MATH 21C FINAL EXAM SOLUTIONS, LINDBLAD, FALL 02

1. (a). $\mathbf{r}(1) = \mathbf{i} + \mathbf{k}$. $\mathbf{r}'(t) = 2t\mathbf{i} - 2t\mathbf{j} + 3t^2\mathbf{k}$. $\mathbf{r}'(1) = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. Tangent line given by $\mathbf{r}_0(t) = (1+2t)\mathbf{i} + (-2t)\mathbf{j} + (1+3t)\mathbf{k}$.
(b).

$$\int_0^1 \sqrt{4t^2+4t^2+9t^4}\,dt \;=\; \int_0^1 t\sqrt{8+9t^2}\,dt \;=\; \frac{1}{27}(8+9t^2)^{3/2}\Big|_0^1 = \frac{17^{3/2}-8^{3/2}}{27}.$$

2. $\overrightarrow{PQ} = \langle 0,1,-1\rangle$, $\overrightarrow{PR} = \langle 2,0,-1\rangle$ Normal to the plane is

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 2 & 0 & -1 \end{vmatrix} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

Equation of plane is $-(x-0) - 2(y-1) - 2(z-1) = 0$ which can be written as $x + 2y + 2z = 4$.
(b). area is $|\overrightarrow{PQ}\times\overrightarrow{PR}|/2 = \sqrt{(-1)^2+(-2)^2+(-2)^2}/2 = 3/2$.
(c).

$$\cos\angle P \;=\; \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} \;=\; \frac{\langle 0,1,-1\rangle\cdot\langle 2,0,-1\rangle}{|\langle 0,1,-1\rangle||\langle 2,0,-1\rangle|} \;=\; \frac{1}{\sqrt{2}\sqrt{5}}.$$

$$\cos\angle Q \;=\; \frac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{|\overrightarrow{QP}||\overrightarrow{QR}|} \;=\; \frac{\langle 0,-1,1\rangle\cdot\langle 2,-1,0\rangle}{|\langle 0,-1,1\rangle||\langle 2,-1,0\rangle|} \;=\; \frac{1}{\sqrt{2}\sqrt{5}}.$$

$$\cos\angle R \;=\; \frac{\overrightarrow{RP}\cdot\overrightarrow{RQ}}{|\overrightarrow{RP}||\overrightarrow{RQ}|} \;=\; \frac{\langle -2,0,1\rangle\cdot\langle -2,1,0\rangle}{|\langle -2,0,1\rangle||\langle -2,1,0\rangle|} \;=\; \frac{4}{5}.$$

3.

$$\nabla T = \left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$\nabla T(1,2,2) \;=\; (1,2,2)\,/27.$$

Direction from $(1,2,2)$ to $(2,1,3)$ is $\langle 1,-1,1\rangle$. Unit vector in this direction is $\mathbf{u} = \langle 1,-1,1\rangle/\sqrt{3}$,

$$D_{\mathbf{u}}T(1,2,2) \;=\; \nabla T(1,2,2)\cdot\mathbf{u} = \frac{1}{27\sqrt{3}}.$$

(b). The direction of greatest increase of T is ∇T which is a positive multiple of $-\langle x,y,z\rangle$. But the vector $-\langle x,y,z\rangle$ is the vector from (x,y,z) to the origin $(0,0,0)$.

4. (a). Two planes are parallel if and only if their normals are parallel. The normal to a level surface $f(x,y,z)$ =constant is ∇f. Hence the normal to the ellipsoid at the point (x,y,z) is $\langle 8x, 2y, 2z\rangle$. The plane $x+2y-z=0$ has normal $\langle 1,2,-1\rangle$. Hence the tangent plane to the ellipsoid is parallel to the plane $x+2y-z=0$ if and only if the vectors $\langle 8x,2y,2z\rangle$ and $\langle 1,2,-1\rangle$ are parallel, that is for some value of λ,

$$8x=\lambda, \qquad 2y=2\lambda, \qquad 2z=-\lambda.$$

This gives $(x,y,z)=(\lambda/8,\ \lambda,\ -\lambda/2)$. Plugging in to the equation of the ellipsoid we get

$$4 \;=\; 4(\lambda/8)^2 \;+\; \lambda^2 \;+\; (\lambda/2)^2 \;=\; \frac{21}{16}\lambda^2,$$

so $\lambda=\pm 8/\sqrt{21}$, and the points are $(1/\sqrt{21},\ 8/\sqrt{21},\ -4/\sqrt{21})$ and $(-1/\sqrt{21},\ -8/\sqrt{21},\ 4/\sqrt{21})$

(b). Since the normal is $\langle 1,2,-1\rangle$, the equation of the tangent plane is $x+2y-z$ =constant. We evaluate the constant by plugging in the points we just found and get tangent plane $x+2y-z=\sqrt{21}$ at the first point and $x+2y-z=-\sqrt{21}$ at the second point.

