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SOLUTIONS.

Exam 2, Mathematics 109
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Name:
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Note: There are 3 questions on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted.

I. (35 points)

Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be the functions given by

$$f(x) = \begin{cases} x-1, & \text{if } x \text{ is even;} \\ x+1, & \text{if } x \text{ is odd.} \end{cases} \qquad g(x) = \begin{cases} x+1, & \text{if } x \text{ is even;} \\ x-1, & \text{if } x \text{ is odd.} \end{cases}$$

(1) Is f bijective? Justify your answer.
(2) If the answer to question (1) above is affirmative compute the inverse function $f^{-1} : \mathbb{Z} \to \mathbb{Z}$.
(3) Compute $f \circ g$.

Important note: You may use the fact that if $x \in \mathbb{Z}$, then x is even if and only if there exists $k \in \mathbb{Z}$ such that $x = 2k$ and x is odd if and only if there exists $k \in \mathbb{Z}$ such that $x = 2k+1$.

(1) <u>f is bijective.</u> The proof is based on the

<u>Remark</u> ~~f(x)~~ $\forall x \in \mathbb{Z}$, $f(x)$ odd $\Leftrightarrow$ x even

We will show that $\forall y \in \mathbb{Z}, \exists! x \in \mathbb{Z}, f(x) = y$.

Let $y \in \mathbb{Z}$. If <u>y is even</u>, then the <u>Remark</u> above implies that

$f(x) = y \Leftrightarrow x$ ~~even~~ odd $\wedge f(x) = y \Leftrightarrow x$ ~~odd~~ $\wedge\ x+1 = y \Leftrightarrow \underline{x = y-1}$.

If <u>y is odd</u>, then the Remark above implies that

$f(x) = y \Leftrightarrow x$ even $\wedge f(x) = y \Leftrightarrow x$ odd $\wedge\ x-1 = y \Leftrightarrow \underline{x = y+1}$

(2) The proof in (1) implies that

$$f^{-1} : \mathbb{Z} \to \mathbb{Z}, \quad f^{-1}(y) = \begin{cases} y-1, & y \text{ even} \\ y+1, & y \text{ odd} \end{cases}$$

(3) $f \circ g : \mathbb{Z} \to \mathbb{Z}$ $\quad (f\circ g)(x) = f(g(x)) = \begin{cases} f(x+1), & x \text{ even} \\ f(x-1), & x \text{ odd} \end{cases} = \begin{cases} (x+1)+1, & x \text{ even} \\ (x-1)-1, & x \text{ odd.} \end{cases}$ $\square$

II. (35 points)

Let $\{F_n\}_{n\geq 1}$ be the Fibonacci sequence defined inductively by

$$F_1 = F_2 = 1, \qquad F_{n+2} = F_{n+1} + F_n, \quad \forall n \in \mathbb{N}.$$

(1) Prove that for all natural numbers n we have

$$\sum_{i=1}^{n} F_i^2 = F_n \cdot F_{n+1}.$$

(2) Prove that for all natural numbers n we have

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

Important note: α and β are the roots of the quadratic equation $x^2 = x + 1$.

Proof. (1) "Weak" Induction. $P(n): \sum_{i=1}^{n} F_i^2 = F_n \cdot F_{n+1}$

Base case: $P(1): F_1^2 = F_1 \cdot F_2 \iff 1^2 = 1 \cdot 1 \iff 1 = 1$ true.

Induction Step: Let $n \in \mathbb{N}$. Assume $P(n)$. Then

$P(n+1) \iff \sum_{i=1}^{n+1} F_i^2 = F_{n+1} \cdot F_{n+2} \iff \left(\sum_{i=1}^{n} F_i^2\right) + F_{n+1}^2 = F_{n+1} \cdot F_{n+2} \iff$

$\underset{P(n)}{\iff} F_n \cdot F_{n+1} + F_{n+1}^2 = F_{n+1} \cdot F_{n+2} \iff F_{n+1} \cdot (F_n + F_{n+1}) = F_{n+1} \cdot F_{n+2}$

$\underset{(\text{definition})}{\iff} F_{n+1} \cdot F_{n+2} = F_{n+1} \cdot F_{n+2}$. true $\square$

(2) "Strong" Induction. $P(n): F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$

Base cases:

$P(1): 1 = \dfrac{\alpha - \beta}{\alpha - \beta} \iff 1 = 1$ ✓

$P(2): 1 = \dfrac{\alpha^2 - \beta^2}{\alpha - \beta} = \dfrac{\cancel{(\alpha - \beta)}(\alpha + \beta)}{\cancel{\alpha - \beta}} \iff 1 = \alpha + \beta = \dfrac{1+\sqrt{5}}{2} + \dfrac{1-\sqrt{5}}{2} = \dfrac{1}{2} + \dfrac{1}{2} = 1$ ✓

Inductive Step: Let $n \in \mathbb{N}$. Assume $P(1) \wedge \ldots \wedge P(n)$. Then

$P(n+1) \iff F_{n+1} = \dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \iff F_n + F_{n-1} = \dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \iff \dfrac{\alpha^n - \beta^n + \alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} = \dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$

$\iff \dfrac{\alpha^{n-1}(1+\alpha) - \beta^{n-1}(1+\beta)}{\alpha - \beta} = \dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \iff$

$\dfrac{\alpha^{n-1}(\alpha^2) - \beta^{n-1} \cdot (\beta^2)}{\alpha - \beta} = \dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$

$\Updownarrow$

$\dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$ $\square$

III. (30 points)

(1) Let A, B, C be three subsets of a universal set X. Prove that

$$(A \cap B = A \cap C) \wedge (A \cup B = A \cup C) \implies B = C.$$

(2) If the following statement is true, prove it; if it is false, negate it and prove its negation.

$$\exists y \in \mathbb{R}, \quad \forall x \in \mathbb{R}, \quad xy = 1.$$

(1) Assume $(A \cap B = A \cap C) \wedge (A \cup B = A \cup C)$.

Step 1 First, we will prove that $B \subseteq C$.

Let $x \in B$.

Case 1 $\quad x \in A \Rightarrow x \in A \cap B \xrightarrow{\text{Hypothesis}} x \in A \cap C \Rightarrow x \in A \wedge x \in C \Rightarrow x \in C$.

Case 2 $\quad x \notin A \Rightarrow x \in A \cup B \wedge x \notin A \xrightarrow{\text{Hyp.}} x \in A \cup C \wedge x \notin A \Rightarrow x \in C$.

Consequently $B \subseteq C$.

Step 2 The proof of $C \subseteq B$ is identical to the proof above because the hypothesis is a statement which is symmetric in B and C.

(2) The statement is false. Its negation is equivalent to:

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, xy \neq 1.$$

We will prove the statement above:

Let $y \in \mathbb{R}$, let $x := 0$. Then $x \in \mathbb{R}$ and $xy = 0 \cdot y = 0 \neq 1$.