This is not a complete list of the solutions. Problems with sufficient solutions in the back of the text were not included.

**Problem 1.94**: *Prove that if $a$ and $b$ are rational numbers with $a < b$, then there exists a rational number $r$ such that $a < r < b$.*

**Proof.** Let $a$ and $b$ be rational numbers, and suppose $a < b$. Since $a$ and $b$ are rational, there exist integers $m_1$, $n_1$, $m_2$, $n_2$, with $n_1 \neq 0$ and $n_2 \neq 0$ such that $a = \frac{m_1}{n_1}$ and $b = \frac{m_2}{n_2}$. Since $n_1 \neq 0$ and $n_2 \neq 0$, $2n_1n_2 \neq 0$. Let $r = \frac{a+b}{2} = \frac{m_1n_2+m_2n_1}{2n_1n_2}$, which is rational since the numerator and denominator are both integers, and the denominator is nonzero.

Note that since $a < b$, $a = \frac{2a}{2} < \frac{a+b}{2} = r$. Similarly, $r = \frac{a+b}{2} < \frac{2b}{2} = b$.

Thus $r$ is a rational number such that $a < r < b$. ∎

**Problem 1.95**: *Prove that if $x$ is a positive real number, then $x + \frac{1}{x} \geq 2$.*

**Proof.** Let $x$ be a positive real number. Note that since $x$ is positive, $\frac{(x-1)^2}{x} \geq 0$. Therefore $x + \frac{1}{x} = \frac{x^2+1}{x} = \frac{x^2-2x+1}{x} + \frac{2x}{2x} = \frac{(x-1)^2}{x} + 2 \geq 2$. ∎

**Problem 1.100**: *How many triplets of the form $p_1 = p$, $p_2 = p+2$, $p_3 = p+4$ are there, where $p_1$, $p_2$, and $p_3$ are prime? Prove your answer.*

**Claim 0.1.** *If $p$ is any natural number, one of $p$, $p+2$, or $p+4$ is divisible by 3.*

**Proof.** Let $p$ be any natural number. The remainder when $p$ is divided by 3, is either 0, 1, or 2. If the remainder is 0, then 3 divides $p$, and we're done. If the remainder is 1, then $p+2$ is divisible by 3. If the remainder is 2, then $p+4$ is divisible by 3. In any case, one of $p$, $p+2$, or $p+4$ is divisible by 3. ∎

The above claim implies that if $p$, $p+2$, are $p+4$ primes, then one of them must be three. It must be $p$, so the only triplet of primes of the given form is 3, 5, and 7.

**Problem 1.104**:

**a)** Yes, the inspector can begin his inspection at one of the towns with 3 roads and end it at the other town with 3 roads.

**b)** No, the two towns do not exist.

If each town has an even number of roads, or if there are exactly two towns with an odd number of roads, then there are two towns such that the inspector can begin his inspection at one town and end it in the second town.

**Problem 2.3**: *Prove that if $A \subseteq B$, $B \subseteq C$ and $C \subseteq A$, then $A = C$.*

**Proof.** Let $A$, $B$, and $C$ be sets, and assume that $A \subseteq B$, $B \subseteq C$ and $C \subseteq A$. Since $A \subseteq B$ and $B \subseteq C$, by Proposition 2.4, $A \subseteq C$. Since we also have that $C \subseteq A$, then $A = C$. ∎

**Problem 2.12**: *Let $X = \{a, b, c, d\}$. List all the members of $\mathcal{P}(X)$. How many sets have you listed?*

Since $|X| = 4$, there should be $2^4 = 16$ elements of $\mathcal{P}(X)$:
$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\},$
$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X = \{a, b, c, d\}\}$.