5. (a). $\nabla f=\langle 2x+3y, 10y+3x\rangle=\langle 0,0\rangle$ if and only if $(x,y)=(0,0)$. This is the only critical point.

$$D \;=\; \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy}\end{vmatrix} \;=\; \begin{vmatrix} 2 & 3 \\ 3 & 10\end{vmatrix} \;=\; 11>0,$$

since $f_{xx}>0$, the point $(0,0)$ is a local minimum.

(b). Using Lagrange multipliers we get the equations

$$\begin{cases} 2x+3y=\lambda 2x \\ 10y+3x=\lambda 2y \\ x^2+y^2=4. \end{cases}$$

We can eliminate λ by multiplying the first equation by y and the second equation by x and subtracting, to get $3y^2+2xy-10xy-3x^2=0$ which becomes $3y^2-8xy-3x^2=0$. This factors as $(3y+x)(y-3x)=0$, so $y=3x$ or $x=-3y$. In the first case we get $x^2+(3x)^2=4$ so $(x,y)=(\sqrt{2/5},3\sqrt{2/5})$ and $f=126/5$, and in the second case we get $(-3y)^2+y^2=4$ so $(x,y)=(-3\sqrt{2/5},\sqrt{2/5}$ and $f=90/5$. The maximum value of f on the circle is $126/5$ and the minimum value is $90/5$.

(c). The absolute max of f is $126/5$ and the absolute min is $f(0,0)=8$.

6. We will maximize $V=xyz$ over the tetrahedron given by $x\geq 0$, $y\geq 0$, $z\geq 0$, and $2x+2y+z\leq 120$. We first look for interior critical points and get $\nabla V=\langle yz,xz,xy\rangle=\langle 0,0,0\rangle$ which implies that at least two of x,y,z vanish, so (x,y,z) is on the boundary of the tetrahedron where $V=0$. The boundary of the tetrahedron consists of four triangles, three on the coordinate planes where $V=0$, and one

on the plane $2x + 2y + z = 120$ with $x \geq 0$, $y \geq 0$ and $z \geq 0$. The edges of this triangle lie in the coordinate planes and so $V = 0$ on the edges. It is now clear that the maximum value of V on the tetrahedron must lie on this triangle $G = 2x + 2y + z = 120$ with $x > 0, y > 0, z > 0$. Using Lagrange multipliers to maximize $V = xyz$ subject to these conditions, we get

$$\begin{cases} yz = 2\lambda \\ xz = 2\lambda \\ xy = \lambda z \\ 2x + 2y + z = 120. \end{cases}$$

Hence

$$yz = xz = 2xy$$

Since x, y and z are positive, we can divide to get $y = x$ and $z = 2x$. Solving the constraint gives $2x + 2x + 2x = 120$ so $x = y = 20$ and $z = 40$. Then $V = 20^2(40) = 16000$ cubic inches.

Comment. In the original version of the test, the additional constraint $0 \leq x \leq y \leq z$ was also imposed. The max point we just found does satisfy these additional inequalities and so gives the correct answer. However, had you worked on the original problem without realizing that there is some symmetry, you would have tried to maximize $V = xyz$ in the tetrahedron $0 \leq x \leq y \leq z$ with $2x+2y+z \leq 120$. Again there is no interior critical point. The boundary of this tetrahedron is made up of the sides

(I)	$x = 0,$	$0 \leq y \leq z,$	$2y + z \leq 120,$
(II)	$x = y,$	$0 \leq y \leq z,$	$4y + z \leq 120,$
(III)	$z = y,$	$0 \leq x \leq y,$	$2x + 3y \leq 120,$
(IV)	$2x + 2y + z = 120$	$0 \leq x \leq y \leq z.$	

On side (I), $V = 0$.

To deal with side (II), we express everything in terms of the variables y and z, which lie in the triangle $0 \leq y \leq z$ and $4y + z \leq 120$. We have $V = y^2 z$. The only critical point of this function is $(y, z) = (0, 0)$ where $V = 0$. To find the max of V on side (II) we have to check the max on its boundary, which consists of three segments in the lines (i) $y = 0$, (ii) $y = z$ and (iii) $4y + z = 120$. For (i), since $y = 0$ we have $V = 0$. For (ii), since $z = y$ we can write V in terms of y alone as $V = y^3$. The range of y is given by $0 \leq y$ and $5y \leq 120$, so $0 \leq y \leq 24$, and the max of V on this line segment is $V = 24^3$. For (iii) we have $4y + z = 120$, and $0 \leq y \leq z$. We can write V in terms of y alone as $V = y^2(120 - 4y)$ and the range of y is $0 \leq y \leq 120 - 4y$, so $0 \leq y \leq 24$. Differentiating we get $V' = 240y - 12y^2$ so $y = 0$ or $y = 20$ which give $V = 0$ and $V = 20^2(40)$. The endpoint $y = 24$ gives $V = 24^3$, but the largest of these values is $V = 20^2(40)$.

