SOLUTIONS

Exam 2, Mathematics 109
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Name:
Student ID:
Section Number:

Note: This exam consists of 3 problems worth a total of 100 points. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted.

I. (40 pts.) For each natural number $n \geq 3$, let $A_n = (n\,,\, n^n\,]$.

(1) Show that, for all natural numbers $n \geq 3$, we have $n! \in A_n$.

(2) Find $\bigcup\limits_{n\geq 3} A_n$ and $\bigcap\limits_{n\geq 3} A_n$. Justify your answers.

Recall: $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, for all $n \in \mathbb{N}$.

(1) We will apply the extended first principle of math. induction to show that

$$\forall n \geq 3, \quad P(n): n < n! \leq n^n.$$

(a) $P(3) \iff 3 < 3! \leq 3^3 \iff 3 < 6 \leq 27$, which is true.

(b) Let $n \geq 3$. Assume $P(n)$ is true. We will show that

$$P(n): (n+1) < (n+1)! \leq (n+1)^{n+1}$$

is also true.

$$n < n! \Longrightarrow \frac{(n+1)}{\cancel{n}} \cdot \cancel{n} < \frac{n+1}{n} \cdot n! \Longrightarrow$$

$$\Longrightarrow (n+1) < \frac{1}{n}(n+1)! < (n+1)! \quad (*).$$

$$n! \leq n^n \Longrightarrow (n+1)\,n! \leq (n+1)\,n^n \Longrightarrow$$

$$\Longrightarrow (n+1)! \leq (n+1)(n+1)^n \Longrightarrow (n+1)! \leq (n+1)^{n+1} \quad (**)$$

$$\left.\begin{matrix}(*)\\(**)\end{matrix}\right\} \Longrightarrow P(n+1) \qquad \left.\begin{pmatrix}(a)\\(b)\end{pmatrix}\right\} \Longrightarrow P(n) \text{ is true, } \forall n \geq 3.$$

(2) a) We will prove that $\bigcup_{n\geq 3} A_n = \{n \in \mathbb{N} \mid n > 3\}$

"$\subseteq$" obvious

"$\supseteq$" Let $m \in \mathbb{N}$, $m > 3$.

Then, we claim that $m \in A_{m-1} = (m-1, (m-1)^{m-1}]$.

In order to show this, one needs to show (by induction) that $m \leq (m-1)^{m-1}$ ....

b) We will prove that $\bigcap_{n\geq 3} A_n = \emptyset$.

Indeed assume $\bigcap_{n\geq 3} A_n \neq \emptyset$. Let $m \in \bigcap_{n\geq 3} A_n$.

This implies that $m \in A_n$, $\forall n \geq 3$; therefore

$$m \geq n, \quad \forall n \geq 3.$$

Since $\lim_{n\to\infty} n = \infty$, this shows that $m \geq \infty$.

False $\square$.

II. (30 pts.) Let $\{f_n\}_{n\in\mathbb{N}}$ be the Fibonacci sequence, defined recursively by $f_1 = f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for all $n \geq 3$.

(1) Prove that, for each natural number $n \geq 1$, f_{3n} is an even number.

(2) Prove that, for each natural number $n > 1$, we have an equality

$$\sum_{i=1}^{n-1} f_i = f_{n+1} - 1\,.$$

(1) We will apply the ~~extended first or second principle~~ first principle of math. induction to show that

$$\forall\, n \geq 1,\quad P(n):\ 2 \mid f_{3n}$$

a) $P(1):\ 2 \mid f_3 \iff 2 \mid (f_1 + f_2) \iff 2 \mid 2$ true.

b) Let $n \geq 1$. Assume $P(n):\ 2 \mid f_{3n}$ is true. We will show that $P(n+1):\ 2 \mid f_{3n+3}$ is true.

$$f_{3n+3} = f_{3n+2} + f_{3n+1} = (f_{3n} + f_{3n+1}) + f_{3n+1} =$$
$$= f_{3n} + 2 \cdot f_{3n+1}$$

$$\left.\begin{array}{l} 2 \mid f_{3n} \\ 2 \mid 2 \cdot f_{3n+1} \end{array}\right\} \Rightarrow 2 \mid f_{3n+3}.$$

$$\left.\begin{array}{l} (a) \\ (b) \end{array}\right\} \Rightarrow \forall\, n \geq 1,\ P(n):\ 2 \mid f_{3n}.$$

(2) We will ~~prove by~~ apply the extended first principle of math. induction ~~that~~ $\forall\, n > 1$, $P(n):\ \sum_{i=1}^{n-1} f_i = f_{n+1} - 1$ is true.

(a) $P(2):\ f_1 = f_3 - 1 \iff f_3 = 1 + f_1 \iff 2 = 1 + 1$ which is true

(b) Let $n \in \mathbb{N}_{>1}$.
Assume that $P(n)$ is true. We will show that $P(n+1)$ is true.

$$\sum_{i=1}^{n} f_i = \left(\sum_{i=1}^{n-1} f_i\right) + f_n \overset{P(n)}{=} (f_{n+1} - 1) + f_n =$$

$$= (f_{n+1} + f_n) - 1 = f_{n+2} - 1 \qquad \square.$$

III. (30 pts.) Give the explicit description of the solution set for the following Diophantine equation.

$$395 \cdot x - 180 \cdot y = 65. \quad (*)$$

Euclidian algorithm for determining $d := \gcd(395, 180)$.

$$395 = 2 \cdot 180 + 35$$

$$180 = 5 \cdot 35 + 5$$

$$35 = 7 \cdot 5 + 0$$

Therefore $5 = \gcd(395, 180)$. Since $5 \mid 65$, eq $(*)$ has infinitely many solutions. The sol. set is given by

$$Sol(*) = \left\{ \left(x_0 + \frac{180}{5} \cdot k, \; y_0 + \frac{395}{5} \cdot k\right) \;\middle|\; k \in \mathbb{Z} \right\},$$

where (x_0, y_0) is a particular solution to $(*)$ to be determined below:

$$5 = -5 \cdot 35 + 180 = -5\,(395 - 2 \cdot 180) + 180 =$$

$$= (-5) \cdot 395 + (-11) \cdot (-180)$$

Therefore $65 = 13 \cdot 5 = (-65) \cdot 395 + (-141) \cdot (-180)$

$$(x_0, y_0) = (-65, -141).$$

$\square$

$$Sol(*) = \{(-65 + 36k, \; -141 + 79k) \mid k \in \mathbb{Z}\}.$$