1

SOLUTIONS

Exam 1, Mathematics 109
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October 17, 2005

Name:
Student ID:
Section Number:

Note: There are 3 problems on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted.

I. (30 points)

(1) Write formally the following statement: " For all strictly positive real numbers x and y, the sum $x/y + y/x$ is at least 2".
(2) Write the formal negation of the statement in (1) above.
(3) Prove or disprove the statement in (1) above.

$x, y \in \mathbb{R}$

(1) $(\forall x)(\forall y)\ (x \in \mathbb{R}_{>0}) \wedge (y \in \mathbb{R}_{>0}) \longrightarrow x/y + y/x \geq 2$

(2) $(\exists x)(\exists y)\ (x \in \mathbb{R}_{>0}) \wedge (y \in \mathbb{R}_{>0}) \wedge (x/y + y/x < 2)$

(3) We will prove the statement in (1).

Let $x, y \in \mathbb{R}_{>0}$.

Then $\frac{x}{y} + \frac{y}{x} - 2 = \frac{x^2 + y^2 - 2xy}{xy} = \frac{(x-y)^2}{xy}$

Since $x, y \in \mathbb{R}_{>0}$, $(x-y)^2 \geq 0$ and $xy > 0$.

Therefore $\frac{(x-y)^2}{xy} \geq 0$. Therefore

$\frac{x}{y} + \frac{y}{x} - 2 \geq 0$. Therefore $\frac{y}{x} + \frac{x}{y} \geq 2$. $\square$

II. (30 points)

(1) Write formally the following statement: "If x is an irrational real number, then $x^2 + x$ is an irrational real number."

(2) Prove or disprove the statement in (1) above.

(1) $x \in \mathbb{R}$

$$(\forall x)\,(x \notin \mathbb{Q}) \rightarrow (x^2 + x \notin \mathbb{Q}).$$

(2) The statement in (1) is false. We will show this by proving that its negation is true. The negation of the statement above is

$$(\exists x)\,(x \notin \mathbb{Q}) \wedge (x^2 + x \in \mathbb{Q}).$$

Proof.

Let $x = \frac{-1+\sqrt{5}}{2}$. We claim that $(x \notin \mathbb{Q})$ and $(x^2 + x \in \mathbb{Q})$.

Step 1 We show that $\frac{-1+\sqrt{5}}{2} \notin \mathbb{Q}$ by contradiction.

Assume that $\frac{-1+\sqrt{5}}{2} \in \mathbb{Q}$. Then

$$\sqrt{5} = 2 \cdot \left(\frac{-1+\sqrt{5}}{2}\right) + 1 \in \mathbb{Q}.$$

Therefore, there exist $m, n \in \mathbb{N}$ with $\gcd(m,n) = 1$, such that $\sqrt{5} = m/n$. Consequently

$$5n^2 = m^2$$

Therefore $5 \mid m^2 \Rightarrow 5 \mid m \rightarrow \exists\, k \in \mathbb{Z},\ m = 5k.$

Therefore $5n^2 = 25k^2$.

Consequently $n^2 = 5k^2 \Rightarrow 5 \mid n^2 \Rightarrow 5 \mid n$.

Therefore $\left.\begin{array}{l} 5 \mid m \\ 5 \mid n \end{array}\right\} \Rightarrow \left.\begin{array}{l} 5 \mid \gcd(m,n) \\ \gcd(m,n) = 1 \end{array}\right\} \Rightarrow 5 \mid 1$.

This is false.

Consequently $\cancel{\sqrt{5} \in}$ $\frac{-1+\sqrt{5}}{2} \notin \mathbb{Q}$.

<u>Step 2</u>

$$\left(\frac{-1+\sqrt{5}}{2}\right)^2 + \left(\frac{-1+\sqrt{5}}{2}\right) = \frac{1-2\sqrt{5}+5}{4} + \frac{-1+\sqrt{5}}{2} =$$

$$= \frac{1-\cancel{2\sqrt{5}}+5+(-2)\cancel{+2\sqrt{5}}}{4} =$$

$$= \frac{4}{4} = 1 \in \mathbb{Q}.$$

$\square$

III. (40 points)

(1) Show that the following propositional expression is a tautology.

$$(P \longrightarrow Q) \longrightarrow ((P \vee Q) \longleftrightarrow Q)$$

(2) Use the tautology in (1) above to show that if A and B are two subsets of a universal set $\mathcal{U}$ such that $A \subseteq B$, then $A \cup B = B$.

(3) Prove or disprove the following statement: "If A and B are two subsets of a universal set $\mathcal{U}$, then $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$."

(1) Draw the truth table for

$$(P \to Q) \to ((P \vee Q) \leftrightarrow Q)$$

and conclude that this propositional expression is true for all possible truth values of the propositional variables P and Q.

(2) Let $x \in \mathcal{U}$. Let $P(x)$ and $Q(x)$ be the propositions

$P(x) : x \in A$; $Q(x) : x \in B$.

Please note that $(A \subseteq B) \Leftrightarrow (\forall x)\, P(x) \to Q(x)$.

$(A \cup B = B) \Leftrightarrow (\forall x)\big(P(x) \vee Q(x)\big) \leftrightarrow Q(x)$.

Therefore, we have the equivalence:

$$\big((A \subseteq B) \to (A \cup B = B)\big) \Leftrightarrow (\forall x)\big(P(x) \to Q(x)\big) \to \big((P(x) \vee Q(x)) \leftrightarrow Q(x)\big)$$

Now, the right hand side is true as a consequence of (1). Consequently the left hand side is also true. Therefore:

$$(A \subseteq B) \to (A \cup B = B). \quad \square$$

(3) Formally, the statement is:

$$(\forall A)(\forall B)\ \mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$$

where A, B are subsets of the universal set $\mathcal{U}$.

The statement above is <u>false</u>. We will show this by proving that its negation is true. Its negation is equivalent to the following.

$$(\exists A)(\exists B)\ \mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B).$$

<u>Proof.</u>

Let $\mathcal{U} = \mathbb{N}$, and let $A = \{1,2,3\}$, $B = \{4,5,6\}$.

Then $A \cup B = \{1,2,3,4,5,6\}$. Consequently

$$\{3,4\} \subseteq A \cup B \Rightarrow \{3,4\} \in \mathcal{P}(A \cup B) \quad (*)$$

However $\{3,4\} \not\subseteq A$ and $\{3,4\} \not\subseteq B$. Therefore $\{3,4\} \notin \mathcal{P}(A)$ and $\{3,4\} \notin \mathcal{P}(B)$.

Consequently $\{3,4\} \notin \mathcal{P}(A) \cup \mathcal{P}(B) \quad (\substack{*\\*})$

$$\left.\begin{array}{l}(*)\\(\substack{*\\*})\end{array}\right\} \Rightarrow \mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B). \Rightarrow$$

$$\Rightarrow \mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B) \quad \square$$