Set $W = DUVUD$. We show that Y is $\mathcal{B}_N^*$-W-irreducible. Suppose $m < \omega$, $y_0, ..., y_{m-1} \in Y$ and $S_0, ..., S_{m-1} \in \mathcal{B}_N^*$ are pairwise W-apart. Suppose for each $i < m$ that $T_i \subseteq G \times F_2$ is a maximal V-spaced set which witness that $y_i \in Y$. Set $B_i = \{g \in T_i : DUg \cap S_i \neq \emptyset\}$. Then $\bigcup_{i<m} B_i$ is V-spaced, so enlarge to a maximal V-spaced set $B \subseteq G \times F_2$.

For each $i < m$, we enlarge $S_i \cup UB_i$ to $J_i \in \mathcal{B}_n^*$ as follows. Suppose $C \subseteq G \times F_2$ is an F_2-coset. Each set of the form $C \cap Ug$ is connected. Since $S_i \in \mathcal{B}_N^*$, it follows that given $g \in B_i$, there is at most one connected component $\Theta_{C,g}$ of $S_i \cap C$ with $Ug \cap \Theta_{C,g} = \emptyset$, but $Ug \cap D_n\Theta_{C,g} \neq \emptyset$. We add the line segment in C connecting $\Theta_{C,g}$ and Ug. Upon doing this for each $g \in B_i$ and each F_2-coset C, this completes the construction of J_i. Observe that $J_i \subseteq D_{n-1}S_i \cap UB_i$.

Claim. Let C be an F_2-coset, and suppose Y_0 is a connected component of $S_i \cap C$. Let Y be the connected component of $J_i \cap C$ with $Y_0 \subseteq Y$. Then $Y \subseteq D_{2r+n}Y_0$. In particular, if $Y_0 \neq Z_0$ are two connected components of $S_i \cap C$, then Y_0 and Z_0 do not belong to the same component of $J_i \cap C$.

Proof. Let $L = \{x_j : j < \omega\} \subseteq C$ be a ray with $x_0 \in Y_0$ and $x_j \notin Y_0$ for any $j \geq 1$. Then $\{j < \omega : x_j \in J_i \cap C\}$ is some finite initial segment of ω. We want to argue that for some $j \leq 2r + n + 1$, we have $x_j \notin J_i \cap C$. First we argue that if $x_n \in J_i \cap C$, then $x_n \in UB_i$. Otherwise, we must have $x_n \in D_{n-1}S_i$. But since $x_n \notin D_{n-1}Y_0$, there must be another component Y_1 of $S_i \cap C$ with $x_n \in D_nY_1$. But this implies that Y_0 and Y_1 are not D_{2n-1}-apart, a contradiction since $2n - 1 \leq 4r - 3n = N$.

Fix $g \in B_i$ with $x_n \in Ug$. Let $q < \omega$ be least with $q > n$ and $x_q \notin U_g$. We must have $q \leq 2r + n + 1$. We claim that $x_q \notin J_i \cap C$. Towards a contradiction, suppose $x_q \in J_i \cap C$. We cannot have $x_q \in UB_i$, so we must have $x_q \in D_{n-1}S_i$. But now there must be some component Y_1 of $S_i \cap C$ with $x_q \in D_{n-1}Y_1$. But then $D_{2r+2n}Y_0 \cap Y_1 \neq \emptyset$, a contradiction as Y_0 and Y_1 are D_N-apart. This concludes the proof that $Y \subseteq D_{2r+n}Y_0$.

Now suppose $Y_0 \neq Z_0$ are two connected components of $S_i \cap C$. Then Y_0 and Z_0 are N-apart. In particular, $Z_0 \not\subseteq D_{2r+n}Y_0$, so cannot belong to the same connected component of $J_i \cap C$ as Y_0. □

Claim. $J_i \in \mathcal{B}_n^*$.

