

Make-up Final Exam
Mathematics 200B
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May 16, 2006

Name:
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Note: There are 4 problems on this exam. Each of them is worth 50 points. You will not receive credit unless you show all your work. No books or lecture notes are permitted.

I. Let R be an arbitrary ring and P a non-trivial projective left R -module.

- (1) Show that for all non-trivial left R -modules M , we have $\text{Hom}_R(P, M) \neq 0$.
- (2) Show that a morphism $A \xrightarrow{j} B$ of left R -modules is surjective if and only if the induced morphism (of abelian groups) $\text{Hom}_R(P, A) \xrightarrow{j_*} \text{Hom}_R(P, B)$ is surjective.
- (3) In the context described in (2) above, is it true that j is an isomorphism if and only if j_* is an isomorphism? Justify.

Note. As usual, the morphism j_* is defined by $j_*(f) = j \circ f$, for all elements $f \in \text{Hom}_R(P, A)$.

II. Let $f \in \mathbb{Z}[X]$ be a non-zero polynomial.

- (1) Show that if the leading coefficient of f is in $\mathbb{Z}^\times := \{+1, -1\}$, then the \mathbb{Z} -module $\mathbb{Z}[X]/(f)$ is free of finite rank.
- (2) Show that if the leading coefficient of f is not in \mathbb{Z}^\times , then the \mathbb{Z} -module $\mathbb{Z}[X]/(f)$ is not finitely generated.
- (3) In the context of (2) above, is the \mathbb{Z} -module $\mathbb{Z}[X]/(f)$ free? Justify.
- (4) In the context of (2) above, is the \mathbb{Z} -module $\mathbb{Z}[X]/(f)$ flat? Justify.

Note. By definition, the leading coefficient of a non-zero polynomial of variable X is the coefficient of the highest degree monomial which appears in f . For example, the leading coefficient of $f = -2X^2 + 1$ is -2 .

III. Let $\mathbb{Z}[i]$ be the usual Gaussian ring. Prove the following.

- (1) The ring $\mathbb{Z}[i]/(2)$ is a local ring which is not a field. Also, prove that the quotient of $\mathbb{Z}[i]/(2)$ by its (unique) maximal ideal is a field of cardinality 2.
- (2) Show that if p is a prime such that $p \equiv 3 \pmod{4}$, then $\mathbb{Z}[i]/(p)$ is a field of cardinality p^2 .
- (3) Show that if p is a prime such that $p \equiv 1 \pmod{4}$, then $\mathbb{Z}[i]/(p)$ is isomorphic to a direct sum of two fields of cardinality p each.

Note. In approaching the above exercise, you may use (without providing a proof) the following (elementary) **Theorem.** *An odd prime p is equal to a sum of two (integral) squares if and only if $p \equiv 1 \pmod{4}$.*

IV. Let X, Y be two independent variables over an arbitrary field F . Prove the following.

- (1) The rings $R_1 := F[X, Y]/(Y^2 - X)$ and $R_2 := F[X, Y]/(Y^2 - X^2)$ are not isomorphic.
- (2) The ring $R_1 \otimes_F L$ is an integral domain for every field L containing F .
- (3) Is it true that the ring $R_1 \otimes_{F[X]} L$ is an integral domain for every field L containing $F[X]$? Justify.