Math 109 | Homework 5 | T.A. Tai Melcher
UCSD Fall 2003 | | Office AP&M 6402E

This is not a complete list of the solutions. Problems with sufficient solutions in the back of the text were not included.

Problem 3.1: *Prove by induction that for each natural number n, each of the following is true.*

m) $4^n - 1$ *is divisible by 3.*

Proof. Let $S = \{n \in \mathbb{N} : 4^n - 1 \text{ is divisible by } 3\}$. Since $4^1 - 1 = 3$ is certainly divisible by 3, $1 \in S$. Now suppose $n \in S$. Then $4^n - 1$ is divisible by 3. Then

$$4^{n+1} - 1 = 4 \cdot 4^n - 1 = 4 \cdot 4^n - 4 + 4 - 1 = 4(4^n - 1) + 3$$

is divisible by 3. Thus $n+1 \in S$, and by the Principle of Mathematical Induction, $S = \mathbb{N}$. ■

q) $\left(1 + \frac{1}{2}\right)^n \geq 1 + \frac{n}{2}$.

Proof. Let $S = \{n \in \mathbb{N} : \left(1 + \frac{1}{2}\right)^n \geq 1 + \frac{n}{2}\}$. Since $\left(1 + \frac{1}{2}\right)^1 = 1 + \frac{1}{2}$, $1 \in S$. Now suppose $n \in S$. Then $\left(1 + \frac{1}{2}\right)^n \geq 1 + \frac{n}{2}$, and

$$\begin{aligned}
\left(1 + \frac{1}{2}\right)^{n+1} &= \left(1 + \frac{1}{2}\right)^n \left(1 + \frac{1}{2}\right) \\
&\geq \left(1 + \frac{n}{2}\right)\left(1 + \frac{1}{2}\right) \\
&= 1 + \frac{1}{2} + \frac{n}{2} + \frac{n}{4} = 1 + \frac{n+1}{2} + \frac{n}{4} > 1 + \frac{n+1}{2}
\end{aligned}$$

Therefore $n+1 \in S$, and so by the Principle of Mathematical Induction, $S = \mathbb{N}$. ■

Problem 3.2: *Prove by induction that for each natural number n, $(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$. Assume that $i^2 = -1$ and also assume the following trigonometric identities:*

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

and

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

Proof. Let $x \in \mathbb{R}$. Let

$$S = \{n \in \mathbb{N} : (\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)\}.$$

$1 \in S$ because $(\cos(x) + i\sin(x))^1 = \cos(x) + i\sin(x)$. Assume $n \in S$. Then

$$\begin{aligned}(\cos(x) + i\sin(x))^{n+1} &= (\cos(x) + i\sin(x))^n(\cos(x) + i\sin(x)) \\ &= (\cos(nx) + i\sin(nx))(\cos(x) + i\sin(x)) \\ &= [\cos(nx)\cos(x) - \sin(nx)\sin(x)] \\ &\quad + i\,[\sin(nx)\cos(x) + \cos(nx)\sin(x)] \\ &= \cos[(n+1)x] + i\sin[(n+1)x]\end{aligned}$$

where the second equality is due to the inductive hypothesis. Hence $n + 1 \in S$. Therefore, by the Principle of Mathematical Induction, $S = \mathbb{N}$. ■

Problem 3.7: *Take an equilateral triangle, divide each side into n equal segments, and connect the division points with all possible segments that are parallel to the original sides. Find a formula for relating the number n to the number of small triangles and prove, by induction, that your formula holds.*

If we divide each side into 1 equal segment – that is, we don't divide it – then we have 1 triangle. If we divide each side into 2 equal segments and connect the division points in the way described, we have 4 triangles. Similarly, for $n = 3$ equal segments we end up with 9 triangles, and $n = 4$ equal segments generates 16 triangles. So we guess that the number of small triangles is n^2, and shall attempt to prove this by induction.

Claim 0.1. *Let $f(n)$ be the number of small triangles created when an equilateral triangle is divided as described above. Then $f(n) = n^2$.*

Proof. Let $S = \{n \in \mathbb{N} : f(n) = n^2\}$. We have seen that $1 \in S$. Now suppose that $n \in S$ and divide each side of an equilateral triangle into $n + 1$ segments and connect the division points with all possible segments that are parallel to the original sides.

Now delete one side of the original triangle, say the bottom side, and all the small line segments that connect this side. We now have a large triangle in which each side has been divided into n equal segments and the division points are connected with all possible segments that are parallel to the original sides. By the induction hypothesis, we have n^2 small triangles, and the question is, how many small triangles have we deleted. The picture is as follows:

(**note:** When dividing an interval into n equal intervals, you get $n+1$ endpoints of those intervals. That's why we have $n+2$ endpoints when we divide one side of the triangle into $n + 1$ equal segments.)

We see that the line segment joining x_1 to y_1 and the one joining x_1 to y_2 create 1 small triangle. Now for each $i = 2, 3, \dots, n+1$, the line segment joining x_i to y_i creates 1 small triangle, and the line segment joining x_i to y_{i+1} creates another small triangle. So the total number of small triangles which were deleted above is 1+2n.

Therefore, the total number of small triangles is $n^2+2n+1 = (n+1)^2$. So $n+1 \in S$, and by the Principle of Mathematical Induction, $S = \mathbb{N}$. $\blacksquare$

Problem 3.9: *What is wrong with the proof in the text that all members of the human race are of the same sex?*

A_3 does not exist if $n = 1$.

