Problem 2.55: *For each natural number n, let $A_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$ and $B_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Find $\cap_{n\in\mathbb{N}} A_n$ and $\cap_{n\in\mathbb{N}} B_n$.*

$$\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} \left[-\frac{1}{n}, \frac{1}{n}\right] = \{0\}$$

and

$$\bigcap_{n\in\mathbb{N}} B_n = \bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}.$$

Problem 2.63: *Give an example of a collection of sets indexed by a set consisting of 5 members.*

There are infinite such examples, but one collection that would work would be

$$\mathcal{A} = \{(n-1, n+1) : n \in \mathbb{N}, n \leq 5\},$$

or rather $\mathcal{A} = \{A_n : n \in \Lambda\}$, where

$$A_n = (n-1, n+1) \text{ and } \Lambda = \{n \in \mathbb{N} : n \leq 5\}.$$

Problem 2.64: *Give an example of a collection of sets indexed by the set of prime numbers.*

Again, an infinite number of solutions exist, but one example would be

$$\mathcal{A} = \{\{p-5, p, p+2\} : p \in \mathbb{N}, p \text{ prime}\},$$

or, if you like, $\mathcal{A} = \{A_p : p \in \Lambda\}$, where

$$A_p = \{p-5, p, p+2\} \text{ and } \Lambda = \{p \in \mathbb{N} : p \text{ prime}\}.$$

Problem 2.65: *Prove Theorem 2.9(c): Let $\{A_\alpha : \alpha \in \Lambda\}$ be an indexed family of sets. Then*

$$\left(\bigcap\{A_\alpha : \alpha \in \Lambda\}\right)' = \bigcup\{A'_\alpha : \alpha \in \Lambda\}.$$

Proof. To show the equality of two sets, one must show that each is a subset of the other. Here, I will show

$$\left(\bigcap\{A_\alpha : \alpha \in \Lambda\}\right)' \subset \bigcup\{A'_\alpha : \alpha \in \Lambda\},$$

and leave the other containment argument to you – it is basically the same.

So let $x \in (\cap\{A_\alpha : \alpha \in \Lambda\})'$. By definition of the complement then, $x \notin \cap\{A_\alpha : \alpha \in \Lambda\}$. Thus, there exists some $\alpha' \in \Lambda$ such that $x \notin A_{\alpha'}$. Thus, $x \in (A_{\alpha'})'$, and so $x \in \cup\{A'_\alpha : \alpha \in \Lambda\}$ as desired. ■

Problem 2.70: *Let* $\{A_\alpha : \alpha \in \Lambda\}$ *be an indexed family of sets. Prove that*

$$\bigcup_{\alpha,\beta\in\Lambda,\alpha\neq\beta}(A_\alpha - A_\beta) \subset \bigcup_{\alpha\in\Lambda} A_\alpha - \bigcap_{\alpha\in\Lambda} A_\alpha.$$

Proof. Let $x \in \cup_{\alpha,\beta\in\Lambda,\alpha\neq\beta}(A_\alpha - A_\beta)$. Then there exist some $\alpha', \beta' \in \Lambda$ such that $\alpha' \neq \beta'$ and $x \in A_{\alpha'} - A_{\beta'}$. By definition of the difference of sets, $x \in A_{\alpha'}$ and $x \notin A_{\beta'}$. $x \in A_{\alpha'}$ implies that $x \in \cup_{\alpha\in\Lambda} A_\alpha$. $x \notin A_{\beta'}$ implies that $x \notin \cap_{\alpha\in\Lambda} A_\alpha$. So

$$x \in \bigcup_{\alpha\in\Lambda} A_\alpha - \bigcap_{\alpha\in\Lambda} A_\alpha.$$

Since x was an arbitrary element of $\cup_{\alpha,\beta\in\Lambda,\alpha\neq\beta}(A_\alpha - A_\beta)$, this implies that

$$\bigcup_{\alpha,\beta\in\Lambda,\alpha\neq\beta}(A_\alpha - A_\beta) \subset \bigcup_{\alpha\in\Lambda} A_\alpha - \bigcap_{\alpha\in\Lambda} A_\alpha.$$

■