MATH 109	HOMEWORK 6	T.A. Tai Melcher
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This is not a complete list of the solutions. Problems with sufficient solutions in the back of the text were not included.

Problem 3.41: *Let $x_1 = 1$, and for $n > 1$, let $x_n = \sqrt{3x_{n-1} + 1}$. Prove that $x_n < 4$ for all $n \in \mathbb{N}$.*

Proof. Let $x_1 = 1$, and for $n > 1$, let $x_n = \sqrt{3x_{n-1} + 1}$. Let $S = \{n \in \mathbb{N} : x_n < 4\}$. $1 \in S$ because $x_1 = 1 < 4$. Assume $n \in S$. Then

$$x_{n+1} = \sqrt{3x_n + 1} < \sqrt{3 \cdot 4 + 1} = \sqrt{13} < 4,$$

where the first inequality is due to the inductive hypothesis. Hence $n + 1 \in S$. Therefore, by the Principle of Mathematical Induction, $S = \mathbb{N}$. ■

Problem 3.51: *Prove by induction that for each natural number n,*

$$\sum_{k=1}^{n} 2^{k-1} = 2^n - 1.$$

Proof. Let $S = \{n \in \mathbb{N} : \sum_{k=1}^{n} 2^{k-1} = 2^n - 1\}$. Note that $1 \in S$ since

$$\sum_{k=1}^{1} 2^{k-1} = 2^{1-1} = 1 = 2^1 - 1.$$

Now suppose $n \in S$. Then $\sum_{k=1}^{n} 2^{k-1} = 2^n - 1$, and

$$\begin{aligned}\sum_{k=1}^{n+1} 2^{k-1} &= \left(\sum_{k=1}^{n} 2^{k-1}\right) + 2^n \\ &= (2^n - 1) + 2^n = 2 \cdot 2^n - 1 = 2^{n+1} - 1.\end{aligned}$$

Thus $n+1 \in S$ and by the Principle of Mathematical Induction, $S = \mathbb{N}$. ■

Problem 3.61: *For each natural number n, let $f(n)$ denote the number of subsets of $\{1, 2, 3, \ldots, n\}$ that do not contain two consecutive numbers.*

a) *Find a pattern for $f(n)$. (Don't forget to count the empty set.)*

First we consider $f(n)$ for small n.

n	subsets of desired form	$f(n)$
1	$\emptyset, \{1\}$	2
2	$\emptyset, \{1\}, \{2\}$	3
3	$\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}$	5
4	$\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{2,4\}$	8
5	$\emptyset, \{1\}, \{2\}, \{3\}, \{3\}, \{4\}, \{5\}, \{1,3\}, \{1,4\}, \{1,5\},$ $\{2,4\}, \{2,5\}, \{3,5\}, \{1,3,5\}$	13

We observe that when $n = 3, 4, 5$, $f(n) = f(n-1) + f(n-2)$. So we make this our guess.

b) *Use the Second Principle of Mathematical Induction to prove that your pattern is correct.*

Proof. Let $S = \{n \in \mathbb{N} : n \geq 3 \text{ and } f(n) = f(n-1) + f(n-2)\}$. We can see from the above that $3 \in S$, since $f(3) = 5 = 3+2 = f(2)+f(1)$. Now suppose that $n \geq 3$ and $\{3, 4, 5, \ldots, n\} \subset S$. The subsets of $\{1, 2, 3, \ldots, n+1\}$ that do not contain two consecutive natural numbers are

(i) those subsets of $\{1, 2, 3, \ldots, n\}$ that do not contain two consecutive natural numbers, and
(ii) those subsets of $\{1, 2, 3, \ldots, n, n+1\}$ that contain $n+1$ and no consecutive natural numbers.

The number of subsets of $\{1, 2, 3, \ldots, n\}$ that do not contain two consecutive natural numbers is $f(n)$. The subsets of $\{1, 2, 3, \ldots, n, n+1\}$ that contain $n+1$ but do not contain two consecutive natural numbers are subsets of the form $A \cup \{n+1\}$, where A is a subset of $\{1, 2, 3, \ldots, n-1\}$ which does not contain two consecutive natural numbers. The number of such sets is $f(n-1)$. Therefore, $f(n+1) = f(n) + f(n-1)$, so $n+1 \in S$. By the Extended Second Principle of Mathematical Induction, $S = \{3, 4, 5, \ldots\}$. ∎

Problem 3.64: *Let $f_1, f_2, f_3, \ldots$ be the Fibonacci numbers. Prove by induction that for each natural number n:*

a) $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$.