Proof. Fix an F_2-coset C and two connected components $Y \neq Z$ of $J_i \cap C$. By the previous claim, each of Y and Z can only contain at most one non-empty component of $S_i \cap C$. The claim will be proven after considering three cases.

(1) First suppose each of Y and Z contain a non-empty component of $S_i \cap C$, say $Y_0 \subseteq Y$ and $Z_0 \subseteq Z$. Then since Y_0 and Z_0 are D_{4r+3n}-apart, the previous claim implies that Y and Z are D_n-apart.
(2) Now suppose Y contains a non-empty component Y_0 of $S_i \cap C$ and that Z does not. Then for some $g \in B_i$, we have $Z = Ug \cap C$. Towards a contradiction, suppose $D_nY \cap Ug \neq \emptyset$. Let $L = \{x_j : j \leq M\}$ be the line segment connecting Y and Ug with $L \cap Y = \{x_0\}$ and $L \cap Ug = \{x_M\}$. We must have $M \leq n$. We cannot have $x_0 \in UB_i$, so we must have $x_0 \in D_{n-1}S_i$. This implies that $x_0 \in D_{n-1}Y_0$. We cannot have $x_0 \in Y_0$, as otherwise, we would have connected Y_0 and $Ug \cap C$ when constructing J_i. It follows that for some $h \in B_i$, we have that x_0 is on the line segment $L' = \{x'_j : j \leq M'\}$ connecting Y_0 and $Uh \cap C$, and we have $M' \leq n$. But this implies that $Ug \cap D_{2n}Uh \neq \emptyset$, a contradiction since $V \supseteq UDU$ and $D \supseteq D_{2n}$.
(3) If neither Y nor Z contain a component of $S_i \cap C$, then there are $g \neq h \in B_i$ with $Y = Uh \cap C$ and $Z = Ug \cap C$. It follows that Y and Z are D_n-apart. □

Claim. Suppose $i \neq j < m$. Then J_i and J_j are D-apart.

Proof. We have that $J_i \subseteq D_{n-1}S_i \cup UB_i$, and likewise for j. As $UB_i \subseteq U^2DS_i$ and as $D \supseteq D_{2n}$, we have $J_i \subseteq U^2DS_i$, and likewise for j. As S_i and S_j are W-apart and as $V \supseteq UDU$, we see that J_i and J_j are D-apart. □

Claim. Suppose $g \in B \setminus \bigcup_{i<m} B_i$. Then Ug and J_i are D-apart for any $i < m$.

Proof. As $g \notin B_i$, we have U_g and S_i are D-apart. Also, for any $h \in B$ with $g \neq h$, we have that Ug and Uh are D-apart. Now suppose $DUg \cap J_i \neq \emptyset$. If $x \in DUg \cap J_i$, then on the coset $C = F_2x$, x must belong on the line between a component of $S_i \cap C$ and Uh for some $h \in B_i$. Furthermore, we have $x \in D_{n-1}Uh$. But since $D_{2n} \subseteq D$, this contradicts that Ug and Uh are D^2-apart (using the full assumption $V \supseteq UD^2U$). □

We can now finish the proof of Proposition 18. The collection $\{J_i : i < m\} \cup \{Ug : g \in B \setminus (\bigcup_{i<m} B_i)\}$ is a pairwise D-apart collection of members of $\mathcal{B}_n^*$. As X is $\mathcal{B}_n^*$-D-irreducible, we can find $y \in X$ with $y|_{J_i} = y_i|_{J_i}$ for each $i < m$ and with $(gy)|_U = \alpha$ for each $g \in B \setminus (\bigcup_{i<m} B_i)$. As $J_i \supseteq UB_i$

and since $B_i \subseteq T_i$, we actually have $(gy)|_U = \alpha$ for each $g \in B$. As B is a maximal V-spaced set, it follows that $y \in Y$ and $y|_{S_i} = y_i|_{S_i}$ as desired. $\square$

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