Math 21C, Midterm 1 Solutions, Winter 2003..

1. A curve is given by the parametric equations

$$x = t^4 - 2t^2, \qquad y = t^3.$$

Determine all the points (x, y) where the curve has vertical tangent, and all the points (x, y) where it has horizontal tangent.

Solution.

$$\frac{dx}{dt} = 4t^3 - 4t = 4t(t^2 - 1), \qquad \frac{dy}{dt} = 3t^2.$$

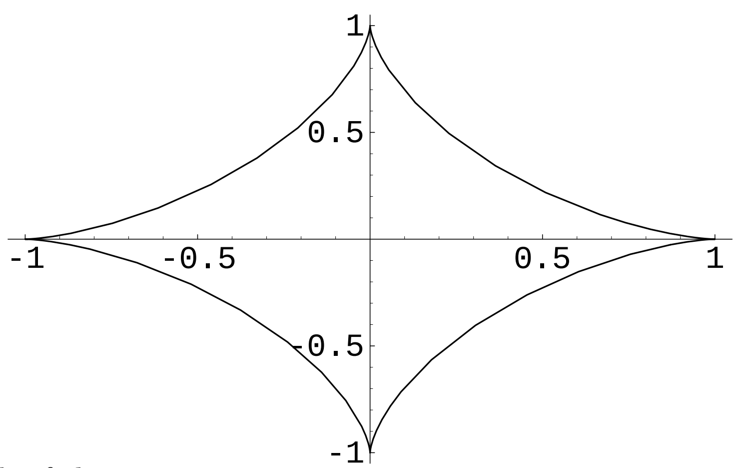
$$\text{Slope} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{4t(t^2-1)} = \frac{3t}{4(t^2-1)}.$$

When $t \to \pm 1$ the slope tends to infinity so the tangent is vertical. When $t = 0$ the slope is zero so the tangent is horizontal. Plugging in $t = -1, +1, 0$ give points:
$(-1, 1)$, $(-1, -1)$ tangent is vertical
$(0, 0)$ tangent is horizontal.

2. Calculate the length of the curve given by parametric equations

$$x = \sin^3 t, \qquad y = \cos^3 t, \qquad 0 \le t \le 2\pi.$$

This curve is shown below.



Solution. The length of the curve is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\left(3\sin^2 t \cos t\right)^2 + (-3\cos^2 t \sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^4 t \cos^2 t + 9\cos^4 t \sin^2 t}\, dt = 3\int_0^{2\pi} |\sin t \cos t| \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= 3\int_0^{2\pi} |\sin t \cos t|\, dt = \frac{3}{2}\int_0^{2\pi} |\sin 2t|\, dt.$$

Now to integrate $|\sin 2t|$ we notice that this function is periodic with period $\pi/2$. This is suggested by the symmetry of the curve shown in the diagram. Hence we have

$$L = \frac{3}{2}\int_0^{2\pi} |\sin 2t|\, dt = \frac{3}{8}\int_0^{\pi/2} \sin 2t\, dt$$
$$= -\frac{3}{16}\cos 2t|_0^{\pi/2} = -\frac{3}{16}(-1-1) = \frac{3}{8}.$$

3. Consider the two lines L_1 and L_2 given parametrically by

$$L_1 : \begin{cases} x = 3+3s \\ y = 1+2s \\ z = 3+s \end{cases} \qquad L_2 : \begin{cases} x = -2+t \\ y = 1-t \\ z = 4-t \end{cases} .$$

(a). Calculate the coordinates (x, y, z) where the lines L_1 and L_2 cross.

Solution. At the point (x, y, z) of intersection

$$x = 3+3s = -2+t, \qquad y = 1+2s = 1-t, \qquad z = 3+s = 4-t.$$

From the equation for x we get $t = 3s+5$, from the equation for y we get $t = 1-(1+2s) = -2s$. Hence combining these $t = 3s+5 = -2s$ and $5s+5 = 0$ so $s = -1$. Plug the value $s = -1$ into the first line to get

$$(x, y, z) = (0, -1, 2).$$

Notice that using the relation that at the intersection, $t = 3s+5 = 2$ we can check that the point we just found is also on the second line, by plugging $t = 2$ into the equation of the second line.

(b). Calculate the angle between the lines L_1 and L_2.

