Final Exam
Mathematics 200A (Group Theory)
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Name:
Student ID:
Section Number:

Note: There are 4 problems on this exam. Each of them is worth 25 points. You will not receive credit unless you show all your work. No books or lecture notes are permitted.

I. Let p be a prime number and let $\mathcal{Z} := \{z \in \mathbb{C}^\times \mid z^{p^n} = 1, \text{ for some } n \in \mathbb{Z}_{\geq 0}\}$. For each $k \in \mathbb{Z}_{\geq 0}$, let $\mathcal{H}_k = \{z \in \mathcal{Z} \mid z^{p^k} = 1\}$. Prove the following.

(1) $\mathcal{Z}$ and $\mathcal{H}_k$ are subgroups of $(\mathbb{C}^\times, \cdot)$, for all $k \in \mathbb{Z}_{\geq 0}$.
(2) $\mathcal{H}_k \leq \mathcal{H}_m$ if and only if $k \leq m$.
(3) $\mathcal{H}_k$ is cyclic for all $k \in \mathbb{Z}_{\geq 0}$. Construct a generator for $\mathcal{H}_k$.
(4) Every proper subgroup H of $\mathcal{Z}$ equals $\mathcal{H}_k$ for some $k \in \mathbb{Z}_{\geq 0}$.
(5) $\mathcal{Z}$ is not finitely generated.

II. Let q and p be two distinct prime numbers.

(1) Show that $\mathrm{Aut}(C_p \times C_q) \xrightarrow{\sim} \mathrm{Aut}(C_p) \times \mathrm{Aut}(C_q)$.

(2) Show that if q divides $p-1$, then there exist non–trivial group morphisms

$$\phi : C_q \longrightarrow \mathrm{Aut}(C_p \times C_q)\,.$$

(3) For each non–trivial group morphism ϕ as in (2) above, let

$$H_\phi := (C_p \times C_q) \rtimes_\phi C_q\,.$$

Show that the groups H_ϕ described above are isomorphic to one another.

(4) If $q = 2$ and p is an odd prime, show that for all ϕ as above we have

$$H_\phi \xrightarrow{\sim} D_{4p}\,.$$

Note: C_n denotes the cyclic group of order n; $\rtimes_\phi$ denotes the semidirect product along ϕ; D_{2n} denotes the dihedral group of order $2n$.

III. For all $n \in \mathbb{N}$, let F_n denote the free group on n generators.

(1) Show that if $n \geq 2$ then the center $Z(F_n)$ of F_n is trivial.
(2) Prove that $F_1 \times F_1$ is not a free group.
(3) Are there non–trivial and non–injective group morphisms $\phi : \mathbb{Z} \longrightarrow F_2$? Justify your answer. (Here, $\mathbb{Z}$ denotes the usual additive group $(\mathbb{Z}, +)$.)

IV. Let G be a group and H a normal subgroup of G, $H \lhd G$, with the property that $[G : H] = 4$. Let K be an arbitrary subgroup of G, $K \leq G$. Show that if the quotient group $K/K \cap H$ is non–trivial, then it is isomorphic to either C_2, or $C_2 \times C_2$, or C_4.