FACT SHEET FOR MATH 20C

1. Motion in Three-Space

•. If $\mathbf{r}(t)$ is the position vector of a particle in three-space at time t, then its velocity and acceleration are given by

$$\mathbf{v}(t) = \mathbf{r}'(t), \qquad \mathbf{a}(t) = \mathbf{v}'(t).$$

Besides, the unit tangent and normal vectors to the trajectory of the particle are given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$$

The arc-length for $t \in [a,\, b]$ is given by

$$\int_b^a ||\mathbf{r}'(t)||\, dt\,.$$

2. Tangent Planes and Linear Approximations

•. The equation of the tangent plane for the graph of a function $z = f(x, y)$ at the point (a, b) is given by

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

•. The tangent plane to the surface $f(x, y, z) = 0$ at the point (a, b, c) is given by the equation

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0$$

•. The linear approximation of a function $f(x, y)$ at the point (a, b) in its domain is given by

$$f(a + h, b + k) \approx f(a, b) + f_x(a, b)h + f_y(a, b)k$$

•. A function $f(x, y)$ is differentiable at a point (a, b) in its domain if $f_x(a, b)$ and $f_y(a, b)$ exist and

$$\lim_{(h,k)\to(0,0)} \frac{f(a + h, b + k) - (f(a, b) + f_x(a, b)h + f_y(a, b)k)}{\sqrt{h^2 + k^2}} = 0.$$

3. Gradients and Directional Derivatives

•. Given a function f and a vector $\mathbf{v}$, the derivative of f with respect to $\mathbf{v}$ at the point P is given by

$$D_{\mathbf{v}} f(P) = \nabla f(P) \cdot \mathbf{v}$$

•. Given a function f, and a vector $\mathbf{v}$, the directional derivative of f in the direction of $\mathbf{v}$ at the point P is given by

$$D_{\mathbf{u}} f(P) = \nabla f(P) \cdot \mathbf{u}$$

where

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

4. Optimization in Several Variables

•. A point P in the domain of a function f is called a critical point if one of the following is satisfied.

(1) $\nabla f(P) = 0$, i.e., all the partial derivatives are zero.
(2) At least one of the partial derivatives does not exist.

•. If (a, b) is a critical point of a function $f(x, y)$ in two variables, we define the *discriminant* as

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$$

Theorem 4.1 (Second Derivative Test)**.** *Let $P = (a, b)$ be a critical point of $f(x, y)$. Assume that the second derivatives f_{xx}, f_{yy}, f_{xy} are continuous near P. Then*

(1) *If $D > 0$, then $f(a, b)$ is a local minimum or local maximum.*
(2) *if $D < 0$, then $f(a, b)$ is a saddle point.*
(3) *if $D = 0$, the test is inconclusive.*

Furthermore,

(4) *if $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.*
(5) *if $D > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum.*

Theorem 4.2 (Lagrange Multipliers)**.** *Assume that f and g are differentiable functions. If f has a local minimum or maximum on the constraint $g = 0$ at the point P, and if $\nabla g(P) \neq \boldsymbol{0}$, then there is a scalar λ such that*

$$\nabla f(P) = \lambda \nabla g(P)$$