MATH 109 HOMEWORK 2 T.A. Tai Melcher
UCSD Fall 2003 Office AP&M 6402E

This is not a complete list of the solutions. Problems with sufficient solutions in the back of the text were not included.

Problem 1.41:

a) There is a cow that does not eat grass.
b) All horses eat grass.
c) All blue cars weigh at least 4000 pounds.
d) There is a math book that is not blue and is easy to read.
e) No cows are spotted.
f) There is a car with 15 cylinders.
g) Each car is not old or is not in good running condition.

Problem 1.48: *Write the negation and the contrapositive of the following statement without using any negative words: "If x is a positive number, then there is an $\epsilon > 0$ such that $x < \epsilon$ and $x > 1/\epsilon$.*

Let P be the statement "x is a positive number" and let Q be the statement "there is an $\epsilon > 0$ such that $x < \epsilon$ and $1/\epsilon < x$." Then P can be written as $x > 0$ and Q can be written as $(\exists\epsilon > 0)(x < \epsilon \wedge 1/\epsilon < x)$. So the given statement is $P \to Q$.

The contrapositive of this statement is $\neg Q \to \neg P$. The negation of Q is the statement $(\forall\epsilon > 0)(x \geq \epsilon \vee 1/\epsilon \geq x)$. The negation of P is the statement $x \leq 0$. Thus, the contrapositive can be written as "If every for every positive ϵ, $x \geq \epsilon$ or $1/\epsilon \geq x$, then x is less than or equal to zero."

The given statement $P \to Q$ is logically equivalent to $\neg P \vee Q$, and the negation of this statement is $P \wedge \neg Q$. Therefore the negation of the given statement is "x is a positive number and $(\forall\epsilon > 0)(x \geq \epsilon \vee 1/\epsilon \geq x)$".

Problem 1.50: *Find the truth set of the propositional function*

$$(x^2 + 1)(x - 3)(x^2 - 2)(2x - 3) = 0,$$

when it is given that the set of meanings of this propositional function is each of the following:

a) $\mathbb{Z}$, $\{3\}$
b) $\mathbf{Q}$, $\{3, \frac{3}{2}\}$
c) $\mathbb{R}$, $\{3, \frac{3}{2}, \sqrt{2}, -\sqrt{2}\}$
d) $\mathbb{C}$, $\{3, \frac{3}{2}, \sqrt{2}, -\sqrt{2}, i, -i\}$

Problem 1.54: *Both of the following statements have the set of the positive real numbers as their set of meanings. Which statement is true?*
(a) $(\forall x)(\exists y)(x < y^2)$.
(b) $(\exists y)(\forall x)(x < y^2)$.

The first statement is true and the second statement is false – neither is hard to prove.

Problem 1.58: *Let A, B, and C be integers. Prove that if A divides B and B divides C, then A divides C.*

Proof. Let A, B, and C be integers, and suppose $A|B$ and $B|C$. Since $A|B$, then there exists an integer m such that $B = Am$. Since $B|C$, then there exists an integer n such that $C = Bn$. Thus $C = Amn$, and since mn is an integer, $A|C$.

Problem 1.61: *Prove that no integer is both even and odd.*

Proof. (by contradiction) Suppose n is an integer that is both even and odd. Then there exist integers p and q such that $n = 2p$ and $n = 2q + 1$. Thus,

$$2p = 2q + 1$$

and so

$$1 = 2(p - q)$$

is even. This is a contradiction, and so no such integer n exists. ■

Problem 1.65: *Prove that if x and y are real numbers such that $x < -2$ and $y < -3$, then the distance between (x, y) and $(1, 4)$ is greater than 7.*

Proof. The distance between (x, y) and $(1, 4)$ is

$$d = \sqrt{(x-1)^2 + (y-4)^2}.$$

Since $x < -2$, then $x - 1 < 3$ and so $(x - 1)^2 > 9$. Similarly, $y < -3$ implies that $(y - 4)^2 > 49$. Thus,

$$d > \sqrt{9 + 49} > 7.$$

■

Problem 1.69: *Let A and B be integers with $B \neq 0$. Prove that if A divides B, then $|A| \leq |B|$.* **Proof.** If A divides B, then there exists some integer P so that $B = PA$. Thus,

$$|B| = |PA| = |P||A|.$$

We know $|P| \geq 1$, and so

$$|B| = |P||A| \geq 1 \cdot |A| = |A|.$$

■

Problem 1.71: *Prove that for any natural number n, n is either prime or a perfect square, or n divides $(n - 1)!$.*

Proof. Suppose n is neither prime nor a perfect square. Then we wish to show that n divides $(n - 1)!$. Since n is not prime, there exist two natural numbers $p, q > 1$ so that $n = pq$. Since n is not a perfect square, $p \neq q$, and thus p and q are distinct divisors of $(n - 1)!$. Hence, their product n divides $(n - 1)!$. ■

Problem 1.75: *Prove, by contradiction, that the sum of two even integers is even.*

Proof. Suppose m and n are two even integers such that $m+n$ is odd. Then there are integers p,q, and r such that $m = 2p$,$n = 2q$, and $m + n = 2r + 1$. But this implies that

$$2r + 1 = m + n = 2p + 2q = 2(p + q)$$

and thus $m + n$ is both even and odd. However, this contradicts Problem 1.61, which states that no such integer exists. ■

Problem 1.76: *Either prove or give a counterexample to the converse of "If x and y are even, then xy is even".*

Converse: If xy is even, then x and y are even.

