Final Exam, Mathematics 109
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Name:
Student ID:
Section Number:

Note: There are 5 problems on this exam, worth 40 points each. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted. Good luck !

I. (40 pts.) Let $X = [0, \infty)$ and let $R \subseteq X \times X$ be the relation on X (i.e. from X to X) defined as follows

$$R = \{(x, y) \mid x, y \in [0, \infty)\,, \quad x^2 + x - y^2 = 0\}\,.$$

(1) Is R an equivalence relation on X? Justify.
(2) Show that R is a functional relation on X.
(3) Write the explicit expression of $f(x)$, $x \in X$, for the function

$$f : X \longrightarrow X$$

determined by the functional relation R above.
(4) Show that the function f is bijective.
(5) Determine the inverse $f^{-1} : X \longrightarrow X$ of the bijective function f.

II. (40 pts.)

(1) Solve (i.e. determine the full solution set of) the following system of linear congruences.

$$\begin{aligned} x &\equiv 1 \mod 2 \\ x &\equiv 2 \mod 3 \\ x &\equiv 4 \mod 5 \end{aligned}$$

(2) Show that if $x \in \mathbb{Z}$ is a solution to the system above, then

$$x^3 \equiv -1 \mod 30 \,.$$

III. (40 pts.) Let $\{f_n\}_{n\geq 1}$ be the Fibonacci sequence given recursively by $f_1 = f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$, for all $n \in \mathbb{N}$.

(1) Prove that f_{3n} is even, f_{3n+1} is odd, and f_{3n+2} is odd, for all natural numbers n.

(2) Prove that for each natural number n, we have

$$\gcd(f_n, f_{n+1}) = 1\,.$$

(3) Prove that for all natural numbers n we have an equality

$$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1\,.$$

IV. (40 pts.) The universe for the variable p below is the set of prime numbers $\mathcal{P}$, while the universe for the variables n and m is the set of natural numbers $\mathbb{N}$.

(1) Write the negation of the following statement

$$(\forall\, p)(\forall\, n)(\exists\, m) \quad (\gcd(p,n) = 1) \longrightarrow (n^m \equiv 1 \mod p)\ .$$

(2) Prove or disprove the statement in (1) above.

V. (40 pts.)

(1) Prove that if A and B are two subsets of a given universal set $\mathcal{U}$, then

$$\overline{(\overline{A} \cup B)} \cap A = A \setminus B \,.$$

Here, as usual, $\overline{C}$ denotes the universal complement of the set C inside $\mathcal{U}$.

(2) For each natural number $n \in \mathbb{N}$, let

$$A_n := \left(-\frac{1}{n}\,,\; \frac{2n-1}{n}\right) = \left\{x \,|\, x \in \mathbb{R}, \quad -\frac{1}{n} < x < \frac{2n-1}{n}\right\} \,.$$

Determine $\cup_{n \in \mathbb{N}} A_n$ and $\cap_{n \in \mathbb{N}} A_n$.