SOLUTIONS

Exam 2, Mathematics 109
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Name:
Student ID:

Note: There are 3 questions on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted.

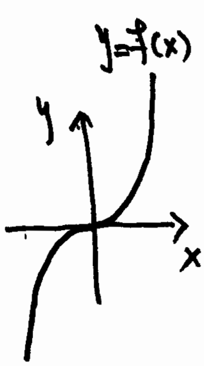
I. (35 points)

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : [0, \pi/2] \to \mathbb{R}$ be the functions given by

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0; \\ -x^2, & \text{if } x < 0. \end{cases} \qquad g(x) = \sin x.$$

(1) Is f bijective? Justify your answer. If the answer is affirmative compute the inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$.
(2) Is g bijective? Justify your answer.
(3) Compute $f \circ g$.
(4) Does $g \circ f$ make sense? Justify your answer.

(1) f is bijective. Proof: We will show that $\forall y \in \mathbb{R}, \exists!\, x \in \mathbb{R}$ such that $f(x) = y$.



Let $y \in \mathbb{R}$. Note that $f(x) = x^2 \geq 0, \forall x \geq 0$ and $f(x) = -x^2, \forall x < 0$.

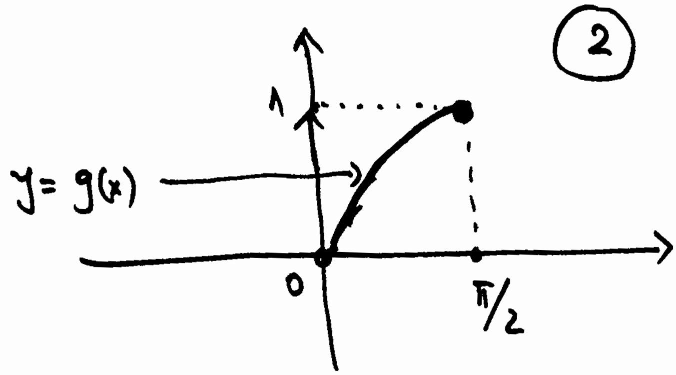
Consequently, if $y \geq 0$, then $f(x) = y \iff (x \geq 0) \wedge (x^2 = y) \iff \boxed{x = \sqrt{y}}$

if $y < 0$, then $f(x) = y \iff (x < 0) \wedge (-x^2 = y) \iff \boxed{x = -\sqrt{-y}}$. $\square$.

The calculations above imply that $f^{-1} : \mathbb{R} \to \mathbb{R}$ is given by $f^{-1}(y) := \begin{cases} \sqrt{y}, & y \geq 0 \\ -\sqrt{-y}, & y < 0 \end{cases}$

(2) g is not bijective because it is not surjective. Proof: Let $y = 2 \in \mathbb{R}$. Then, since $g(x) = \sin x \in [-1, 1], \forall x \in [0, \pi/2]$, $(\cancel{\exists})\, x \in \mathbb{R}$, such that $g(x) = 2$. $\square$

(2)



Remark.

g is injective.

(3)

$$\begin{tikzcd} {[0, \pi/2]} \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$f \circ g : [0, \pi/2] \longrightarrow \mathbb{R}.$, $(f \circ g)(x) = f(g(x)) = \begin{cases} g(x)^2, & g(x) \geq 0 \\ -g(x)^2, & g(x) < 0 \end{cases}$

However, since $g(x) = \sin x \geq 0$, $\forall\, x \in [0, \pi/2]$,

we have $(f \circ g)(x) = g(x)^2 = (\sin x)^2$, $\forall\, x \in [0, \pi/2]$

(4) $g \circ f$ does not make sence because

$\text{Codomain}(f) = \mathbb{R} \neq [0, \pi/2] = \text{Domain}(g).$

Remark. (exercise). Replace g by $G : \mathbb{R} \to \mathbb{R}$
$G(x) = \sin x$.

Compute $f \circ G$ and $G \circ f$.

II. (30 points)

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows.

$$f(x) = \begin{cases} x^2 \cdot \cos x, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

(1) Use the ε–δ definition of continuity written in quantifier language to prove that f is continuous at $x = 0$.

(2) Write down (without proof) a quantified ε–δ statement saying that "the limit of $f(x)$ when x approaches $\pi/2$ is not equal to 1."

(1) "f is continuous at $x=0$ if $\lim_{x\to 0} f(x) = f(0) = 0$".

$\Updownarrow$

$\forall \varepsilon > 0, \exists \delta > 0, \forall x, 0 < |x-0| < \delta \Rightarrow |f(x) - 0| < \varepsilon.$

Proof.

$|\cos x| \leq 1 \; \forall x \in \mathbb{R}$

Let $\varepsilon > 0$. Let $\delta := \sqrt{\varepsilon}$. Let $x \in \mathbb{R}$, such that $0 < |x| < \sqrt{\varepsilon}$. Then $|f(x) - 0| = |x^2 \cdot \cos x| = |x|^2 \cdot |\cos x| \leq |x|^2 < (\sqrt{\varepsilon})^2 = \varepsilon$. $\square$.

(2) First, we will write a quantified ε-δ statement saying that $\lim_{x \to \pi/2} f(x) = 1$.

$P: \forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R} \; 0 < |x - \pi/2| < \delta \Rightarrow |f(x) - 1| < \varepsilon.$

Now, we negate statement P:

$\neg P: \exists \varepsilon > 0, \forall \delta > 0, \exists x \in \mathbb{R} \; (0 < |x - \pi/2| < \delta) \wedge (|f(x) - 1| \geq \varepsilon)$

$\square$

III. (35 points)

Let $f : X \to Y$ be a function. For any subset $A \subseteq X$, we define

$$f(A) := \{f(x) \mid x \in A\}.$$

Note that $f(A)$ is a subset of Y, for all $A \subseteq X$.

(1) Prove that $f(A \cup B) = f(A) \cup f(B)$, for any two sets $A, B \subseteq X$.

(2) Is it true that for any function f as above and any sets $A, B \subseteq X$, we have $f(A \cap B) = f(A) \cap f(B)$? If the answer is affirmative, give a proof; if the answer is negative, provide a counterexample.

(1) Let $y \in Y$.

$\underline{y \in f(A \cup B)} \iff \exists x \in A \cup B,\ f(x) = y \iff$

$\iff (\exists x \in A,\ f(x) = y) \vee (\exists x \in B,\ f(x) = y)$

$\iff (y \in f(A)) \vee (y \in f(B)) \iff \underline{y \in f(A) \cup f(B)}.$

Consequently $f(A \cup B) = f(A) \cup f(B)$. $\square$

(2) The answer is negative. Here is a counterexample:

$X = \{1, 2\}$, $Y = \{1\}$

$f : X \to Y$ $f(1) = 1$, $f(2) = 1$

$A = \{1\}$, $B = \{2\}$.

$\Downarrow$ $\Downarrow$

$f(A) = \{1\}$ $f(B) = \{1\}$ $\Rightarrow f(A) \cap f(B) = \{1\}$

On the other hand, $A \cap B = \emptyset \Rightarrow f(A \cap B) = \emptyset$

$\Rightarrow f(A \cap B) \neq f(A) \cap f(B).$

<u>Remark</u> The inclusion $f(A \cap B) \subseteq f(A) \cap f(B)$ holds always. The inclusion $f(A) \cap f(B) \subseteq f(A \cap B)$ is false, in general (see counterexample above.)