If xy is even, this only implies that x or y is even. So there are many counterexamples. Take $x = 2$, $y = 3$.

Problem 1.84: *Let A,B, and C be integers such that $A^2 + B^2 = C^2$. Prove that at least one of A and B is even.*

Proof. Let A, B, and C be integers, and assume that $A^2 + B^2 = C^2$. Assume for the sake of contradiction that both A and B are odd. Since A and B are odd, there exist integers m and n such that $A = 2m + 1$ and $B = 2n+1$. Thus $(2m+1)^2+(2n+1)^2 = C^2$, or $2(2m^2+2m+2n^2+2n+1) = C^2$. Thus C^2 is even, and by a theorem proved in lecture (I think. If it wasn't let me know, and I'll include it to complete the solution), C is even. Since C is even, then there exists an integer k such that $C = 2k$. Thus $2(2m^2 + 2m + 2n^2 + 2n + 1) = (2k)^2 = 4k^2$. Dividing both sides of this equation by 2 yields $2m^2 + 2m + 2n^2 + 2n + 1 = 2k^2$. The left hand side of this equation is odd, whereas the right hand side is even, which is a contradiction. Therefore at least one of A and B must be even. ■

Problem 1.86: *Everyone knows that $3^2 + 4^2 = 5^2$. Prove that there do not exist three consecutive natural numbers such that the cube of the largest is equal to the sum of the cubes of the other two.*

Proof. Assume for the sake of contradiction that there is a natural number n for which $n^3+(n+1)^3 = (n+2)^3$. Then $2n^3+3n^2+3n+1 = n^3+6n^2+12n+8$. Simplifying this gives us that $n^3 = 3n^2 + 9n + 7$, or $n(n^2 - 3n - 9) = 7$. Thus n divides 7. Since 7 is prime, it must be the case that $n = 1$ or $n = 7$. If $n = 1$, then $(1)(1 - 3 - 9) = -11 \neq 7$, and if $n = 7$ then $7(49 - 21 - 9) = 133 \neq 7$. This is a contradiction, so it must be the case that no such natural number n exists. ■

Problem 1.90:

Definition 0.1. A prime is *average* provided it is the average of two different prime numbers.

Consider the following propositions:
P: Every prime greater than 3 is average.
Q: Every even number other than 2 can be written as $x + y$*, where* x *and* y *are prime, and possibly* $x = y$*.*
R: Every even number greater than 6 can be written as the sum of two different prime numbers.

a) *Prove that* $R \to P \wedge Q$*.*
Proof. Assume for the sake of contradiction that R is true and that $P \wedge Q$ is false, that is, that $\neg P \vee \neg Q$ is true.

Case I: Assume $\neg P$ is true. Then there exists a prime number greater than 3 that is not average. Let p be such a number, that is, p is prime, greater than 3, but not average. Then $2p$ is even, and since R is true, there exist two different prime numbers x and y for which $2p = x + y$. But then $p = \frac{x+y}{2}$, which means that p is average. This contradicts the fact that p is not average.

Case II: Assume $\neg Q$ is true. Then there exists an even number other than 2 which cannot be written as the sum of two primes. Let n be such a number, that is, n is even, greater than 2, and cannot be written as the sum of two primes. Note that $n \neq 4$, because $4 = 2 + 2$, and $n \neq 6$, because $6 = 3 + 3$. Thus $n > 6$. Since R is true, there exist two different prime numbers x and y such that $n = x + y$. This now contradicts the fact that n cannot be written as the sum of two primes.

In either case, we reached a contradiction. Therefore $R \to (P \wedge Q)$. ■

Remark 0.2. Given a choice, I would have proved the above statements directly, as opposed to using the method of contradiction, which seems unnecessary here.

b)*Prove that* $P \wedge Q \to R$*.*
Proof. Suppose $P \wedge Q$ holds. Let n be an even number greater than 6. Q implies that we may write n as the sum of two primes, that is, $n = p + q$. If $p \neq q$, then we're done. Otherwise, $p = q$ and so $n = 2p$, where p is a prime greater than 3. Then by P, p is average and so $p = \frac{p_1+p_2}{2}$ for two distinct primes p_1 and p_2. Thus,

$$n = 2p = 2\left(\frac{p_1 + p_2}{2}\right) = p_1 + p_2$$

as desired. ■