Math 20F - Linear Algebra

Midterm Examination #1 - ANSWERS

February 5, 2003

Write your name or initials on every page before beginning the exam.

You have 50 minutes. There are six problems. You may not use notes, textbooks, or other materials during this exam. You must show your work in order to get credit. Good luck!

Name:

Student ID:

Tuesday section time:

I	
II	
III	
IV	
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Total	

The first problem asks you to circle all the correct answers. In each case, there are four possibilities (a)-(d). You should circle each of (a)-(d) that is a correct answer. ***This could be none of them, or one, or two, or three, or all four of them.***

I. For this problem, A, B and C are arbitrary, **non-singular** matrices.

1. $(ABC)^{-1}$ must be equal to which of the following?

(a) $C^{-1}B^{-1}A^{-1}$ (b) $(BC)^{-1}A^{-1}$ (c) $(C^TB^TA^T)^{-1}$ (d) $A^{-1}B^{-1}C^{-1}$
ANSWERS: (a) and (b)

2. $\det(A^TB)$ must equal which of the following?

(a) $\det(A)\det(B)$ (b) $\dfrac{\det(B)}{\det(A)}$ (c) $\det(A^{-1})\det(B)$ (d) $\dfrac{\det(B)}{\det(A^{-1})}$
ANSWERS: (a) and (d)

3. $A^{-1}AB$ must equal which of the following?

(a) ABA^{-1} (b) $AA^{-1}B$ (c) B (d) $BB^{-1}B$.
ANSWERS: (b), (c) and (d)

II. Indicate whether the statements are true or false. (To be true, they must *always* be true.).

<u>**F**</u> **1.** If A is an $m \times n$ matrix and B is an $n \times r$ matrix, then the matrix products AB and A^TB^T are defined.

<u>**T**</u> **2.** If A and B are both non-singular, then AB is non-singular.

<u>**Not graded**</u> **3.** If AB is non-singular, then both A and B are non-singular. *(Answer: true if A and B are also assumed to be square. False otherwise.)*

<u>**F**</u> **4.** If A is a square matrix and $\det(A) = 0$, then $A^2 = 0$, where 0 is the matrix containing all zeros.

<u>**T**</u> **5.** If A is a square matrix and $A^2 = 0$, then $\det(A)$ equals zero.

III.

1. State the definition of "$\mathbf{x}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$."

ANSWER:
"There are scalars $a_1, \ldots, a_n$ such that $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$."

2. State the definition of "$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a spanning set for V."

ANSWER: "Every $\mathbf{x}$ in V is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$"

IV. 1. Find the solution set S to the following set of equations. How many solutions does it have?

$$\begin{aligned} x_1 + x_3 + x_4 &= 1 \\ 2x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_2 + x_3 &= 1 \end{aligned}$$

ANSWER: (Sketch) Set up the augmented matrix as

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \end{array}\right)$$

Row operations show this has row echelon form

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 \end{array}\right)$$

(You might instead derive RREF if you wish.) Back substitution gives that the solution set is

$$S = \{(1+\alpha, \alpha/2, -2\alpha, \alpha) : \alpha \in \mathbb{R}\}$$

The equations have infinitely many solutions.

V. Compute the determinant of A, where $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$.

How many solutions does the homogenous equation $A\mathbf{x} = \mathbf{0}$ have?

ANSWER: (Sketch) The determinant is equal to 3. (Best way to show this is by row operations!) Because the determinant is non-zero, the homogenous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. That is, it has only one solution, namely $\mathbf{x} = \mathbf{0}$.

IV. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$.

Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a spanning set for $\mathbb{R}^3$?

ANSWER: Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 5 \\ 1 & 3 & 0 \end{pmatrix}$$

Using row operations, this is equivalent to

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 5 \\ 1 & 3 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 3 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & -5 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Since, all three variables are lead variables (or, since the determinant of A can be seen to be non-zero), the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution. Thus, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent. Since $\mathbb{R}^3$ has dimension three, these three linearly independent vectors are a basis for $\mathbb{R}^3$.