Proof. Let $S = \{n \in \mathbb{N} : f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}\}$. Since $f_1 = 1$ and $f_2 = 1$, we have $f_1 = f_{2\cdot 1}$, which shows that $1 \in S$. Now assume that $n \in S$, which means $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$. Then

$$\begin{aligned} f_1 + f_3 + f_5 + &\cdots + f_{2n-1} + f_{2(n+1)-1} \\ &= f_1 + f_3 + f_5 + \cdots + f_{2n-1} + f_{2n+1} \\ &= f_{2n} + f_{2n+1} = f_{2n+2} = f_{2(n+1)}, \end{aligned}$$

where the penultimate equality holds by definition of the Fibonacci sequence. Therefore, $n+1 \in S$. By the Principal of Mathematical Induction, $S = \mathbb{N}$, and we have that $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$ holds for all n. ■

b) $f_2 + f_4 + \ldots + f_{2n} = f_{2n+1} - 1$.

Proof. Let $S = \{n \in \mathbb{N} : f_2 + f_4 + \ldots + f_{2n} = f_{2n+1} - 1\}$. Then $1 \in S$, since $f_{2\cdot 1} = 1 = 2 - 1 = f_{2\cdot 1+1} - 1$. Suppose $n \in S$. That is, $f_2 + f_4 + \ldots + f_{2n} = f_{2n+1} - 1$. Then

$$\begin{aligned} f_2 + f_4 + \ldots + f_{2n} + f_{2(n+1)} &= (f_{2n+1} - 1) + f_{2n+2} \\ &= f_{2n+3} - 1 = f_{2(n+1)+1} - 1, \end{aligned}$$

where the second equality holds by definition of the Fibonacci sequence. Thus, $n+1 \in S$, and by the Principal of Mathematical Induction, $S = \mathbb{N}$. ■

Problem 3.71: *The "name-one-thousand" game is a two-player game. The first player names* 1, 2, *or* 3. *Thereafter, each player in turn adds* 1, 2, *or* 3 *to the previous total. The first player to name* 1000 *wins. Prove by induction that the second player has a winning strategy.*

Proof. Let $S = \{n \in \mathbb{N} : 1000 - 4n \text{ is a winning position for the second player.}\}$. $1 \in S$ because if the first player adds $k \in \{1, 2, 3\}$ to the value 996, the second player responds by adding $4 - k$, which is also in $\{1, 2, 3\}$, to bring the total to 1000, thereby winning the game.

Assume $n \in \mathbb{N}$, where $n \leq 250$, that is, $1000 - 4n$ is a winning position for the second player. Consider playing from the position $1000 - 4(n+1)$. If the first player adds $k \in \{1, 2, 3\}$ to the value $1000 - 4(n+1)$, then the second player responds by adding $4 - k$, which is also in $\{1, 2, 3\}$, thereby bringing the total to $1000 - 4n$, which by the inductive hypothesis is a winning position for that player. Hence $n + 1 \in S$. Therefore, by the Principal of Mathematical Induction, $S = \mathbb{N}$ and in particular, $250 \in S$, so 0 (the starting position) is a winning position for the second player. ■

Problem 3.75: *For each natural number i, let f_i be the ith Fibonacci number and let*

$$a_i = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^i - \left(\frac{1-\sqrt{5}}{2}\right)^i}{\sqrt{5}}.$$

Prove by induction that for each $i \in \mathbb{N}$, $f_i = a_i$.

Proof. Let $S = \{i \in \mathbb{N} : f_i = a_i\}$. Note that

$$a_1 = \frac{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 = f_1$$

and so $1 \in S$. We also have that

$$a_2 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} = \frac{\frac{1+2\sqrt{5}+5-1+2\sqrt{5}-5}{4}}{\sqrt{5}} = 1 = f_2$$

and so $2 \in S$. Now suppose $\{1, \ldots, i\} \subset S$ where $i > 2$. We would like to show that $a_{i+1} = f_{i+1}$ and so $i+1 \in S$. Note that this is equivalent to proving

$$a_{i+1} = a_i + a_{i-1} \tag{1}$$

because $i-1, i \in S$ implies that $a_{i-1} = f_{i-1}$ and $a_i = f_i$, and so (1) implies

$$a_{i+1} = a_i + a_{i-1} = f_i + f_{i-1} = f_{i+1}$$

by definition of the Fibonacci sequence. So if we can prove (1), then we're done.

