Final Exam, Math. 20C
Prof. Cristian D. Popescu
March 15, 2010

Name:
Student ID:
Section number or TA name:

Note: There are 6 problems on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted.

I. (40 points) Let $f(x, y)$ be the function given by

$$f(x, y) = x^4 + y^4 - 4xy + 1 .$$

(1) Determine the critical points of $f(x, y)$.
(2) Classify the critical points of $f(x, y)$ into local maxima, local minima, and saddle points, respectively.
(3) Determine the global maximum and minimum points of $f(x, y)$ on the (closed and bounded) domain

$$D = \{(x, y) \,|\, 0 \leq x \leq 3 , \quad 0 \leq y \leq 2\} .$$

(4) Use the method of Lagrange multipliers to find the minimum value of $f(x, y)$ along the curve given by the equation $4xy = 1$. Is there a maximum value for f along the given curve ?

II. (30 points)

(1) Compute the integral

$$\iint_D f(x,y)dxdy\,,$$

where $f(x,y) = y\cos(x^2)$ and D is the plane region situated above the x–axis and bounded by the curves $y=0$, $x=y^2$, and $x=9$.

(2) Find the volume of the solid which is inside the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=4$.

III. (30 points) Let $\mathcal{C}$ be the space curve whose vectorial equation is given by

$$\vec{r}(t) = t^2\,\vec{i} + \ln t\,\vec{j} + 2t\,\vec{k}\,, \qquad t \in (0,\,\infty)\,.$$

(1) Find the length of the arc of $\mathcal{C}$ corresponding to $1 \le t \le 2$.
(2) Find the unit tangent vector $\vec{T}$, unit normal vector $\vec{N}$ to the curve $\mathcal{C}$ at the point $P(1,0,2)$.
(3) Find the equation of the plane passing through $P(1,0,2)$ and containing the vectors $\vec{T}$ and $\vec{N}$ computed in (2) above.

IV. (30 points) Let $g : \mathbb{R}^3 \to \mathbb{R}$ be the function given by given by

$$g(x, y, z) = e^{(x+y+z)} \cdot \cos(x + y + z).$$

(1) Find the rate of change in the value of g at the point $P(0, 0, 0)$ in the direction toward the point $Q(1, 1, 1)$

(2) Find the direction in which the value of g decreases the fastest at the point $P(0, 0, 0)$ and find the rate at which it decreases in that direction.

(3) Find the equation of the plane tangent to the surface $g(x, y, z) = 1$ at the point $P(0, 0, 0)$.

V. (30 points) Let $(\pi_1) : 2x + y + z - 1 = 0$ and $(\pi_2) : x + 2y - 2z = 0$ be the equations of two planes (π_1) and (π_2).

(1) Find the angle $\theta \in [0, \pi)$ determined by the two planes above.
(2) Find the point of intersection between the plane (π_1) and the line which passes through $P_0(1, 0, 1)$ and is perpendicular on (π_1).
(3) Find the distance between the point $P_0(1, 0, 1)$ and the plane (π_1).

VI. (40 points) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function given by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

(1) Compute $\lim_{(x,y)\to(0,0)} f(x,y)$ (if the limit exists) and determine whether the function f is continuous at $(0,0)$ or not.

(2) Compute $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ (if they exist) and determine whether the function f is differentiable at $(0,0)$ or not.