To deal with side (III), we express everything in term of the variables x, y, which lie in the triangle $0 \leq x \leq y$ and $2x + 3y \leq 120$. We have $V = xy^2$. The only critical point of this function is $(0,0)$. The boundary of the triangle consists of three segments in the lines (i) $x = 0$, (ii) $x = y$ and (iii) $2x + 3y = 120$. On (i) $V = 0$. On (ii) we write V in terms of x alone as $V = x^3$, and the bounds on x are given by $0 \leq x \leq (120 - 2x)/3$ so $0 \leq x \leq 24$. The max value of V on this line segment is 24^3. On (iii) $2x + 3y = 120$ and we will write V in terms of y. (You can write V in terms of x if you prefer.) The bounds on y are $0 \leq 60 - 3y/2 \leq y$ so $20 \leq y \leq 40$. We have $V = (60 - 3y/2)y^2$ and $V' = 120y - 9y^2/2$. This vanishes at $y = 0$ and $y = 80/3$, which give $V = 0$ and $V = (20)(80/3)^2 = 20^2(40)(8/9)$.

To deal with the triangle side (IV), we will look for max points using Lagrange multipliers, but we also have to check the edges of the triangle. However, each edge of side (IV) is also an edge of one of the other sides, and we have already computed the max of V on all these, so we just need to find the "interior critical points" on side (IV) using Lagrange multipliers. We want to maximize $V = xyz$ subject to the constraint $G = 2x + 2y + z = 120$ and satisfying the inequalities $0 < x < y < z$. The equations are

$$\begin{cases} yz = 2\lambda \\ xz = 2\lambda \\ xy = \lambda z \\ 2x + 2y + z = 120. \end{cases}$$

so

$$yz = xz = 2xy$$

Since x, y, z are positive, we have $y = x$ and $z = 2x$. Solving the constraint gives $2x + 2x + 2x = 120$ so $x = y = 20$ and $z = 40$. Then $V = 20^2(40)$.

Taking the maximum over the interior of the tetrahedron and all the faces, we get maximum of V is $20^2(40) = 16000$ cubic inches.

7. (a). Set

$$D \;=\; \{(x,y) : 0 \leq y \leq 1, \quad 1 \leq x \leq 1/y\}.$$

The integral equals $\iint_D x^2 e^{-x^2}\, dA$.

(b). After sketching the domain, we can write it as

$$D \;= \{(x,y) : 1 \leq x \leq \infty, \quad 0 \leq y \leq 1/x\}.$$

Then

$$\iint_D x^2 e^{-x^2}\, dA \;=\; \int_1^\infty \int_0^{1/x} x^2 e^{-x^2}\, dxdy \;=\; \int_1^\infty x^2 e^{-x^2} y\Big|_0^{1/x}\, dx$$

$$= \int_1^\infty x e^{-x^2/2}\, dx \;=\; -\frac{e^{-x^2}}{2}\Big|_1^\infty \;=\; \frac{e}{2}.$$

8. Notice that

$$\iint_R (1+y)\cos(x^2+y^2)\,dA = \iint_R \cos(x^2+y^2)\,dA \ + \ \iint_R y\cos(x^2+y^2)\,dA,$$

and the second integral vanishes by symmetry. This is helpful, but we don't need it. Using polar coordinates. The integral we want to compute is

$$\int_0^{2\pi}\int_1^2 (1+r\sin\theta)\ \cos(r^2)\,r\,drd\theta \ = \ \int_0^{2\pi}\int_1^2 r\cos(r^2) \ + \ \sin\theta r^2\cos(r^2)\,drd\theta.$$

We cannot integrate this, so we switch the order of integration to get

$$\int_1^2\int_0^{2\pi} r\cos(r^2)+\sin\theta\,r^2\cos(r^2)\,d\theta dr \ = \ \int_1^2 \left(r\cos(r^2)\theta-\cos\theta\,r^2\cos(r^2)\right)\Big|_0^{2\pi} dr$$
$$= \int_1^2 2\pi r\cos r^2\,dr \ = \ \pi\sin(r^2)\big|_1^2 = \pi(\sin 4-\sin 1).$$

9. The surface is $z = 8 - x - 2y$ and its area is

$$\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA \ = \ \iint_D \sqrt{1+(-1)^2+(-2)^2}\,dA \ = \ \iint_D \sqrt{6}\,dA$$
$$= \int_0^1\int_{2x^2}^{1+x^2} \sqrt{6}\,dydx \ = \ \sqrt{6}\int_0^1 1+x^2-2x^2\,dx \ = \ \sqrt{6}(x-x^3/3)\big|_0^1 = \frac{2\sqrt{6}}{3}$$

10. Describing D as a type I region we get $0 \le y \le 1,\ 2x \le y \le 4-2x$. Then the volume of E is

$$\iint_D x-y+20\,dA \ = \ \int_0^1\int_{2x}^{4-2x} x+20-y\,dydx$$
$$= \int_0^1 (x+20)y-y^2/2\big|_{2x}^{4-2x} dx = \int_0^1 (x+20)(4-4x)-(2-x)(4-2x)+2x^2\,dx$$
$$= \int_0^1 -4x^2-76x+80\,dx \ = \ \frac{-4}{3}-38+80.$$