Solutions. The directions of the lines can be read off from the equations

$$\text{first line } : \langle 3, 2, 1\rangle, \qquad \text{second line } : \langle 1, -1, -1\rangle.$$

Since $\langle 3, 2, 1\rangle \cdot \langle 1, -1, -1\rangle = 3-2-1 = 0$, the lines are perpendicular; the angle is $90°$.

4. Calculate the equation of the plane which contains the three points $P = (2, 1, 0)$, $Q = (0, 1, 1)$, $R = (-3, 0, 0)$.

Solutions. $\overrightarrow{PQ} = \langle -2, 0, 1\rangle$ and $\overrightarrow{PR} = \langle -5, -1, 0\rangle$. Hence A normal to the plane is given by

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 1 \\ -5 & -1 & 0 \end{vmatrix} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}.$$

Hence the equation of the plane is

$$\langle 1, -5, 2\rangle \cdot \langle x-2, y-1, z\rangle \;=\; 0,$$

or

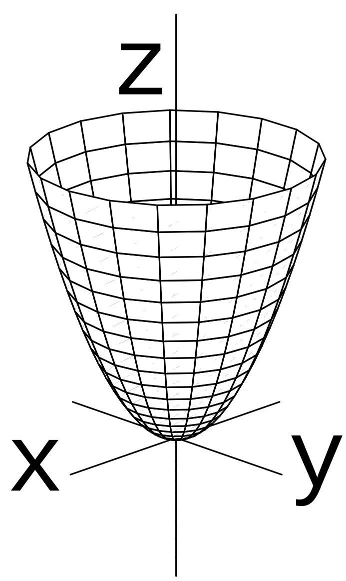
$$x - 5y + 2z + 3 = 0.$$

You can check the answer by plugging in the three points.

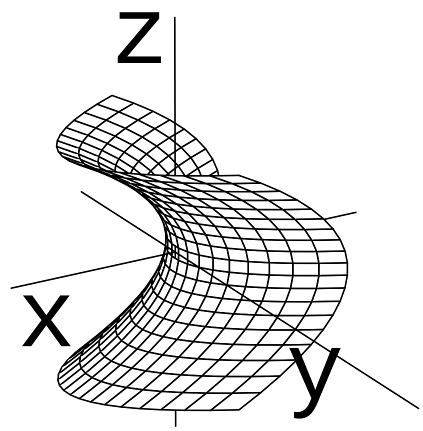
5. For the four surfaces below, choose the correct equation from the following list and write it next to the surface.

$$\begin{array}{lll}
x^2 - y^2 + z^2 = 1, & x = y^2 + z^2, & x^2 + y^2 = 1 \\
x^2 - y^2 - z^2 = 1, & y = x^2 + z^2, & z = x^2 \\
x^2 + y^2 - z^2 = 1, & z = x^2 - y^2, & y = z^2 \\
x^2 + y^2 - z^2 = 1, & x = z^2 - y^2, & x^2 - y^2 = 1
\end{array}$$

(a). This equation does not appear on the list, but a good guess would be $z = x^2 + y^2$.

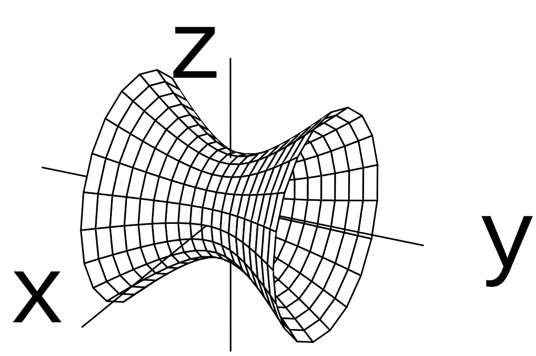


(a)



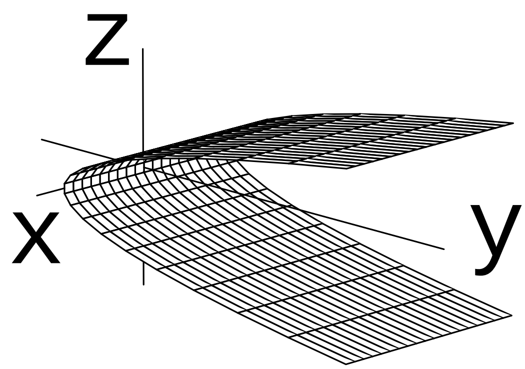
(b)

(b). $x = z^2 - y^2$.



(c)

(c). $x^2 - y^2 + z^2 = 1$.



(d)

(d). $y = z^2$.