Exam 2, Mathematics 103B
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Name:
Student ID:

Note: There are 3 problems on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted.

I. (35 points)

It is easy to check that the map $\phi : \mathbb{R}[X] \to \mathbb{C}$, given by $\phi(f) = f(i)$, for every polynomial $f \in \mathbb{R}[X]$, is a ring morphism.

(1) Use long division in $\mathbb{R}[X]$ to show that $\ker(\phi) = \langle x^2 + 1 \rangle$.
(2) Use the fundamental isomorphism theorem to show that $I := \langle x^2 + 1 \rangle$ is a maximal ideal in $\mathbb{R}[X]$.
(3) Give an example of a prime ideal in $\mathbb{R}[X]$ which is not maximal. Justify your answer.

II. (30 points)

Let $\mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ and $\mathbb{Q}[\sqrt{3}] := \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$. It is easy to show that these two sets are subrings of the field of complex numbers, with the usual addition and multiplication.

(1) Show if $\phi : \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{3}]$ is a ring morphism, then

$$\phi(x) = x, \text{ for all } x \in \mathbb{Q}.$$

(2) Show that $\sqrt{2} \notin \mathbb{Q}[\sqrt{3}]$.

(3) Show that there does not exist a ring morphism $\phi : \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{3}]$.

III. (35 points)

Let F be a field and R be a commutative ring.

(1) Prove that any ring morphism $\phi : F \to R$ must be injective.
(2) Prove that the kernel of any ring morphism $\phi : R \to F$ must be a prime ideal of R.
(3) Prove that for any ring morphism $\phi : R \to F$, the field of quotients $Q(R/\ker\phi)$ is isomorphic to a subfield of F.