Problem 3.19: *Prove that for each odd natural number $n \geq 3$,*

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{(-1)^n}{n}\right) = 1.$$

Proof. Let $S = \{n \in \mathbb{N} : n \text{ is even or } n \geq 3, n \text{ is odd and } (1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{4})\cdots(1+\frac{(-1)^n}{n}) = 1\}$. 2 is even, and so $2 \in S$. Note also that $(1+\frac{1}{2})(1-\frac{1}{3}) = \frac{3}{2}\cdot\frac{2}{3} = 1$, and so $3 \in S$. Suppose $n \in \mathbb{N}$, $n \geq 3$ and $\{2, 3, \dots, n\} \subset S$.

Case 1: n is odd, and so $n+1$ is even. Thus $n+1 \in S$.

Case 2: $n > 2$ is even, and so $n-1 \geq 3$ is odd and $n-1 \in S$. Then

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{(-1)^{n-1}}{n-1}\right) = 1.$$

So

$$\begin{aligned}
&\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{(-1)^{n+1}}{n+1}\right)\\
&\qquad= \left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{(-1)^{n-1}}{n-1}\right)\\
&\qquad\qquad\times\left(1+\frac{(-1)^n}{n}\right)\left(1+\frac{(-1)^{n+1}}{n+1}\right)\\
&\qquad= 1\cdot\frac{n+1}{n}\cdot\frac{(n+1)-1}{n+1} = 1,
\end{aligned}$$

where in the last equality we have used that n even implies $(-1)^n = 1$ and $n+1$ odd implies that $(-1)^{n+1} = -1$. Thus $n+1 \in S$, and by the Extended Second Principle of Mathematical Induction, $\{n \in \mathbb{N} : n \geq 2\} \subset S$. $\blacksquare$

Problem 3.20: *Prove that for each natural number $n \geq 6$, $(n+1)^2 \leq 2^n$.*

Proof. Let $S = \{n \in \mathbb{N} : n \geq 6 \text{ and } (n+1)^2 \leq 2^n\}$. $6 \in S$ because $(6+1)^2 = 49 \leq 64 = 2^6$. Assume $n \in S$, for $n \geq 6$. Note that since $n \geq 6$, $n^2 \geq 2$. Then

$$\begin{aligned}((n+1)+1)^2 = (n+2)^2 &= n^2 + 4n + 4 \\ &= (n^2 + 2n + 1) + (2 + 2n + 1) \\ &\leq (n^2 + 2n + 1) + (n^2 + 2n + 1) \\ &= 2(n+1)^2 \leq 2 \cdot 2^n = 2^{n+1},\end{aligned}$$

where the second inequality is due to the inductive hypothesis. Hence $n+1 \in S$. Therefore, by the Extended Principle of Mathematical Induction, $S = \{n \in \mathbb{N} : n \geq 6\}$. ■

Problem 3.37:
a) *Prove that for each natural number n, any set with n members has $\frac{n(n-1)}{2}$ 2-element subsets.*

Proof. Let $S = \{n \in \mathbb{N} :$ any set with n elements has $\frac{n(n-1)}{2}$ 2-element subsets$\}$. Then clearly $1 \in S$ because a set with 1 element has no 2-element subsets. Now suppose $n \in S$. To show that $n+1 \in S$, consider an $(n+1)$-element set A, and let $a \in A$. Then the set $A - \{a\}$ has n elements, and so has $\frac{n(n-1)}{2}$ 2-element subsets. Clearly these are all the 2-element subsets of A which do not contain a. There are n 2-element subsets of A which contain a. Thus, there are $n + \frac{n(n-1)}{2} = \frac{(n+1)n}{2}$ 2-element subsets of A, which was a random set of $(n+1)$-elements. Thus, $n+1 \in S$, and since S is inductive, $S = \mathbb{N}$. ■

b) *Use the result of (a) to show that for each natural number n, any set with n members has $\frac{n(n-1)(n-2)}{6}$ 3-element subsets.*

Proof. If $n = 1$ or $n = 2$, then there are no 3-element subsets. So the base case should start with $n = 3$, which has exactly one subset, which agrees with the formula $\frac{3(3-1)(3-2)}{6} = 1$.

Now suppose any set with n members has $\frac{n(n-1)(n-2)}{6}$ 3-element subsets. Let $A = \{a_1, a_2, \ldots, a_n, a_{n+1}\}$. There are two types of 3-element subsets, those that contain the element a_{n+1} and those that don't.

Case 1: a_{n+1} is an element of the 3-element subset. In this case, the other two elements must be in $\{a_1, a_2, \ldots, a_n\}$. From part (a), there are exactly $\frac{n(n-1)}{2}$ 2-element subsets in $\{a_1, a_2, \ldots, a_n\}$.

Case 2: a_{n+1} is not an element of the 3-element subset. Then the 3-element subset must be contained in $\{a_1, a_2, \ldots, a_n\}$. By our inductive hypothesis, there are $\frac{n(n-1)(n-2)}{6}$ 3-element subsets of $\{a_1, a_2, \ldots, a_n\}$.

Thus, there are a total of $\frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} = \frac{3n^2-3n}{6} + \frac{n^3-3n^2+2n}{6} = \frac{n^3-n}{6} = \frac{(n+1)n(n-1)}{6}$ 3-element subsets of $\{a_1, a_2, \ldots, a_n, a_{n+1}\}$. By induction, we have that the number of 3-elements subsets of a set with n members is $\frac{n(n-1)(n-2)}{6}$ for all $n \geq 3$. ■