Final Exam
Mathematics 200B
Dr. Cristian D. Popescu
March 21, 2006

Name:
Student ID:

Note: There are 4 problems on this exam. Each of them is worth 50 points. You will not receive credit unless you show all your work. No books or lecture notes are permitted.

I.

(1) Prove that for any $\mathbb{Z}$–module $K \neq 0$, one has $\mathrm{Hom}_{\mathbb{Z}}(K, \mathbb{Q}/\mathbb{Z}) \neq \{0\}$.
(2) Use (1) above to show that a morphism of $\mathbb{Z}$–modules

$$A \xrightarrow{j} B$$

is injective if and only if the corresponding morphism

$$\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \xrightarrow{j^*} \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$$

is surjective.
(3) Use (1) and (2) above to show that if R is an arbitrary ring ($0_R \neq 1_R$) and M is a left R–module, then M is a flat left R–module if and only if $M' := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is an injective right R–module.
(4) Use (3) above to show that $\mathbb{Z}/n\mathbb{Z}$ is an injective $\mathbb{Z}/n\mathbb{Z}$–module.

Note. The module $M' := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is endowed with the right R–module structure canonically induced by the given left R–module structure of M.

II. Let $f = X^4 + X^3 + X^2 + X + 1$ in $\mathbb{Z}[X]$.

(1) Show that the $\mathbb{Z}$–algebra $(\mathbb{Z}[X]/(f)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a field.
(2) Is $\mathbb{Z}[X]/(f)$ a projective $\mathbb{Z}$–module ? Justify.
(3) Is $\mathbb{Z}[X]/(f)$ a flat $\mathbb{Z}$–module ? Justify.
(4) Is $\mathbb{Z}[X]/(f)$ a projective $\mathbb{Z}[X]$–module ? Justify.
(5) Is $\mathbb{Z}[X]/(f)$ a flat $\mathbb{Z}[X]$–module ? Justify.

Note. The $\mathbb{Z}$– and $\mathbb{Z}[X]$–module structures on $\mathbb{Z}[X]/(f)$ to which (1)–(5) above refer are the canonical ones.

III. Let R be the quadratic integer ring $\mathbb{Z}[\sqrt{-5}]$. Consider the following ideals in R: $I_2 = (2, 1+\sqrt{-5})$, $I_3 = (3, 2+\sqrt{-5})$, and $I_3' = (3, 2-\sqrt{-5})$.

(1) Prove that 2, 3, $1+\sqrt{-5}$ and $1-\sqrt{-5}$ are irreducible elements in R, no two of which are associated in divisibility in R.
(2) Use (1) to produce two essentially distinct factorizations of 6 as a product of irreducible elements in R and conclude that R is not a UFD.
(3) Prove that I_2, I_3 and I_3' are prime ideals in R.
(4) Is I_2 a principal ideal in R ? Justify.

Hint. It may be useful to show that R and $\mathbb{Z}[X]/(X^2+5)$ are isomorphic as rings.

IV.

(1) Let R be a PID. Show that any prime ideal $\mathfrak{P}$ in the polynomial ring $R[X]$ is either principal or of the form $\mathfrak{P} = (\pi, f)$, where π is an irreducible element in R and f is a polynomial in $R[X]$ whose image $\widehat{f}$ via the canonical surjective ring morphism

$$R[X] \twoheadrightarrow R/(\pi)[X]$$

is irreducible in $R/(\pi)[X]$.

(2) Use (1) above to construct a non–principal prime ideal in $\mathbb{Z}[X]$. Is the ideal you have just constructed a maximal ideal ? Justify.

(3) Use (1) above to show that every prime ideal in the ring $\mathbb{Z}[X, Y]/(X^2Y - Y)$ is generated by at most three elements.

Hint. It may be useful to view $R[X]$ as a subring of its ring of fractions $Q[X]$, where $Q := Q(R)$ is the total field of fractions of R.