MATH 109 HOMEWORK 9 T.A. Tai Melcher
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This may not be a complete list of the solutions. Problems with sufficient solutions in the back of the text were not included.

Problem 4.26: *Let* $R = \{(a, b) \in \mathbb{R}\times\mathbb{R} :$ *there is an integer* k *such that* $a-b = 2k\pi\}$.

a) *Prove that* R *is an equivalence relation on* $\mathbb{R}$.

Proof. To prove R is an equivalence relation, we must prove R is reflexive, symmetric, and transitive. So let $a, b, c \in \mathbb{R}$. Then $a - a = 0 = 0 \cdot 2\pi$ where $0 \in \mathbb{Z}$. Thus $(a, a) \in R$ and R is reflexive.

Now suppose $(a, b) \in R$. Then there exists $k \in \mathbb{Z}$ such that $a - b = 2k\pi$. Then $b - a = -2k\pi = 2(-k)\pi$, and $-k \in \mathbb{Z}$. Thus $(b, a) \in R$ and R is symmetric.

If $(a, b) \in R$ and $(b, c) \in R$, then there exist integers k and n such that $a - b = 2k\pi$ and $b - c = 2n\pi$. Then

$$a - c = (a - b) + (b - c) = 2k\pi + 2n\pi = 2(k + n)\pi$$

where $k + n \in \mathbb{Z}$. Thus $(a, c) \in R$ and R is transitive. ■

b) *List three members of* $\left[\frac{\pi}{4}\right]$.

The elements of $\left[\frac{\pi}{4}\right]$ are real numbers b such that $\frac{\pi}{4} - b = 2k\pi$ for some integer k. That is, $b = \frac{\pi}{4} - 2k\pi$ for some integer k. Thus, we may generate elements of the equivalence class $\left[\frac{\pi}{4}\right]$ simply by plugging integers into the previous equation. For example, for $k = 0, 1, 2$, we have $\frac{\pi}{4}$, $\frac{\pi}{4} - 2\pi = -\frac{7\pi}{4}$, and $\frac{\pi}{4} - 4\pi = -\frac{15\pi}{4}$ are elements of $\left[\frac{\pi}{4}\right]$.

c) *List three members of* $[1]$.

As in part b) above, we have 1, 1-2π, and 1-4π are three members of $[1]$.

d) *Which numbers, if any, belong to* $\left[\frac{\pi}{4}\right] \cap [1]$?

None. The intersection of these equivalence classes is empty. To see this, suppose there exists some $x \in \left[\frac{\pi}{4}\right] \cap [1]$. Then there exist some integers k and n such that $\frac{\pi}{4} - 2k\pi = x$ and $1 - 2n\pi = x$. Thus,

$$\frac{\pi}{4} - 2k\pi = 1 - 2n\pi.$$

That is, $k - n = \frac{1}{2\pi}\left(\frac{\pi}{4} - 1\right)$. However, $k - n$ is clearly an integer and the right hand side of this equation is not an integer. Thus, we have a contradiction, and there exists no such element x that belongs to both equivalence classes.

Problem 4.27: *Let $\mathbb{Q}$ be the set of all rational numbers, and let R be the set of ordered pairs (x, y) in $\mathbb{Q} \times \mathbb{Q}$ such that when x and y are represented by fractions in lowest terms these fractions have the same denominator.*

(a) *Prove that R is an equivalence relation on $\mathbb{Q}$.*

Proof. Let $x, y, z \in \mathbb{Q}$. Then there exist some integers m, n, p, q, j, k such that $\gcd(m, n) = 1$, $\gcd(p, q) = 1$ and $\gcd(j, k) = 1$, and $x = \frac{m}{n}$, $y = pq$, and $z = \frac{j}{k}$.

Certainly, $n = n$ and so $(x, x) \in R$. Thus R is reflexive.

Suppose $(x, y) \in R$. Then $n = q$ implies $q = n$ and so $(y, x) \in R$. Thus R is symmetric.

Now suppose $(x, y) \in R$ and $(y, z) \in R$. Then $n = q$ and $q = k$ implies $n = k$ and so $(x, z) \in R$. Therefore R is transitive. ■

(b) *Prove that $[1/6] = [5/6]$.*

Proof. $y \in [1/6]$ iff there exists some integer n such that $\gcd(n, 6) = 1$ and $y = n/6$ iff $y \in [5/6]$. ■

(c) *Are $[4/6]$ and $[5/6]$ disjoint sets? Prove your answer.*

Yes. $[4/6]$ and $[5/6]$ are disjoint because $4/6$ and $5/6$ have different denominators when represented in lowest terms.