**Problem 2.16**: *Find three sets $A$, $B$, and $C$ such that $A \in B$, $B \in C$, and $A \in C$.*

There are many solutions to this problem, but three easy sets that would satisfy the above conditions would be

$$A = \{a\}, B = \{\{a\}, b\}, \text{ and } C = \{\{a\}, \{\{a\}, b\}\}.$$

Note then that $B = \{A, b\}$ and $C = \{A, B\}$.

**Problem 2.20**: *We give two "proofs" of the following conjecture:*
*If $A$ and $B$ are sets such that $\mathcal{P}(A) \subset \mathcal{P}(B)$, then $A \subset B$.*
*Which, if any, are correct? Justify your answer.*

**"Proof" 1** is incorrect. We are *given* that $\mathcal{P}(A) \subset \mathcal{P}(B)$, and wish to *prove* that $A \subset B$. This proof accomplishes nothing by assuming information we don't have. **"Proof" 2** is correct.

**Problem 2.22**: *Let $X = \{1, 2, 3, 4\}$. List all pairs $A$, $B$ of subsets of $X$ such that $A$ and $B$ are disjoint and $A \cup B = X$.*

We can present the answer in a table:

| $A$ | $B$ |
|---|---|
| $\emptyset$ | $\{1,2,3,4\}$ |
| $\{1\}$ | $\{2,3,4\}$ |
| $\{2\}$ | $\{1,3,4\}$ |
| $\{3\}$ | $\{1,2,4\}$ |
| $\{4\}$ | $\{1,2,3\}$ |
| $\{1,2\}$ | $\{3,4\}$ |
| $\{1,3\}$ | $\{2,4\}$ |
| $\{1,4\}$ | $\{2,3\}$ |
| $\{2,3\}$ | $\{1,4\}$ |
| $\{2,4\}$ | $\{1,3\}$ |
| $\{3,4\}$ | $\{1,2\}$ |
| $\{1,2,3\}$ | $\{4\}$ |
| $\{1,2,4\}$ | $\{3\}$ |
| $\{1,3,4\}$ | $\{2\}$ |
| $\{2,3,4\}$ | $\{1\}$ |
| $\{1,2,3,4\}$ | $\emptyset$ |

**Problem 2.30**: *Prove or find a counterexample to the following statement. For any sets $P$, $Q$, and $R$, $(P \cap Q) \cup R = P \cap (Q \cup R)$.*

The statement is false. For a counterexample, let $P = \emptyset$, $Q = \emptyset$, and $R = \{1\}$. Then $P \cap Q = \emptyset$, so $(P \cap Q) \cup R = \{1\}$. Also, $Q \cup R = \{1\}$, so $P \cap (Q \cup R) = \emptyset$. Therefore $(P \cap Q) \cup R \neq P \cap (Q \cup R)$.

**Problem 2.31**: *A student is asked to prove that for any sets $A$, $B$, and $C$, $A-(B\cup C) = (A-B)\cap(A-C)$. The student writes: "Let $x \in A-(B\cup C)$. Then $x \in A$ and $x \notin B$ or $x \notin C$. Therefore, $x \in A - B$ and $x \in A - C$. Thus, $A - (B \cup C) = (A - B) \cap (A - C)$." What, if anything, is wrong with this proof?*

The first conclusion is wrong, the "or" in that statement should be an "and". From that, the student incorrectly concludes that $x \in A-B$ and $x \in A-C$. Finally, the student only showed subset inclusion in one direction.

**Problem 2.36**: *In former times, it was customary to denote interesection by $\cdot$ and union by $+$. To what extent do $\cap$ and $\cup$ mimic the behavior of $\cdot$ and $+$?*

Both operations are commutative and associative, and the distributive laws hold, that is, for three sets $A$, $B$, and $C$,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

However, one should note that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

while the analogous identity does not hold for $\cdot$ and $+$. The empty set mimics 0 with respect to $\cup$ and $+$, since

$$A \cup \emptyset = A,$$

and the universal set U mimics 1 with respect to $\cap$ and $\cdot$,

$$A \cap U = A.$$