To do this, we first make the following observations.

$$\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) = -1, \tag{2}$$

$$1 - \frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} \text{ and } 1 - \frac{1-\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}. \tag{3}$$

We also have that

$$\begin{aligned} \frac{1+\sqrt{5}}{2} + 1 &= \frac{1+\sqrt{5}}{2} - \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) \\ &= \frac{1+\sqrt{5}}{2}\left(1 - \frac{1-\sqrt{5}}{2}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^2. \end{aligned} \tag{4}$$

If we let $A = \frac{1+\sqrt{5}}{2}$ and $B = \frac{1-\sqrt{5}}{2}$, we may rewrite the above calculations as follows:

$$AB = -1 \tag{2'}$$

$$1 - A = B \text{ and } 1 - B = A \tag{3'}$$

$$A + 1 = A + (-AB) = A(1-B) = A^2. \tag{4'}$$

Similarly, we could show that $B + 1 = B^2$. Given these identities, we consider the following:

$$\begin{aligned} a_i + a_{i-1} &= \frac{A^i - B^i}{\sqrt{5}} + \frac{A^{i-1} - B^{i-1}}{\sqrt{5}} \\ &= \frac{A^{i-1}(A+1) - B^{i-1}(B+1)}{\sqrt{5}} \\ &= \frac{A^{i-1}A^2 - B^{i-1}B^2}{\sqrt{5}} \\ &= \frac{A^{i+1} - B^{i+1}}{\sqrt{5}} = a_{i+1}. \end{aligned}$$

Therefore, since $a_i = f_i$, $a_{i-1} = f_{i-1}$, and $f_{i+1} = f_i + f_{i-1}$, we have $a_{i+1} = f_{i+1}$ and $i + 1 \in S$. By the Second Principle of Mathematical Induction, $S = \mathbb{N}$. ■

Problem 3.83: *A certain rare goblet is supposed to weigh 43 ounces. Explain how to check the weight of this goblet given a balance scale and 1000 each of 7-oz and 11-oz weights.*

First note that

$$\begin{aligned} 11 &= 1(7) + 4 \\ 7 &= 1(4) + 3 \\ 4 &= 1(3) + 1 \\ 3 &= 3(1) + 0, \end{aligned}$$

and so

$$\begin{aligned} 1 &= 4 - 1(3) \\ &= 4 - 1[7 - 1(4)] \\ &= 2(4) - 1(7) \\ &= 2[11 - 1(7)] - 1(7) \\ &= 2(11) - 3(7). \end{aligned}$$

Then we have that

$$43 = 43[2(11) - 3(7)] = 86(11) - 129(7)$$

So put 86 11-oz weights on one side of the scale, and put 129 7-oz weights together with the goblet on the other side of the scale.

Problem 3.84(a): *Use the Euclidean algorithm to find gcd(901,952) and to find integers m and n such that $901m + 952n =$ gcd(901,952). Show your work.*

$$
\begin{aligned}
952 &= 1(901) + 51 \\
901 &= 17(51) + 34 \\
51 &= 1(34) + 17 \\
34 &= 2(17) + 0
\end{aligned}
$$

So gcd(952,901) = 17, and

$$
\begin{aligned}
17 &= 51 - 34 \\
&= 51 - [901 - 17(51)] \\
&= 18(51) - 901 \\
&= 18(952 - 901) - 901 \\
&= 18(952) - 19(901).
\end{aligned}
$$

Problem 3.93: *Suppose a, b, and c are integers with a and b not both 0 and that $d = \gcd(a, b)$. Prove that if d does not divide c, then the equation $ax + by = c$ has no integer solutions for x and y.*

Proof. Suppose a, b, and c are integers with a and b not both 0 and that $d = \gcd(a, b)$. Suppose x and y are integers satisfying the equation $ax + by = c$. Since $d|a$, and $d|b$, there exist integers m and n such that $a = dm$ and $b = dn$. Then $d(mx + ny) = dmx + dny = c$. Therefore $d|c$. ■

NOTE: The above is a proof of the contrapositive, NOT a proof by contradiction, of the desired statement.