Problem 4.36: *For any two points (a, b) and (c, d) of the plane, define $(a, b) \cong (c, d)$ provided that $a^2 + b^2 = c^2 + d^2$.*

a) *Prove that $\cong$ is an eqivalence relation on $\mathbb{R} \times \mathbb{R}$.*

Proof. Let $(a, b), (c, d), (e, f) \in \mathbb{R} \times \mathbb{R}$. Clearly, $a^2 + b^2 = a^2 + b^2$ and so $(a, b) \cong (a, b)$ and $\cong$ is reflexive.

Now suppose $(a, b) \cong (c, d)$. Then $a^2 + b^2 = c^2 + d^2$ implies $c^2 + d^2 = a^2 + b^2$. So $(c, d) \cong (a, b)$ and $\cong$ is symmetric.

If $(a, b) \cong (c, d)$ and $(c, d) \cong (e, f)$, then we have

$$a^2 + b^2 = c^2 + d^2 = e^2 + f^2$$

implies $(a, b) \cong (e, f)$ and $\cong$ is transitive. ■

b) *List all members of $[(0, 0)]$.*

$[(0, 0)] = \{(0, 0)\}$ because there are no two non-zero real values a and b such that $a^2 + b^2 = 0$.

c) *Give a geometric description of $[(5, 11)]$.*

$[(5, 11)] = \{(a, b) \in \mathbb{R} \times \mathbb{R} : a^2 + b^2 = 146\}$ is the set of all points on the circle with center at the origin and radius $\sqrt{146}$.

Problem 4.45: *For some $n > 1$, let S denote the set of all real $n \times n$ matrices with real entries and let T denote the set of all invertible $n \times n$ matrices. Define a relation $\sim$ on S by $A \sim B$ provided there is a matrix $M \in T$ such that $A = MBM^{-1}$. Prove that $\sim$ is an equivalence relation on S.*

Proof. Let $A, B, C \in S$. Note that the identity matrix $I = I_n$ (the $n \times n$ matrix with 1's on the diagonal and 0's everywhere else) is certainly invertible, and that $IAI^{-1} = IAI = A$ implies $A \sim A$. So $\sim$ is reflexive on S.

Now suppose $A \sim B$. Then there exists $M \in T$ such that $A = MBM^{-1}$. Recall that for any invertible matrix M, M^{-1} is also invertible and has inverse $(M^{-1})^{-1} = M$. Thus,

$$M^{-1}AM = M^{-1}\left(MBM^{-1}\right)M = \left(M^{-1}M\right)B\left(M^{-1}M\right) = B,$$

and so $B = M^{-1}AM$ with $M^{-1} \in T$ implies $B \sim A$. Thus, $\sim$ is symmetric on S.

If $A \sim B$ and $B \sim C$, then there exist invertible matrices M and N such that $A = MBM^{-1}$ and $B = NCN^{-1}$. Thus,

$$A = MBM^{-1} = M\left(NCN^{-1}\right)M^{-1} = (MN)\,C\,(MN)^{-1}$$

since the product of two invertible matrices is invertible and $(MN)^{-1} = N^{-1}M^{-1}$. Thus, $A \sim C$ and $\sim$ is transitive. ∎

Problem 4.62: *Prove Theorem 4.8: If $n \in \mathbb{N}$, congruence modulo n is an equivalence relation on the set of integers.*

Proof. Let $n \in \mathbb{N}$ and $a, b, c \in \mathbb{Z}$. Then $a - a = 0$ and 0 is divisible by n. So $a \equiv a \mod n$, and the relation is reflexive.

Now suppose $a \equiv b \mod n$. Then $n|(a-b)$ so there exists an integer k such that $a - b = kn$. Then $b - a = -kn$ where $-k$ is an integer, and so $n|b - a$ and $b \equiv a \mod n$. Thus, the relation is symmetric.

If $a \equiv b \mod n$ and $b \equiv c \mod n$, then $n|(a-b)$ and $n|(b-c)$. Note that $a - c = (a-b) + (b-c)$ and so is the sum of two terms divisible by n. Thus $n|(a-c)$, and $a \equiv c \mod n$ implies the relation is transitive. ∎

Problem 4.64: *For the equivalence relation $a \equiv b \mod 9$ we have that for each natural number n, $[10^n] = [1]$.*

Proof. Since the given relation is an equivalence relation, it suffices to show that for each natural number n, $10^n \equiv 1 \mod 9$.

Let $S = \{n \in \mathbb{N} : 9|(10^n - 1)\}$. $1 \in S$ because 9 divides $10^1 - 1$. Assume $n \in S$. Then 9 divides $10^n - 1$, and so there exists an integer k such that $10^n - 1 = 9k$.

$$\begin{aligned} 10^{n+1} - 1 &= 10 \cdot 10^n - 1 \\ &= 9 \cdot 10^n + 10^n - 1 \\ &= 9 \cdot 10^n + 9k \\ &= 9(10^n + k), \end{aligned}$$

where the third inequality is due to the statement derived from the induction hypothesis. Hence $n + 1 \in S$. Therefore, by the Principle of Mathematical Induction, $S = \mathbb{N}$. ■