This may not be a complete list of the solutions. Problems with sufficient solutions in the back of the text were not included.

Problem 4.1: *Prove that the ordered pair* $(17, 17) = \{\{17\}\}$.

Proof. By definition, the ordered pair $(17, 17)$ is the set $\{\{17\}, \{17, 17\}\}$. Since $\{17, 17\} = \{17\}$, $\{\{17\}, \{17, 17\}\} = \{17\}$. ■

Problem 4.2: *Prove that* $(a, b) = (c, d)$ *iff* $a = c$ *and* $b = d$.

Proof.
($\Leftarrow$) Suppose $a = c$ and $b = d$. Then $\{a\} = \{c\}$ and $\{a, b\} = \{c, d\}$. Therefore,

$$(a, b) = \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} = (c, d).$$

($\Rightarrow$) Now suppose $(a, b) = (c, d)$. Then $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. Thus $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$, and in either case, $a = c$. Also, $\{a, b\} = \{c\}$ or $\{a, b\} = \{c, d\}$. If $\{a, b\} = \{c\}$, then $a = b = c$ and $\{c, d\} = \{c\}$ so that $b = c = d$. If $\{a, b\} = \{c, d\}$, then $b = c$ or $b = d$. Suppose that $b = c$. Then $\{c, d\} = \{a, b\} = \{c\}$ and so $b = c = d$. In all cases, $a = c$ and $b = d$. ■

Problem 4.4: *Sketch the graph of each of the following relations. For each relation, state its domain and range.*

a) $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 16\}$
$\text{Dom}(S) = \{x \in \mathbb{R} : |x| \leq 4\}$ and $\text{Ran}(S) = \{y \in \mathbb{R} : |y| \leq 4\}$
b) $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y^2 = 2x\}$
$\text{Dom}(S) = \{x \in \mathbb{R} : x \geq 0\}$ and $\text{Ran}(S) = \mathbb{R}$
c) $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 = 2y^2\}$
$\text{Dom}(S) = \mathbb{R}$ and $\text{Ran}(S) = \mathbb{R}$
d) $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 = y^2\}$
$\text{Dom}(S) = \mathbb{R}$ and $\text{Ran}(S) = \mathbb{R}$
e) $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| \leq 1 \textit{ and } |y| > 3\}$
$\text{Dom}(S) = \{x \in \mathbb{R} : |x| \leq 1\}$ and $\text{Ran}(S) = \{y \in \mathbb{R} : |y| > 3\}$
f) $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| = 1 \textit{ and } 3 < y \leq 5\}$
$\text{Dom}(S) = \{x \in \mathbb{R} : |x| = 1\} = \{-1, 1\}$ and $\text{Ran}(S) = \{y \in \mathbb{R} : 3 < y \leq 5\}$

Problem 4.12: *Let* $\mathcal{A} = \{A_i : i \in \Lambda\}$ *be a collection of relations and suppose that* x *belongs to the domain of* A_i *for each* $i \in \Lambda$. *Prove that*

a) $(\cup\{A_i : i \in \Lambda\})[x] = (\cup\{A_i[x] : i \in \Lambda\})$

Proof.

$$\begin{aligned} y \in (\cup\{A_i : i \in \Lambda\})[x] &\iff (x,y) \in \cup\{A_i : i \in \Lambda\} \\ &\iff (x,y) \in A_i \text{ for some } i \in \Lambda \\ &\iff y \in A_i[x] \text{ for some } i \in \Lambda \\ &\iff y \in (\cup\{A_i[x] : i \in \Lambda\}). \end{aligned}$$

Thus, $(\cup\{A_i : i \in \Lambda\})[x] = (\cup\{A_i[x] : i \in \Lambda\})$. ∎

b) $(\cap\{A_i : i \in \Lambda\})[x] = (\cap\{A_i[x] : i \in \Lambda\})$

Proof.

$$\begin{aligned} y \in (\cap\{A_i : i \in \Lambda\})[x] &\iff (x,y) \in \cap\{A_i : i \in \Lambda\} \\ &\iff (x,y) \in A_i \text{ for all } i \in \Lambda \\ &\iff y \in A_i[x] \text{ for all } i \in \Lambda \\ &\iff y \in (\cap\{A_i[x] : i \in \Lambda\}). \end{aligned}$$

Thus, $(\cap\{A_i : i \in \Lambda\})[x] = (\cap\{A_i[x] : i \in \Lambda\})$. ∎

Problem 4.13: *Let $\mathbb{N}$ denote the set of all natural numbers. Let $R = \{(a,b) \in \mathbb{N} \times \mathbb{N} : a \text{ divides } b\}$. List five members of $R[7]$, and list five members of $R[14]$. For which $n \in \mathbb{N}$ is it true that $R[n] = \mathbb{N}$?*

$7, 14, 21, 28, 35 \in R[7]$
$14, 28, 42, 56, 70 \in R[14]$
Note that $R[n] = \mathbb{N}$ iff n divides every element of $\mathbb{N}$. Thus, the only natural number for which this is true is $n = 1$.

Problem 3.25: *Let R be a relation such that $R^{-1} \subseteq R$. Must R be symmetric? Prove your answer.*

Yes.

Proof. Suppose R is a relation such that $R^{-1} \subseteq R$. We wish to show that $R \subseteq R^{-1}$, and thus $R = R^{-1}$ and so is symmetric. So let $(x,y) \in R$. Then $(y,x) \in R^{-1}$. Since $R^{-1} \subseteq R$, we then have that $(y,x) \in R$. Hence, $(x,y) \in R^{-1}$ and $R \subseteq R^{-1}$ as desired. ∎